CHAPTER 89

STRUCTURE OF FREQUENCY DOMAIN MODELS FOR RANDOM WAVE BREAKING

James T. Kirby[1] and James M. Kaihatu[2]

Abstract

We consider the form of a breaking wave dissipation term for use in spectral or stochastic wave evolution models. A time-domain Boussinesq model is tested for accuracy in modelling evolution of second and third moment statistics in shoaling and breaking waves. The structure of the dissipation term in the time domain is then used to infer the corresponding structure of the term in the frequency domain. In general, we find that the dissipation coefficient is distributed like $1/S_\eta(f)$, where $S_\eta(f)$ is the spectral density of the surface displacement η. This implies an f^2 dependence for the coefficient in the inner surfzone, as opposed to a constant distribution over frequency as suggested by Eldeberky and Battjes (1996).

Introduction

Recently, there have been several suggestions on how to structure the breaking wave dissipation term in spectral or stochastic wave evolution models, with the principle question being how to structure the dissipation coefficient as a function of frequency. As an example, Mase and Kirby (1992) developed evolution equations for the shoreward (x direction) evolution of component amplitudes $A_n(x)$, where the A_n are related to surface displacement η according to

$$\eta = \sum_{n=1}^{\infty} \frac{A_n(x)}{2} e^{i \int k_n(x)dx - \omega_n t} + c.c. \tag{1}$$

where k_n is related to ω_n through a suitable wave dispersion relation. Index n is the analog in the discrete spectral representation to a continuous dependence on frequency f in the continuous spectrum representation, and the two representations will be used interchangeably below. Restricting our attention here to wave breaking effects, the evolution equations may be written as

$$A_{n,x} = -\alpha_n A_n + \cdots \tag{2}$$

[1] Professor, Center for Applied Coastal Research, University of Delaware, Newark, DE 19716, USA. kirby@coastal.udel.edu

[2] Oceanographer, Oceanography Division (Code 7322), Naval Research Laboratory, Stennis Space Center, MS 39529-5004

where the omitted terms are related to shoaling and nonlinear interaction effects. Mase and Kirby (1992) proposed a form for α_n given by

$$\alpha_n = \alpha_0 + (f_n/\bar{f})^2\alpha_1 \tag{3}$$

with

$$\alpha_0 = F\beta \tag{4}$$

$$\alpha_1 = (1 - F)\beta\frac{\bar{f}^2\sum_n |A_n|^2}{\sum_n f_n^2|A_n|^2} \tag{5}$$

and where β is determined from a bulk dissipation model such as the one of Thornton and Guza (1983). Based on analysis of laboratory data and numerical results, Mase and Kirby chose to set $F = 0.5$, indicating a dissipation term with a partial dependence on the square of the frequency. They also found that choosing $F = 0.0$, corresponding to an f^2 dependence for the entire dissipation term, destroyed the tail of the computed power spectrum in very shallow water but had little impact on the evolution of spectral shape away from the shallowest measuring gages.

Eldeberky and Battjes (1996) have suggested that a similar formulation corresponding to the choice $F = 1.0$, spreading the dissipation term uniformly over all frequencies, should be utilized, and showed that an adequate description of power spectrum evolution was obtained in several simulations of field data. Eldeberky and Battjes did not consider the effect of this choice on the evolution of higher statistical moments. More recently, Chen et al (1996; referred to as CGE) have examined a number of laboratory and field cases. They have shown that the estimates of power spectrum evolution are relatively insensitive to the choice of F, with error-minimizing F values occupying the entire range $0 \leq F \leq 1$ for various field and laboratory cases. Aside from the Mase and Kirby case, error measures changed by as little as 20% over the entire range of values. In contrast, all data sets support the choice of $F = 0.0$ when error measures based on third-moment statistics are introduced, with the exception of the Mase-Kirby data set. (This last discrepancy is fairly weak, however). There is a clear trend towards increasing error with increasing F in most data sets.

In this talk, the problem of determining the form of the spectral dissipation term is approached from a different direction. Instead of considering a bulk energy decay and an arbitrary distribution of dissipation over f, we instead consider the structure of the dissipation term in a time-dependent Boussinesq model setting, and consider the contribution of the term to energy loss and the structure of that loss in the frequency domain. This loss is then related to the spectral evolution equations, and the form of the dissipation term is deduced. We find that, in general, the dissipation coefficient is distributed more or less as $S_\eta(f)^{-1}$, where $S_\eta(f)$ is the power spectrum of the surface displacement η. For the case of a smooth spectrum, this result indicates an f^2 dependence in the dissipation coefficient in the surfzone, consistent with CGE's results with $F = 0.0$. The conclusions here are based on an examination of the Mase and Kirby (1992) data set, which is well modelled by the chosen time-domain breaking wave model.

Time-Domain Model for Wave Breaking

The literature includes examples of a number of formulations for computing wave breaking in the context of Boussinesq wave models. Three examples include an eddy viscosity model (Zelt, 1991), a surface roller model with an assumed two-layer velocity profile (Schäffer et al, 1993), and a surface roller model with a computed horizontal profile (Svendsen et al, 1996). Schematically, the eddy viscosity model may be written as

$$u_t + uu_x + g\eta_x + \text{dispersive terms} - (\nu_b u_x)_x = 0 \tag{6}$$

The simple roller model of Schäffer et al (1993) is written in the context of a model for total volume flux, and may be written schematically as

$$P_t + \left(P^2/H\right)_x + gH\eta_x + \text{dispersive terms} + R_x = 0 \tag{7}$$

where

$$R = \int_{-h}^{\eta} \left(u_{\text{rot}}^2 - u_{\text{irrot}}^2\right) dz \tag{8}$$

and

$$H = h + \eta \tag{9}$$

The more complex roller model of Svendsen et al (1996) is essentially of the same form. While these models differ in both form and theoretical intent, it may be shown that the numerical representations of each dissipative term are similar. In particular, the contribution of each dissipation term is highly localized in space and time, since it is concentrated on the front face of the breaking wave. Further, the contribution of each dissipation term is about the same size, since each successfully calibrated model must extract the same amount of energy. An illustration of this fact is given by Figure 9 of Svendsen et al (1996).

Since our primary goal below is to examine the spectral signature of the wave energy decay, we can conclude that, due to the structural similarity of the various breaking models in the time domain, it should not matter which of the existing models is used to perform the analysis. The analysis will be based on the eddy viscosity model of Zelt (1991) for two reasons: it is already incorporated in an existing time-domain Boussinesq model (Wei and Kirby, 1996), and it is simple to interpret the terms in the eddy viscosity model in terms of measured sea surface elevations, as described below.

Leading-Order Energy Balance

Let $r = R/H$ or $-\nu_b u_x$ represent the breaking-induced momentum deficit per unit depth. Then, each of the models above may be written in the form

$$u_t + uu_x + g\eta_x + \text{dispersive terms} + r_x = 0 \tag{10}$$

Neglecting nonlinear and dispersive effects, we have

$$u_t + g\eta_x + r_x = 0 \tag{11}$$

$$\eta_t + (hu)_x = 0 \tag{12}$$

Multiplying (11) by $\rho h u$, (12) by $\rho g \eta$ and adding then gives

$$E_t + F_x = -\epsilon_b \tag{13}$$

where

$$E = \frac{1}{2}\rho g \eta^2 + \frac{1}{2}\rho h u^2 \tag{14}$$

is the local energy density/unit surface area,

$$F = \rho g h u \eta \tag{15}$$

is the flux of energy in the x direction, and

$$\epsilon_b = \rho h u r_x \tag{16}$$

is the local rate of energy decay. Each of these quantities may be averaged in time, yielding (for a stationary wave process)

$$\langle F \rangle_x = -\langle \epsilon_b \rangle \tag{17}$$

The average or bulk energy decay $\langle \epsilon_b \rangle$ can be specified according to models such as the one of Thornton and Guza (1983), which models the data set considered below quite well. Each of the quantities in (17) may be thought of as the sum of contributions from each frequency to the total value; i.e.,

$$\langle F \rangle = \sum_n F_n; \qquad \langle \epsilon_b \rangle = \sum_n \epsilon_{bn} \tag{18}$$

We will attach a meaning to each of these component terms in the analysis below.

In order to proceed further, we need to choose a model to evaluate ϵ_b. This will be done using the Zelt (1991) eddy viscosity model.

Eddy Viscosity Model

The eddy viscosity appearing in (6) is written by Zelt (1991) as

$$\nu_b = -\ell^2 u_x; \qquad \ell = B\gamma(h + \eta) \tag{19}$$

where $\gamma = 2$ is a mixing length parameter determined by Heitner & Housner (1970) and chosen so that the resulting model correctly predicts the width of a hydraulic jump. The factor B is given by

$$B = \begin{cases} 1 & ; \quad u_x \le 2u_x^* \\ u_x/u_x^* - 1 & ; \quad 2u_x^* < u_x \le u_x^* \\ 0 & ; \quad u_x > u_x^* \end{cases} \tag{20}$$

and provides a somewhat smoothed onset of breaking dissipation when the local breaking criterion is exceeded. The breaking criterion is given in terms of a critical velocity divergence, chosen to be

$$u_x^* = -0.3\sqrt{g/h} \tag{21}$$

The accuracy of this formulation in modelling breaking random waves will be illustrated below. The model is particularly useful to us in the present context because the terms in the model may be evaluated (to leading order) using time derivatives of the surface elevation. Using $u_x \approx -\eta_t/h$, we get

$$\nu_b = B\gamma^2 h^2 u_x \approx B\gamma^2 h\eta_t \tag{22}$$

$$B = \begin{cases} 1 & ; \quad \eta_t \geq 2\eta_t^* \\ \eta_t/\eta_t^* - 1 & ; \quad \eta_t^* \leq \eta_t < 2\eta_t^* \\ 0 & ; \quad \eta_t < \eta_t^* \end{cases} \tag{23}$$

$$\eta_t^* = 0.3\sqrt{gh} \tag{24}$$

Finally, the instantaneous energy dissipation may be written as

$$\epsilon_b = -\rho h u \left(\nu_b u_x\right)_x \approx -\rho \left(\frac{\eta}{h}\right) \left(\nu_b \eta_t\right)_t \tag{25}$$

The advantage of this formulation is clear in the context of evaluating experimental results, since the energy loss term that would be predicted by the numerical model may be deduced directly from the measured data. Thus, in order to evaluate dissipation effects, We may proceed without actually running the model in the majority of cases, provided that the model as formulated is known to be an accurate predictor of the wave field in sample representative cases.

Laboratory Data

The experimental data considered here is taken from Run 2 of Mase and Kirby (1992). The present results have been reproduced for a number of other data sets, and a more comprehensive view of the study will be published elsewhere. The single case shown here suffices as an indication of the results for a wide range of conditions studied to date.

Figure 1 shows a schematic of the experimental facility. The experimental wave conditions correspond to a Pierson-Moskowitz spectrum generated in $47cm$ of water, with a peak frequency $\bar{f} = 1Hz$ and a significant wave height of $6cm$. Waves were measured using capacitance wave gages at twelve stations across the 1:20 beach profile. Data for the analysis below is taken from the measurement at the $h = 10cm$ depth. This depth corresponds to a point where the probability of breaking is increasing rapidly but the saturated inner surfzone has not yet been established.

A sample of 20 seconds of measured and computed time series of surface elevations at $h = 10cm$ is shown in Figure 2, which indicates an accurate reproduction of wave heights and phases in the numerical model. Computations were performed using the extended fully-nonlinear Boussinesq model code of Wei et al (1995), which is capable of propagating waves in the large water depths used in this experiment. Second and third moment statistics were computed based on the entire experimental run, covering about 800 wave periods. Figure 3 shows the evolution of significant wave height up the beach slope and through the surfzone, while Figure 4 shows the evolution of skewness and asymmetry. Reproduction of measured values by the numerical model is good in both cases.
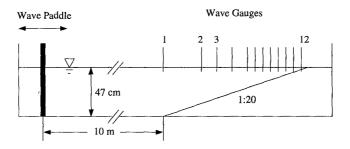
Figure 1: Bottom configuration and wave gage locations for experiments of Mase and Kirby (1992).
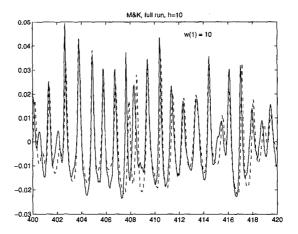

Figure 2: Sample of measured (solid) and predicted (dashed) time series of elevation η for Run 2 of Mase and Kirby (1992). Measurements at $h = 10cm$, corresponding to analysis of dissipation rates below.
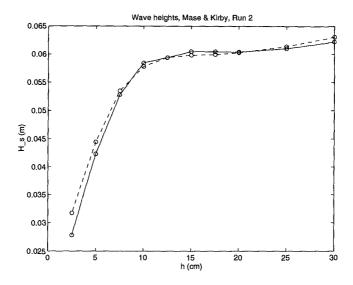
Figure 3: Measured (solid) and predicted (dash) significant wave heights for Run 2 of Mase and Kirby (1992).
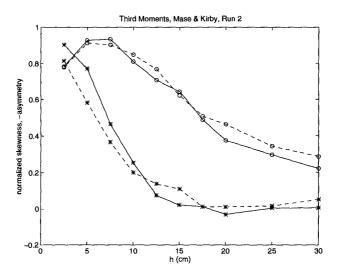

Figure 4: Measured (solid) and predicted (dash) skewness (circles) and asymmetry (stars) for Run 2 of Mase and Kirby (1992).

Analysis of Data

Having verified that the numerical model is capable of correct reproduction of second and third moment statistics in the evolution of a shoaling and breaking random wave train, we may now examine the results for energy dissipation in the model based on a direct analysis of the laboratory data. Figure 5 shows a short segment of the record of surface displacement $\eta(t)$ and dissipation $\epsilon_b(t)$ at $h = 10cm$. Dissipation $\epsilon_b(t)$ is computed directly from the measured data using (25). We compute the smoothed power spectrum of each of these quantities for the entire data run, according to the definitions

$$S_\eta(n) = \frac{\langle |A_n|^2 \rangle}{2\Delta f} \tag{26}$$

$$S_{\epsilon_b}(n) = \frac{\langle |\hat{\epsilon}_{bn}|^2 \rangle}{2\Delta f} \tag{27}$$

where $\hat{\epsilon}_{bn}$ is the Fourier transform of the dissipation term and where brackets here indicate ensemble averaging. Results for the Run 2 data at $h = 10cm$ are shown in Figure 6.
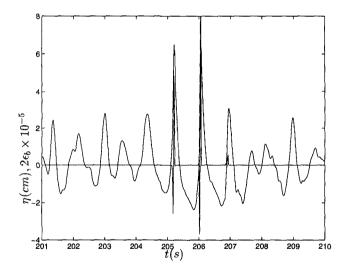

Figure 5: Time history of surface elevation and computed loss $\epsilon_b(t)$ for 9 seconds of Run 2, showing two strong breaking events at a 10cm depth.

Returning to the schematic frequency domain model, we may rearrange the original model equation

$$A_{n,x} + \cdots = -\alpha_n A_n \tag{28}$$

into the form of an energy equation

$$\left\{ \frac{1}{2}\rho g |A_n|^2 \sqrt{gh} \right\}_x = -2\sqrt{gh}\alpha_n \left(\frac{1}{2}\rho g |A_n|^2 \right) \tag{29}$$
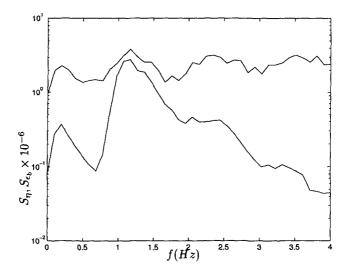
Figure 6: Power spectrum $S_\eta(f)$ of surface displacement (lower trace) and $S_\epsilon(f)$ of energy loss (upper trace) for Mase and Kirby (1992) Run 2 data.

The quantity in brackets on the left hand side of (29) represents the contribution to the wave energy flux from the n'th frequency component, or F_n. The quantity on the right is the contribution to the loss of wave energy at that frequency, or ϵ_{bn}. Using the definitions of power spectral densities (26) and (27), we may write the dissipation coefficient in the form

$$\alpha_n = \frac{1}{\rho g \sqrt{gh}} \frac{1}{\sqrt{2\Delta f}} \frac{[S_{\epsilon_b}(n)]^{1/2}}{S_\eta(n)} \tag{30}$$

Results similar to those presented in Figure 6 may now be utilized to determine the form of α_n. Analysis of a number of data sets has indicated that the spectral tail of $S_\eta(f)$ tends to have an f^{-2} dependence on frequency after breaking is established. This is consistent with the notion that the wave form tends towards a sawtooth shape, with a vertical front face and a linear back slope. We note that this is somewhat simplistic representation of the waves, as it implies that the wavefield would have significant asymmetry and zero skewness, whereas the measured data exhibits a balance between skewness and asymmetry in the inner surfzone. Nevertheless, the f^{-2} dependence seems to characterize several of the data sets which have been examined extremely well.

In contrast, Figure 6 shows that the dependence of $S_{\epsilon_b}(f)$ on f is relatively weak, so that this quantity can be taken to be constant. This is consistent with the notion that the dissipation term has the character of a sequence of isolated, spike-like processes. Taken together, these results indicate that the dominant frequency dependence of α_n is given by

$$\alpha_n \propto S_\eta(n)^{-1} \tag{31}$$

and we may further infer that this dependence translates into a dependence on f^2 in the inner surfzone. Figure 7 shows a sample distribution of $\alpha(f)$ as calculated from (30), together with a best quadratic fit to the calculated distribution. The fit is centered fairly close to the frequency of the spectral peak at $1Hz$, and the dependence of α on f is clearly quadratic towards higher frequencies.
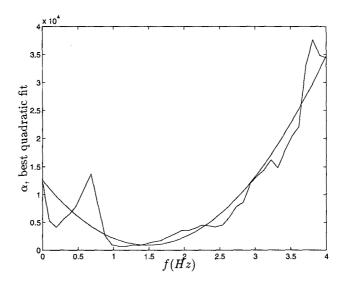

Figure 7: Sample $\alpha(f)$ deduced from data (according to (30)) together with a best quadratic fit.

Effect of Choice $F = 1$ on Modelled Waves

The results of the previous section strongly imply that a value of $F = 0$ should be chosen in the model (3)-(5). CGE have shown that the choice of F does not introduce a strong bias in the prediction of evolving power spectra. It appears that there is a preferred spectral shape that is obtained in shallow water, for which the reasons are still unclear. The effect of redistributing the loss differently across the spectrum serves mainly to enhance or suppress the nonlinear transfer of energy needed to maintain the target spectral shape.

In particular, the choice $F = 1$ implies that the rate of energy loss is the same in all spectral components, which accounts for the entire pattern of overall energy loss in evolving waves (Mase and Kirby, 1992; Eldeberky and Battjes, 1996). However, if the dissipation term is chosen to account for all changes in spectral energy density, there is necessarily a parallel suppression of all nonlinear energy transfer across the spectrum during the breaking process. This loss of an active transfer should suppress the imaginary part of the bispectrum, and would be evidenced by a loss of front-to-back asymmetry in the wave form, or a loss of statistical asymmetry. CGE have shown the consequence of choosing $F = 1$ on the prediction of third moment statistics, which

are severely damaged. A more graphic example is provided by Liu (1990, see pages 55-57), who modelled the breaking decay of a periodic wave using a dissipation coefficient which was uniform across the modelled spectrum. The resulting wave crests in the surfzone clearly lose their asymmetry, which provides direct evidence that nonlinear energy transfer across the spectrum has been suppressed.

In contrast, the choice $F = 0$ strongly concentrates dissipation at higher frequencies. As a result, there is necessarily a high rate of nonlinear transfer of energy across the spectrum from low to high frequencies, in oder to maintain the tail of the spectrum. This accounts for the presence of a strong, negative asymmetry. This choice is much more sensible in light of our understanding of the nonlinear processes going on in this region, and agrees with a direct analysis of the behavior of a well-tested time domain model of the breaking process.

Conclusions

The results of this study indicate that dissipation due to wave breaking should be biased strongly towards higher frequencies in spectral calculations, and that an f^2 dependence in the dissipation coefficent comes closest to matching the desired structure of the breaking terms using a simple functional dependence. This result supports the conclusions of Chen et al (1996) and is obtained by an entirely different route, being based on an anlysis of the frequency structure of the dissipation term in a time-domain wave evolution model.

A much more comprehensive description of the results of this study is currently being prepared, and will consider the form of the dissipation coefficient in the outer surfzone as well as in the established inner surfzone. In particular, there is some indication that, in regions where the breaking events are infrequent, that there is a tendency for α_n to be slightly negative at frequencies below the wind wave peak. This result could be thought of as the effect of distributing a set of localized momentum sources in the domain. Each breaking wave event would impart a kick to the water column in the manner described by Rapp and Melville (1990). The irregular, widely-spaced-in-time nature of these kicks translates into a wave generating mechanism at low frequency, and could contribute to the growth of the low-frequency wave climate in the surfzone.

References

Chen, Y., Guza, R. T. and Elgar, S., 1996, "Modelling breaking surface waves in shallow water", *J. Geophys. Res.*, submitted.

Eldeberky, Y. and Battjes, J. A., 1996, "Spectral modeling of wave breaking: Application to Boussinesq equations", *J. Geophys. Res.*, **102**, 1253-1264.

Heitner, K. L. and Housner, G. W., 1970, "Numerical model for tsunami runup", *J. Waterway, Port, Coast. and Ocean Engrng.*, **96**, 701-719.

Liu, P. L.-F., 1990, "Wave transformation", in *The Sea. Ocean Engineering Science*, **9**, (eds. B. Le Mehaute and D. M. Hanes), Part A, 27-63. Wiley-Interscience.

Mase, H. and Kirby, J. T., 1992, "Hybrid frequency-domain KdV equation for random wave transformation", *Proc. 23d Intl. Conf. Coast. Engrng.*, Venice, 474-487.

Rapp, R. J. and Melville, W. K., 1990, "Laboratory measurements of deep-water breaking waves", *Phil. Trans. Roy. Soc. London A*, **331**, 735-800.

Schäffer, H. A., Madsen, P. A. and Deigaard, R., 1993, "A Boussinesq model for waves breaking in shallow water", *Coastal Engrng.*, **20**, 185-202.

Svendsen, I. A., Yu, K. and Veeromony, J., 1996, "A Boussinesq type breaking wave model with vorticity", *Proc. 25th Intl. Conf. Coastal Engrng.*, this volume.

Thornton, E. B. and Guza, R. T., 1983, "Transformation of wave height distribution", *J. Geophys. Res.*, **88**, 5925-5938.

Wei, G., Kirby, J. T., Grilli, S. T. and Subramanya, R., 1995, "A fully nonlinear Boussinesq model for surface waves. Part 1. Highly nonlinear unsteady waves", *J. Fluid Mech.*, **294**, 71-92.

Wei, G. and Kirby, J. T., 1996, "A coastal processes model based on time-domain Boussinesq equations", Res. Report CACR-96-01, Center for Applied Coastal Research, University of Delaware. Paper presented at *Coastal Dynamics '95*, Gdansk, Sept. 1995.

Zelt, J. A., 1991, "The runup of nonbreaking and breaking solitary waves", *Coastal Engrng.* **15**, 205-246.