CHAPTER 34

COUPLED VIBRATION EQUATIONS FOR IRREGULAR WATER WAVES

Masao Nochino*

Abstract

A new system of equations is proposed for calculating wave deformation of irregular waves under the assumption of small amplitude wave and of mild slope of bottom configuration. The system is composed of a vibration equation for water surface elevation and three elliptic equations defined in horizontal plane. These equations are coupled each other. The coupled vibration equations are capable of calculating water surface elevation of irregular waves in a time domain.

1 Introduction

Some equations have been proposed for the deformation of water waves; the mild slope equation by Berkhoff(1972), the unsteady mild slope equation considering wave-current interaction by Liu(l983), Boussinesq equation by Pregrine(1978). The mild slope equations are able to apply only for a simple harmonic wave. The Boussinesq equation is derived for nonlinear long waves. The equation to calculate the deformation of irregular waves including long waves is expected for the coastal structure design and the analysis of coastal process.

Kubo et. al.(1992) modified the time-dependent mild slope equation by applying the Fourier expansion technique to the coefficent in the mild slope equation. The modified mild equation is capable of simulating random waves without long waves. Nadaoka et. al.(1993) proposed the fully-dispersive wave equation capable of simulating random waves with long waves.

In this paper, a new system of equations is proposed for calculating the deformation of the linear and irregular water waves including long waves. The waves are expressed as motions of coupled vibrations. The system of equations are applicable for the waves in the range of deep water depth to very shallow water depth.

*Associate Professor, Osaka Institute of Technology, Department of Civil Engineering 5-16-1 Omiya Asahi-ku, Osaka, 565, Japan

2 Theory

2.1 Basic equation

The motion equations and the equation of continuity are expressed as follows under the assumption that the motion due to waves be small and the nonlinear terms in the motion equations be negligible.

$$\frac{\partial u}{\partial t} + \frac{\partial \tilde{p}}{\partial x} = 0 \tag{1}$$

$$\frac{\partial v}{\partial t} + \frac{\partial \tilde{p}}{\partial y} = 0 \tag{2}$$

$$\frac{\partial w}{\partial t} + \frac{\partial \tilde{p}}{\partial z} = 0 \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4}$$

where, $\tilde{p}$; the fluctuating pressure defined as

$$\tilde{p} = p/\rho + gz, \tag{5}$$

x,y ;the coordinates in horizontal axes , z ; the vertical coordinate in upward direction with the origin at still water surface, p ; the pressure, ρ; the water density, g; the gravitational acceleration, u,v,w; the water particle velocity in x, y, z direction, respectively. Taking divergence of the motion equation, the Laplace equation of the fluctuating pressure $\tilde{p}$ are given as follows.

$$\triangle \tilde{p} = 0 \tag{6}$$

In this paper, Equation(6) is treated as the basic equation and $\tilde{p}$, the main variable as same as the water surface elevation η.

2.2 Boundary conditions

There are three boundary conditions; two boundary conditions at free surface and one at bottom. These boundary conditions should be expressed by the main variable and linearized as the motion equations are linearized.

The dynamic and kinematic free surface boundary conditions are expressed as follows.

$$\tilde{p} = g\eta \qquad \text{at } z = 0 \tag{7}$$

$$\frac{\partial \eta}{\partial t} = w \qquad \text{at } z = 0 \tag{8}$$

The motion equation should be satisfied at free surface. Eliminating the vertical velocity w in above equation by using the motion equation in vertical direction, Eq. (3), the kinematic boundary condition, Eq.(8), can be expressed as follow.

$$\frac{\partial^2 \eta}{\partial t^2} = -\frac{\partial \tilde{p}}{\partial z} \qquad \text{at } z = 0 \tag{9}$$

The boundary condition at bottom ($z = -h$) is commonly expressed as follow.

$$u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + w = 0 \qquad \text{at } z = -h \tag{10}$$

where, $h = h(x, y)$ indicates the water depth. Derivating above equation with respect to time and applying the motion equations (1)-(3), the following equation is given as the boundary condition at bottom expressed by the fluctuation pressure.

$$\left(\nabla\tilde{p}\cdot\nabla h + \frac{\partial\tilde{p}}{\partial z}\right) = 0 \qquad \text{at } z = -h \tag{11}$$

where $\nabla = (\partial/\partial x, \partial/\partial y)$ indicate the differential operator in horizontal plane.

2.3 Expansion series of fluctuating pressure

The fluctuating pressure $\tilde{p}$ is expanded by using the series of Legendre's Polynomials $P_n(z)$ as follows.

$$\tilde{p} = \sum_{m=1}^{\infty} q_m P_{2(m-1)}(\tilde{z}) \tag{12}$$

where $q_m = q_m(x, y, t)$ is the coefficient of mth term, and

$$\tilde{z} = 1 + z/h. \tag{13}$$

The variable $\tilde{z}$ is defined in the interval of $[0, 1]$ as the coordinate z is defined in [-h,0] for the linearized theory. The Legendre's Polynomials $P_m(\tilde{z})$ are defined as follows.

$$P_0(\tilde{z}) = 1, \quad P_m(\tilde{z}) = \frac{1}{2^m m!}\frac{d^m}{d\tilde{z}^m}(\tilde{z}^2 - 1)^m, \quad m = 1, 2, \cdots \tag{14}$$

For example,

$$P_2(\tilde{z}) = \frac{1}{2}(3\tilde{z}^2 - 1), \quad P_4(\tilde{z}) = \frac{1}{8}(35\tilde{z}^4 - 30\tilde{z}^2 + 3), \quad etc.. \tag{15}$$

The series of Legendre's Polynomials $P_{2(m-1)}(z)$ have the property of the series of the orthogonal functions.

$$\int_0^1 P_{2(m-1)}(z)P_{2(n-1)}(z)dz = \begin{cases} 0, & \text{for } m \neq n \\ \frac{1}{4m-3}, & \text{for } m = n \end{cases} \qquad (m, n = 1, 2, \cdots) \tag{16}$$

The above relation implies that all coefficients q_m in Eq.(12) be uniquely determined.

The expression by the infinite series in Eq.(12) is not suitable for a numerical calculation. Let's suppose, in this paper, that the fluctuating pressure $\tilde{p}$ be expressed by the series of the first 4 terms in the right side of Eq.(12), such as

$$\tilde{p} = q_1 P_0(\tilde{z}) + q_2 P_2(\tilde{z}) + q_3 P_4(\tilde{z}) + q_4 P_6(\tilde{z}). \tag{17}$$

The dynamic and kinematic boundary condition at free surface are rewritten as follows by substituting Eq.(17) into Eqs.(7) and (9).

$$g\eta = q_1 + q_2 + q_3 + q_4 \tag{18}$$

$$\frac{\partial^2 \eta}{\partial t^2} = -\frac{1}{h}(3q_2 + 11q_3 + 21q_4) \tag{19}$$

2.4 Depth integrated equation

There are five unknown variables; $\eta(x,y,t), q_m(x,y,t), (m = 1,2,3,4)$. All variables are the functions defined in horizontal plane. There are two equations, Eqs.(18) and (19) which relate the five unknown variables. Therefore, three more new equations are required in order to solve the five unknown variables, instead of Eq.(6) defined in three dimensional space.

The Galerkin method is applied to Eq.(6) to make up the three new equations.

$$\int_{-h}^{0} P_{2(m-1)}(\tilde{z}) \triangle \tilde{p} dz + P_{2(m-1)}(0) \left(\nabla h \cdot \nabla \tilde{p} + \frac{\partial \tilde{p}}{\partial z} \right) \Bigg|_{z=-h} = 0 \tag{20}$$
$$(\text{for} \quad m = 1,2,3)$$

The second term in left hand side of the above equation is added so that the boundary condition at bottom, Eq.(11) is satisfied. Integrating the above equation and eliminating the coefficient q_4 by using Eq(18), the following equations are given.

$$\begin{aligned} \nabla^2 q_1 + \frac{\nabla(88q_1 + 24q_2 - 88q_3)}{128h} \nabla h - \frac{21q_1 + 18q_2 + 11q_3}{h^2} \\ = \frac{40g\nabla\eta\nabla h}{128h} - \frac{21g\eta}{h^2} \end{aligned} \tag{21}$$

$$\begin{aligned} \nabla^2 q_2 + \frac{\nabla(-190q_1 + 258q_2 + 330q_3)}{128h} \nabla h - \frac{90q_1 + 90q_2 + 55q_3}{h^2} \\ = -\frac{130g\nabla\eta\nabla h}{128h} - \frac{90g\eta}{h^2} \end{aligned} \tag{22}$$

$$\begin{aligned} \nabla^2 q_3 + \frac{\nabla(99q_1 - 693q_2 - 205q_3)}{128h} \nabla h - \frac{99q_1 + 99q_2 + 99q_3}{h^2} \\ = \frac{333g\nabla\eta\nabla h}{128h} - \frac{99g\eta}{h^2} \end{aligned} \tag{23}$$

When deriving the above equations, the assumption for the mild slope of the bottom configuration is applied; the terms related $\nabla^2 h$ and $|\nabla h|^2$ are neglected.

The Eq(19) can be rewritten as follows by eliminating q_4 in a same manner.

$$\frac{\partial^2 \eta}{\partial t^2} = -\frac{1}{h}(21g\eta - 21q_1 - 18q_2 - 10q_3) \tag{24}$$

Eq.(24) shows the form of a vibration equation for the water surface elevation η with the exciting force as a function of q_1, q_2 and q_3. The variables q_1, q_2 and q_3 are determined by the Eqs. (21)-(23) which has the form of the elliptic equation defined in the horizontal plane with the exciting term as a function of η. Eqs.(21)-(24) are coupled each other. The variables η and q_m are in phase. Therefore, the system of Eqs.(21)-(24) expresses the water waves as the system of motions of the coupled vibrations, not as a system of wave equations.

The system of coupled vibration equations are expressed by the water depth h and the constant coefficients. The system is independent to both the wave frequency and the wave length L.

3 Dispersion Relation of Coupled Vibration Equation

The system of coupled vibration equations satisfy the dynamic and kinematic boundary condition which are linearized. Applying the coupled vibration equations for water waves, the dispersion relation of the coupled vibration equation should be coincide with the one of the small amplitude theory (Airy's wave theory).

Suppose the monochromatic wave progressing in the x direction with angular frequency of σ, wave number of k on the water of uniform depth h. The water surface elevation η and the coefficients q_m can be expressed as follows,

$$\eta = \Re\left\{\hat{\eta} e^{i(kx-\sigma t)}\right\} \tag{25}$$

$$q_m = \Re\left\{\hat{q_m} e^{i(kx-\sigma t)}\right\} \qquad m = 1,2,3,4 \tag{26}$$

as η and q_m are in phase, where, $\Re$ denotes the real part and $\hat{}$ the complex amplitude. Substituting above equations to Eqs.(21)-(23), the coefficients q_m are obtained as function of η.

$$q_1 = \frac{21g\eta(495 + 60(kh)^2 + (kh)^4)}{10395 + 4725(kh)^2 + 210(kh)^4 + (kh)^6} \tag{27}$$

$$q_2 = \frac{45g\eta(kh)^2(77 + 2(kh)^2)}{10395 + 4725(kh)^2 + 210(kh)^4 + (kh)^6} \tag{28}$$

$$q_3 = \frac{99g\eta(kh)^4}{10395 + 4725(kh)^2 + 210(kh)^4 + (kh)^6} \tag{29}$$

$$q_4 = \frac{g\eta(kh)^6}{10395 + 4725(kh)^2 + 210(kh)^4 + (kh)^6} \tag{30}$$

Substituting the above equations into Eq.(24), the relation between the angular frequency σ and the wave number k is finally obtained as follows.

$$\frac{\sigma^2 h}{g} = \frac{21(kh)^2(495 + 60(kh)^2 + (kh)^4)}{10395 + 4725(kh)^2 + 210(kh)^4 + (kh)^6} \tag{31}$$

This relation is the dispersion relation for the system of coupled vibration equations and corresponds to the one in the Airy's wave theory.

$$\frac{\sigma^2 h}{g} = kh \tanh kh \tag{32}$$

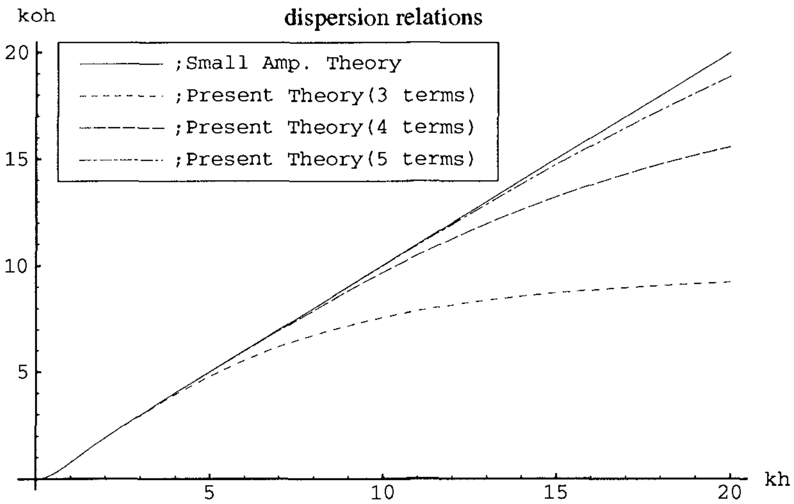

Figure 1: Comparison of dispersion relations

Figure 1 shows the relation of Eq.(32) by the solid line and of Eq.(31) by the broken line. The left hand side of these equations $\sigma^2 h/g$ are expressed by $k_o h$ in the figure. The left two lines in the figure indicate the dispersion relation in the case that the first 3 and 5 terms are adopted in the Eq.(12)(see Appendix).

The dispersion relation of Eq.(31) coincide very well with the one of Airy's wave theory in the rage of kh less than $7(h/L \simeq 1)$. This results implies that the system of coupled vibration equations be capable of expressing the waves on the very shallow water depth to the deep water depth.

Group velocity is important factor for waves deforming in a shallow water region and wave groups progressing in a deep water region. The group velocity C_g is defined by the gradient of the angular frequency σ with respect to the wave number k, that is,

$$C_g = \frac{\partial \sigma}{\partial k}. \tag{33}$$

The Eq.(31) gives the C_g/C of the coupled vibration equation by differentiating the equation with respect to k, where C denotes the phase velocity.

$$\frac{C_g}{C}=\frac{30}{495+60kh^2+kh^4}\frac{343035+83160kh^2+14049kh^4+564kh^6+10kh^8}{10395+4725kh^2+210kh^4+kh^6} \quad (34)$$

The Figure-2 shows the C_g/C due to the Airy's wave theory and to the present

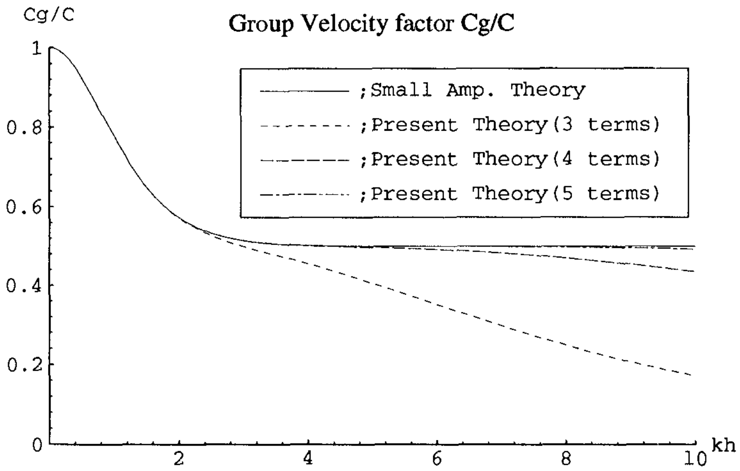

Figure 2: Comparison of group velocity

theory with the first 3, 4 and 5 terms as same as Figure-1. The group velocities due to the present theories in Figure-2 show less accuracy as comparing to the Airy's theory than the dispersion relation in Figure-1. This decrease of the accuracy is caused by differential calculus of the dispersion relation, Eq.(31) in order to obtain the group velocity.

Eq.(31) indicates the form of a *Padé* approximation for the right side of Eq.(32). The accuracy of differential coefficient of the approximated function is generally less than of the approximated function.

4 Calculation Method and Results

4.1 Incident boundary

The time series of water surface elevation is commonly measured in both field measurement and laboratory test. The position at measuring point of off-shore waves is generally treated as the incident boundary in most of calculations. In the calculation of the coupled vibration equations for irregular water waves, the water surface elevation η and the coefficients q_m should be given at the incident boundary. The measured data give the water surface elevation η_{in} at incident boundary. The coefficient q_m is calculated by Eqs.(27)-(30) when the wave number k is given.

The wave number k is determined by the following method. The wave number k and the angular frequency σ for irregular waves are assumed to be a function of time and satisfy the following equations.

$$\{\sigma(t)\}^2 = gk(t)\tanh k(t)h \tag{35}$$

$$\frac{d^2\eta_{in}}{dt^2} = -\{\sigma(t)\}^2\eta_{in} \tag{36}$$

The second equation gives the angular frequency changing with time and the wave number is determined by the first equation. The angular frequency, however, becomes infinity when the η_{in} is nearly equals to zero in the computer calculation. In order to avoid this problem, the complex water surface elevation ζ is used instead of η. the complex water surface elevation ζ is defined as

$$\zeta(t) = \eta_{in} + i\tilde{\eta}_{in}, \tag{37}$$

where, $\tilde{\eta}_{in}$ is the Hilbert transform of η_{in} and i is the imaginary unit. $\tilde{\eta}_{in}$ is calculated by the following relation.

$$i\tilde{H}(\omega) = \begin{cases} H(\omega) & , \text{ for } \omega > 0 \\ 0 & , \text{ for } \omega = 0 \\ -H(\omega) & , \text{ for } \omega < 0 \end{cases}, \tag{38}$$

where $H(\omega)$ and $\tilde{H}(\omega)$ are the complex Fourier transforms of η_{in} and $\tilde{\eta}_{in}$, respectively. Finally, the angular frequency $\sigma(t)$ is redefined by the complex water surface elevation $\zeta(t)$.

$$\{\sigma(t)\}^2 = -\Re\left\{\frac{d^2\zeta/dt^2}{\zeta}\right\} \tag{39}$$

4.2 Calculation results

Eqs.(21)-(24) are applied to the waves progressing in one direction on the constant water depth, $7m$. The waves is composed of the two wave groups; WAVE-A of which the dominant wave period is $9s$ and WAVE-B, $3s$. Each wave group has a very narrow spectra, but is not monochromatic waves. That is, the waves calculated in this paper are parts of the components of the uni-directional irregular waves.

The finite difference method is applied for the equations. The time interval is $0.183s$, the distance of calculation nodes; $0.73m$, the number of nodes; 2800. The water channel is $2km$ long and one end of the channel is a reflection boundary.

The calculated results are shown in Figure 3 as the snapshots of the water surface elevation per $37.5s$. In the Figure, each wave group progresses with the each group velocity. Each wave group stretches with progress because of the frequency dispersion. The WAVE-A started later catches up with, passes the WAVE-B started earlier, then, reflects at the end, again meets with, and passes through the WAVE-B.

These results are a matter of course as the liner wave theory. A point is that the results are calculated in the time domain by the system of the coupled vibration equations.

5 Discussion

In the present theory, the fluctuation pressure $\tilde{p}$ is expanded to 4 terms in Eq.(17) with the coefficients from q_1 to q_4. There is, however, no q_4 appeared in the final form of the coupled vibration equation in Eqs.(21)-(24). The evolution problem is calculated without q_4.

The vertical distribution of fluctuation pressure $\tilde{p}$ in Airy's theory is given as follows.

$$\tilde{p} = g\eta\frac{\cosh k(h+z)}{\cosh kh} = g\eta\frac{\cosh kh\tilde{z}}{\cosh kh} \tag{40}$$

The right side of above equation can be exressed by the Taylor expansion.

$$\tilde{p} = \frac{g\eta}{\cosh kh}\left(1 + \frac{(kh)^2}{2!}\tilde{z}^2 + \frac{(kh)^4}{4!}\tilde{z}^4\right) + O\left(g\eta\frac{(kh)^6}{6!}\right) \tag{41}$$

Comparing Eq.(17) and above equation, the coefficient q_4 corresponds to the resident term in the right side of above equation. The coefficient q_4 indicates the truncation error of the coupled vibration equations.

The combined kinematic-dynamic free surface boundary condition is given as

$$\frac{\partial^2\Phi}{\partial t^2} = -g\frac{\partial\Phi}{\partial z} \tag{42}$$

in the linear water wave problem based on the velocity potential theory. Assuming that the velocity potential $\Phi(x, y, z, t)$ is expressed as the following form,

$$\Phi = \phi(x, y, t)\frac{\cosh k(h+z)}{\cosh kh}, \tag{43}$$

Eq.(42) is rewritten as follow.

$$\frac{\partial^2\phi}{\partial t^2} = -gk\tanh kh\phi \tag{44}$$

This equation has the form of a vibration eqauiton. Expressing water waves as the form of vibration equation is a traditional and natural method. The coupled vibration equations, however, express the any shape of vertical distribution of the fluctuating pressure instead of an unique shape of vertical distribution such like the velocity potential in Eq.(43).

6 Conclusion

The system coupled vibration equations is derived for water waves progressing on the water with mild slope of the bottom configuration. The motion of water surface elevation is expressed by the vibration equation with the coupled exciting force. The new system of equations is able to be applied for the random waves composed by the monochoromatic waves whose relative depth h/L is less than 1.

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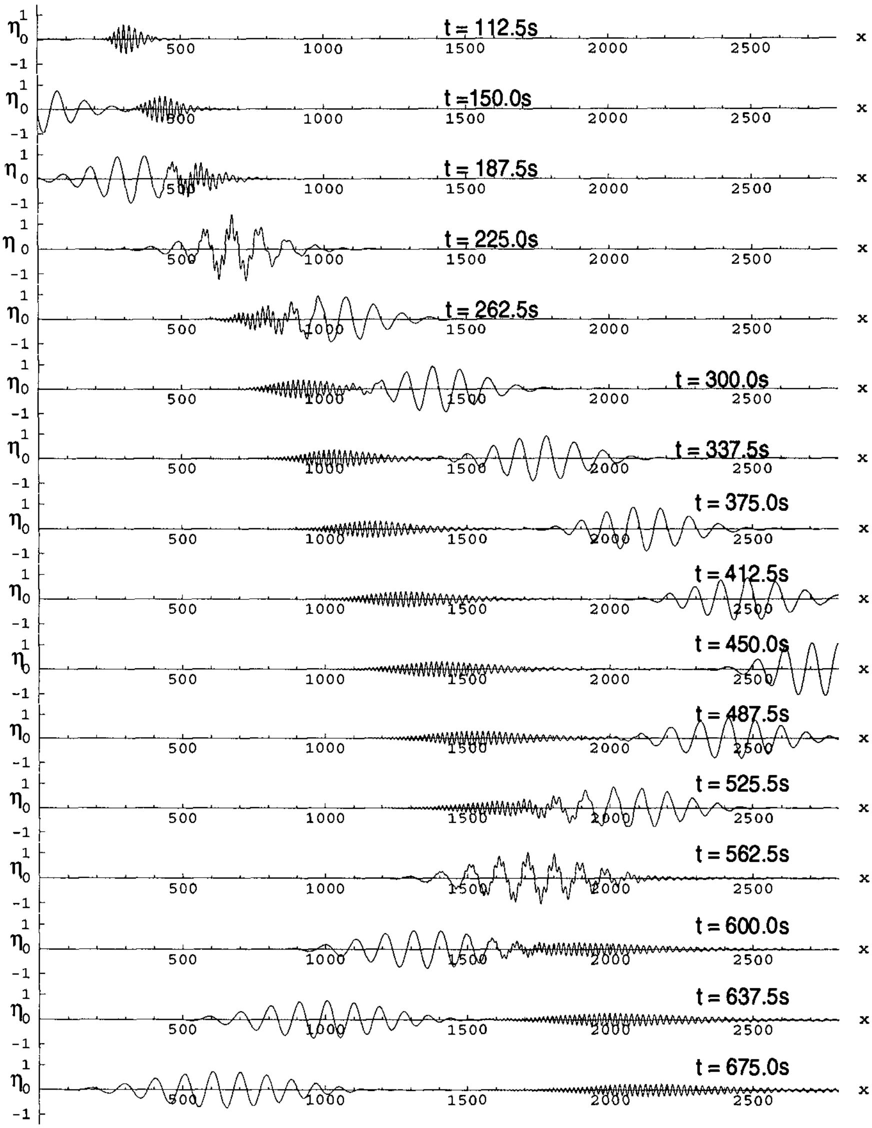

Figure 3: Propagation of two wave groups

APPENDIX

Equations for 3 Terms

$$g\eta = q_1 + q_2 + q_3 \tag{45}$$

$$\nabla^2 q_1 + \frac{\nabla(3q_1 + 7q_2)}{4h}\nabla h - \frac{10q_1 + 7q_2}{h^2} = \frac{3g\nabla\eta\nabla h}{4h} - \frac{10g\eta}{h^2} \tag{46}$$

$$\nabla^2 q_2 + \frac{\nabla(-10q_1 - 11q_2)}{4h}\nabla h - \frac{35q_1 + 35q_2}{h^2} = -\frac{5g\nabla\eta\nabla h}{2h} - \frac{35g\eta}{h^2} \tag{47}$$

$$\frac{\partial^2\eta}{\partial t^2} = -\frac{1}{h}(10g\eta - 10q_1 - 7q_2) \tag{48}$$

Equations for 5 Terms

$$g\eta = q_1 + q_2 + q_3 + q_4 + q_5 \tag{49}$$

$$\nabla^2 q_1 + \frac{\nabla(35q_1 + 99q_2 - 13q_3 + 75q_4)}{64h}\nabla h - \frac{36q_1 + 33q_2 + 26q_3 + 15q_4}{h^2} = \frac{35g\nabla\eta\nabla h}{64h} - \frac{36g\eta}{h^2} \tag{50}$$

$$\nabla^2 q_2 + \frac{\nabla(-95q_1 - 111q_2 + 65q_3 - 210)}{64h}\nabla h - \frac{165q_1 + 165q_2 + 130q_3 + 75q_4}{h^2} = -\frac{95g\nabla\eta\nabla h}{64h} - \frac{165g\eta}{h^2} \tag{51}$$

$$\nabla^2 q_3 + \frac{\nabla(639q_1 + 351q_2 + 571q_3 + 1575q_4)}{256h}\nabla h - \frac{234q_1 + 234q_2 + 234q_3 + 135q_4}{h^2} = \frac{639g\nabla\eta\nabla h}{256h} - \frac{234g\eta}{h^2} \tag{52}$$

$$\nabla^2 q_4 - \frac{\nabla(1222q_1 + 1066q_2 + 1794q_3 + 1291q_4)}{256h}\nabla h - \frac{195q_1 + 195q_2 + 195q_3 + 195q_4}{h^2} = -\frac{1222g\nabla\eta\nabla h}{256h} - \frac{195g\eta}{h^2} \tag{53}$$

$$\frac{\partial^2\eta}{\partial t^2} = -\frac{1}{h}(36g\eta - 36q_1 - 33q_2 - 26q_3 - 15q_4) \tag{54}$$