CHAPTER 23

Modelling Moveable Bed Roughness and Friction for Spectral Waves

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Abstract

The present paper is concerned with the simulation of turbulent boundary layer dynamics over a moveable seabed in random waves. A new theoretical approach for the evaluation of moveable bed roughness in spectral waves based on the grain-grain interaction idea is presented and tested against data from the laboratory and field. The new approach is combined with the methodology which assumes that the spectral wave condition can be represented by a monochromatic representative wave. Good results have been obtained, although further testing against data gathered in the North Sea is required.

1. Introduction

For a moveable sandy bed, one may distinguish three general seabed conditions due to the action of surface gravity waves: a flat bed, rippled bed and sheet flow. If we consider the latter condition, the need to study sediment transport under wave-induced sheet flow conditions is necessary in the understanding of beach profile changes in the surf zone. The understanding of nearbed sediment dynamics is also of great importance for the mathematical description of cross-shore sediment transport.

To understand the effect of changing bed roughness by the hydrodynamic forces requires knowledge of the dynamic behaviour of sand grains in the collision-dominated, high concentration nearbed region. At high shear stresses and sediment transport intensities, the nearbed sediment transport appears to take place in a layer with a thickness that is large compared to the grain size. It is therefore not possible to properly describe flow in this layer by conventional engineering models which assume

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that bed load transport occurs in a layer that has a thickness of the order of one or two grain diameters.
The present paper is concerned with developing an iterative procedure for the estimation of the effective bed roughness for a monochromatic wave, as characterised by the roughness parameter k_s , and extending this to the case of a spectral sea. The nearbed sediment dynamics are modelled in two regions with continuous profiles of stress and velocity. Namely (i) a granular fluid region and (ii) a wall bounded turbulent fluid shear region.
In sheet flow conditions it is assumed that the external drag produced by the boundary layer flow is related to the particle interactions within the sub-bed layer and hence to the effective roughness at the boundary.

2. The sheet flow model

2.1 Formulation of the problem

A typical velocity distribution with depth of a rough bed is supposed to be characterised (Kaczmarek & O'Connor 1993a,b) by a sub-bottom flow and a main or outer flow, as shown in Figure 1.

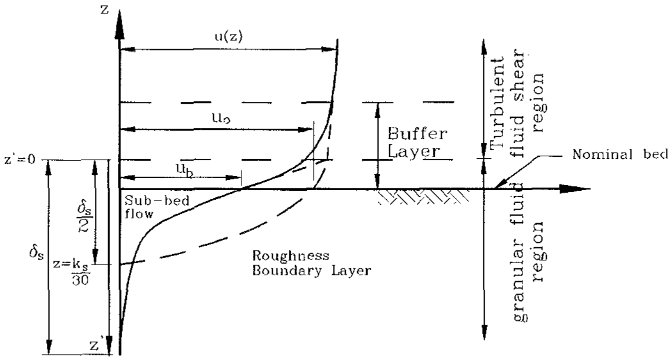

Figure 1: Definition sketch of turbulent flow over a moveable bed.

The velocity distribution is supposed to be continuous. Its intersection with the nominal bottom is the apparent slip velocity u_b . The downward extension of the velocity distribution in the outer zone of the main flow yields a fictitious slip velocity, u_o at the nominal bed, which is necessarily larger than u_b because of the supposed asymptotic transition in the buffer layer between the sub-bed flow and the fully turbulent flow in the turbulent-fluid shear region.
The velocity distribution in the roughness layer depends on the type of geometric roughness pattern and the bed permeability. There must be some transition between both parts of the velocity distribution bridged by the buffer zone. However, for present

purposes it is assumed that the velocity distribution in the turbulent-fluid shear region can be determined by parameters dependent on the geometric roughness properties of the bed and the outer flow parameters, such as the free-stream orbital velocity. It is proposed to extend the sub-bed granular-fluid flow region to the matching point with the velocity distribution in the turbulent-fluid shear region. Thus, shear stress velocities in the two layers are set equal at the theoretical bed level, as it is shown in Figure 1, point A.

The sub-bed flow region has a high sediment concentration. For sheet flow conditions in this layer, chaotic collisions of grains are the predominant mechanism. In this case water does not really transfer shear stresses at all. The dynamic state of such a mixture is characterised by stresses σ_{ij} which are the sum of dynamic σ^*_{ij} and plastic σ°_{ij} stresses.

The first problem, therefore, is to determine the velocity profile distribution in the upper turbulent layer, which means determining the effective roughness height of the bed k_s as well as the lower grain-fluid flow. The intersection of these two profiles will determine point A (See Figure 1).

2.2 Mathematical description of flow in the turbulent upper region

It is assumed that flow in the upper layer is governed by the simplified equation of motion:

$$\frac{\partial u}{\partial t} = \frac{\partial U}{\partial t} + \frac{1}{\rho}\frac{\partial \tau}{\partial z} \tag{2.1}$$

in which u(z,t) is, in general, a combined wave-period-averaged "steady" current and wave velocity and U(t) is the free-stream wave velocity at the top of the wave boundary layer.

The present work uses an eddy viscosity model, which is an extension of Kajiura's (1968) and Brevik's (1981) model. Thus the eddy viscosity over the flow depth is assumed to be given by the equations.

$$v_t(z) = \kappa u_{f\max} z \qquad \text{for} \quad \frac{k_s}{30} \le z \le \frac{\delta_m}{4} + \frac{k_s}{30} \tag{2.2}$$

$$v_t(z) = \kappa u_{f\max}\left(\frac{\delta_m}{4} + \frac{k_s}{30}\right) \quad \text{for} \quad \frac{\delta_m}{4} + \frac{k_s}{30} < z \le 2\delta_m + \frac{k_s}{30} \tag{2.3}$$

in which κ is von Karmen's constant; u_{fmax} is the maximum value of bed shear velocity ($u_f(\omega t)$) during the wave period that is max $[u_f(\omega t)]$; δ_m is the maximum value of δ_1 and δ_2 , that is, max (δ_1 , δ_2) where δ_1 and δ_2 are the boundary layer thickness at the moments corresponding to maximum and minimum velocity (of the combined wave and current flow) at the top of the turbulent boundary layer.

The quantities u_{fmax} , δ_m are determined from the solution of the integral equation derived from equation (2.1) as used by Fredsøe (1984):

$$\frac{\tau(\delta)}{\rho} - \frac{\tau_o}{\rho} = -\int_{\frac{k_s}{30}}^{\delta+\frac{k_s}{30}} \frac{\partial}{\partial t}(U-u)dz \tag{2.4}$$

Fredsøe (1984) assumed a logarithmic velocity profile in the boundary layer

$$\frac{u}{u_f} = \frac{1}{\kappa}\ln\frac{30z}{k_s} \tag{2.5}$$

The solution of equation (2.4) using Fredsøe's (1984) approach enables the value of u_{fmax} to be determined, if k_s is specified. Equation (2.1) can then be solved to provide the velocity distribution in the wave boundary layer.

2.3 Mathematical description of the flow in the granular-fluid region

Particle interactions in the shear-grain-fluid flow are assumed to produce two distinct types of behaviour. The Coulomb friction between particles gives rise to rate-independent stresses (of the plastic type) and the particle collisions give rise to stresses that are rate-dependent (of the viscous type). We assume the co-existence of both types of behaviour and the stress tensor is divided into two parts.

$$\sigma_{ij} = \sigma^0_{ij} + \sigma^*_{ij} \tag{2.6}$$

Where σ^0_{ij} is the plastic stress and σ^*_{ij} is the viscous stress.

For two-dimensional deformation in the rectangular Cartesian co-ordinates x' and z' the Coulomb yield criterion is satisfied by employing the following stress relations:

$$\sigma^0_{x'x'} = -\sigma'(1+\sin\varphi\cos 2\psi) \tag{2.7}$$

$$\sigma^0_{z'z'} = -\sigma'(1-\sin\varphi\cos 2\psi) \tag{2.8}$$

$$\sigma^0_{x'z'} = -\sigma'\sin\varphi\cos 2\psi \tag{2.9}$$

Where φ is the quasi-static angle of internal friction, while ψ , denoting the angle between the major principal stress and the x'-axis is equal to:

$$\psi = \frac{\pi}{4} - \frac{\varphi}{2} \tag{2.10}$$

For the average normal stress:

$$\sigma' = -\left(\frac{\sigma^0_{x'x'} + \sigma^0_{z'z'}}{2}\right) \tag{2.11}$$

we employ the following approximate expression (Sayed and Savage 1983).

$$\sigma' = \alpha^0 \left(\frac{c - c_0}{c_m - c} \right) \tag{2.12}$$

where α^0 is a constant and c_0 and c_m are the solid concentrations corresponding to fluidity and closest packing respectively.
The viscous part of the stress tensor according to Sayed and Savage (1983) is assumed to have the following form:

$$\sigma^*_{x'x'} = \sigma^*_{z'z'} = -(\mu_0 + \mu_2)\left(\frac{\partial u}{\partial z'}\right)^2 \tag{2.13}$$

$$\sigma^*_{x'z'} = \sigma^*_{z'x'} = \mu_1 \left|\frac{\partial u}{\partial z'}\right| \frac{\partial u}{\partial z'} \tag{2.14}$$

in which the viscous stress coefficients μ_0 , μ_1 and μ_2 are functions of the solids concentration c:

$$\frac{\mu_1}{\rho_s d^2} = \frac{0.03}{(c_m - c)^{1.5}} \tag{2.15}$$

$$\frac{\mu_0 + \mu_2}{\rho_s d^2} = \frac{0.02}{(c_m - c)^{1.75}} \tag{2.16}$$

Considering steady fully developed two-dimensional shear-grain-flow, the balance of linear momentum according to Kaczmarek & O'Connor (1993a,b) yields:

$$\alpha^0 \left[\frac{c - c_0}{c_m - c} \right] \sin\varphi \sin 2\psi + \mu_1 \left[\frac{\partial u}{\partial z'} \right]^2 = \rho u_f^2 \tag{2.17}$$

$$\alpha^0 \left[\frac{c - c_0}{c_m - c} \right] (1 - \sin\varphi \cos 2\psi) + (\mu_0 + \mu_2) \left[\frac{\partial u}{\partial z'} \right]^2 = \left[\frac{\mu_0 + \mu_2}{\mu_1} \right]_{c=c_0} \rho u_f^2 + (\rho_s - \rho) g \int_0^{z'} c dz \tag{2.18}$$

where ρ is the density of the fluid.
Eliminating $(\partial u/\partial z')$ from equations (2.16) and (2.17) allows the calculation of the profiles of the sub-bed sediment concentration c and velocity u in relation to known maximum shear stress ($\rho u^2_{f\,max}$) at the theoretical bed level (z'=0).
In Kaczmarek & O'Connor (1993a,b) equation (2.18) was solved for c as a function of depth (z') by using an iteration method in conjunction with numerical integration. Integration started at the theoretical bed level (z'=0) with $c=c_0$. Proceeding downwards at each step the iteration method was used to evaluate c. The integration was stopped

when c was equal to c_{ms}. For the calculations the following numerical values were recommended for the various sand beds.

$$\frac{\alpha^0}{\rho_s g d} = 1 \; ; \; c_0 = 0.32 \; ; \; c_m = 0.53 \; ; \; c_{ms} = 0.50 \; ; \; \varphi = 24.4°$$

3. Results for monochromatic waves

3.1 Plane bed

The above procedure was used to compare computations for the model with the experimental results of Horikawa *et al.* (1982). The conditions for Horikawa *et al.*'s test 1 were used for the model calculations: d = 0.2mm, $s = \rho_s/\rho = 2.66$, $\varphi = 24.4°$, T = 3.64s and U = 127 cm/s. A value of k_s = 7.3mm was found for the roughness parameter.

Having obtained the roughness parameter it is then easy to obtain the instantaneous profiles both in the turbulent layer and the sub-bottom flow zone without reference to empirical formulas of any kind. Knowing u_f and solving equations 2.17 and 2.18 the velocity and concentration distributions at any time inside the entire sub-bottom layer can be found.

The results are shown in Figure 2. A reasonable agreement is obtained between the model and the laboratory data.

The model was then run for a range of conditions including those outside its range of application. The results of these tests are shown in Figure 3. The calculations were obtained using the simplified iteration procedure to determine k_s by introducing a simple logarithmic distribution (2.5) instead of the numerical solution of equation (2.1). Such a simplification makes the calculations much more efficient. It is seen that the roughness parameter, k_s, decreases with increasing dimensionless bed shear stress θ_{max} and k_s is seen to attain its greatest value for small dimensionless shear stresses where $\theta \approx 1$ (the transition from plane bed to ripples).

The trend shown in the present results, that is, that the roughness parameter increases drastically with decreasing dimensionless maximum shear stress, is similar to that shown by Nielsen (1992). Nielsen (1992) showed that the hydraulic roughness for equilibrium ripple formations is of the order $100d_{50}$ to $1000d_{50}$. However, for artificial flat beds where measurements were taken before ripples had time to form Nielsen (1992) found that the hydraulic roughness decreased with decreasing grain roughness Shields parameter.

Next, calculations were carried out for a moveable sandy bed (d = 0.2mm, s = 2.66, $\varphi = 24.4°$) with a variety of wave heights with a mean water depth of 5.0m. The wave period was kept constant at T = 3.6s. The maximum shear stresses were calculated on the basis of equation (2.4) and using the simplified iteration procedure to determine k_s.

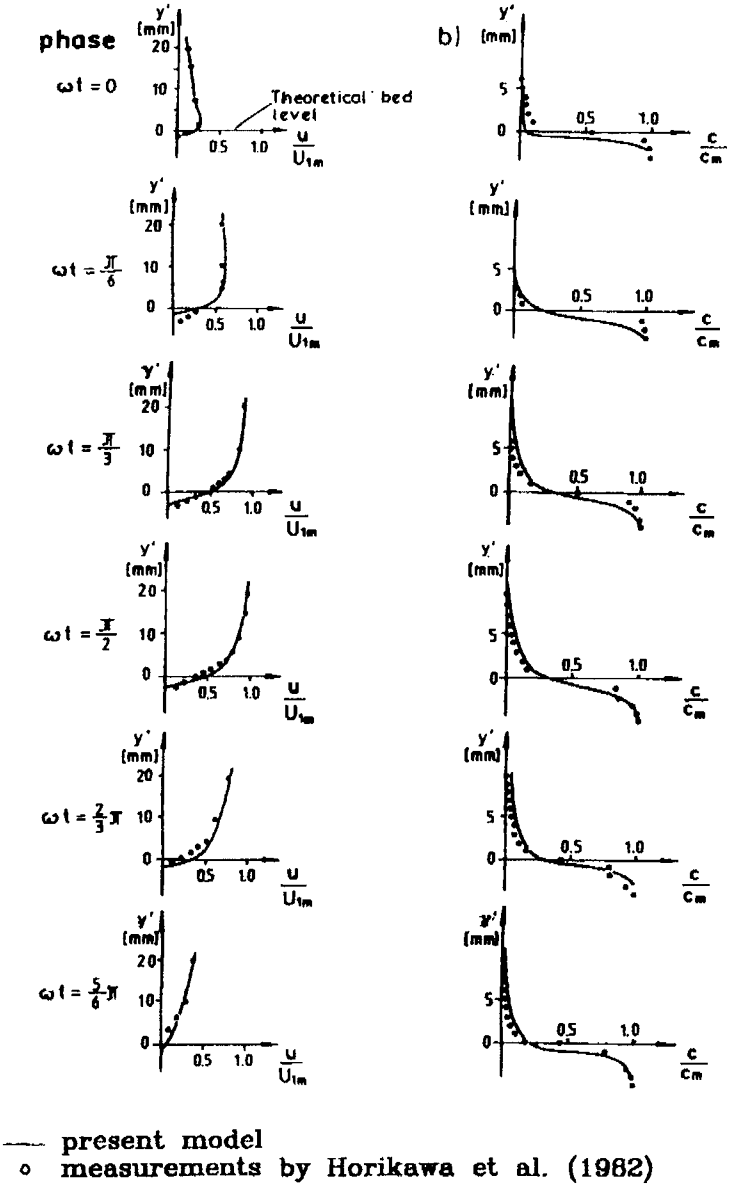

Figure 2: Theoretical and experimental distributions of velocity (a) and concentration (b) below and above the bed.

The results of the analysis of friction for wave-induced sheet flow, shown in Figure 4, suggest that the present approach restricted to the sheet flow regime may be extended to lower flow regimes and on the basis of analogy used to investigate lower flow conditions involving bed ripples.

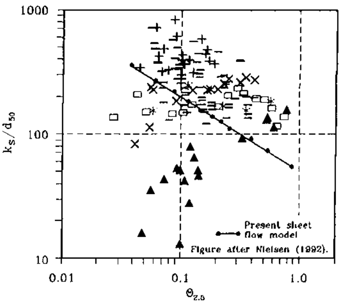

Figure 3: Nikuradse sand roughness by present theory along with results of Nielsen .

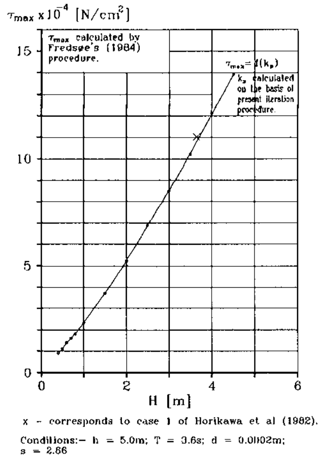

Figure 4: Calculation of maximum shear stresses

3.2 Rippled bed

Calculations for a rippled bed were performed for two different sediment sizes (0.2mm and 0.12mm diameter quartz sands). The calculations were carried out in two steps. Firstly, the values of the bed roughness k_s were obtained using the proposed iterative scheme. Then, the friction factors were calculated on the basis of an adjusted version of the semi-empirical formula of Jonsson and Carlsen (1976) in order to include the effects of the vortices formed in the lee of the roughness element crest due to turbulent mixing.

The theoretical results are shown in Figure 5. Presented alongside these results are the experimental results over a moveable bed reported by Madsen *et al.* (1990). The values of wave friction factor f_w are plotted against the representative value of a fluid-sediment interaction parameter, defined as:

$$S_r = \frac{\theta'}{\theta_c} \tag{3.1}$$

in which the skin Shields parameter is defined for a monochromatic wave as:

$$\theta' = \frac{u'^2_{f\,max}}{(s-1)gd} \tag{3.2}$$

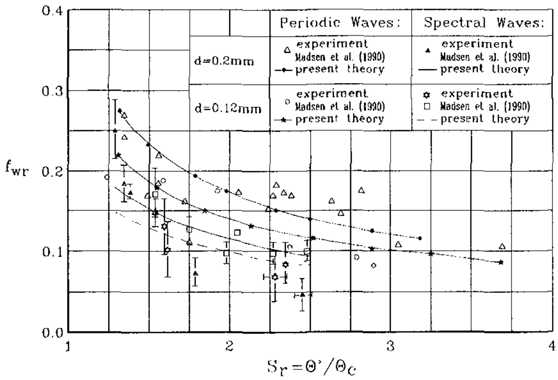

Figure 5: Moveable bed friction factors.

The agreement between theoretical and experimental results appears quite reasonable. It therefore appears that the sheet flow model can be used to investigate rippled bed conditions.

If the model can be used to investigate rippled bed conditions then it might also be possible to extend the analogy to include spectral wave conditions.

4 Spectral sheet flow model

4.1 Introduction

In the real world the Sea's motion is a random process. To describe a real sea it is usual to use spectral methods. However, it is possible to simplify the process by using appropriate representative values for the spectral components (See O'Connor *et al.* 1992).

The effect of random waves on bed roughness needs to be studied, since it is known that the bed friction changes between mono-frequency and random wave conditions. It is hypothesized by Madsen *et al* (1990) that the larger waves in a spectral simulation shave off the sharp ripple crests thereby causing the observed reduction in dissipation and friction factors for spectral waves. In an attempt to explain this reduction of spectral wave friction factors a new theoretical approach for predictive evaluation of moveable bed roughness for spectral waves is proposed. The new approach is based on the methodology which assumes that the spectral wave condition can be represented by a monochromatic wave and is combined with the theoretical grain-grain interaction ideas.

4.2 Modified iterative method

Following on from the iterative method used for monochromatic waves, a modified iterative procedure to evaluate the moveable bed roughness under spectral waves is proposed as shown in Figure 6.
Representative values are used in the calculation routine for the free stream velocity and the angular frequency. Previously for monochromatic waves, the maximum value of shear stress was the maximum value of shear stress during a wave period. For spectral waves the maximum value of the random shear stress time series is used:

$$\tau_{max} = \frac{3\tau_{rms}}{\sqrt{2}} = 3\sigma_{\tau} \tag{4.1}$$

The choice of this maximum value of the random shear stress time series was checked using the simple Rayleigh Method as well as a through running a more sophisticated one dimensional through depth (1DV) k-ε boundary layer model.

4.3 Spectral shear stress

Using the Rayleigh method, it is possible to quickly determine a value for the shear stress for a random time series. Assuming a Rayleigh distribution then:

$$\frac{\tau_{max}}{\tau_{rms}} = [\ln(N)]^{\frac{1}{2}} = R \tag{4.2}$$

T_z /sec	10	10	10
Time /min	10	20	40
N	60	120	240
R	2.02	2.18	2.34

The assumed value of R is:

$$3/\sqrt{2} = 2.12$$

The 1DV k-ε boundary layer model provides a method to directly simulate a random from shear stress from a known random velocity field. The method is based on the previous work of O'Connor *et al.* (1992) where a zero equation mixing length model was used to simulate a random sea.
The two equation k-ε model uses the standard equations to represent the momentum, the turbulent energy, k and the dissipation rate , ε.
Momentum:

$$\frac{\partial u}{\partial t} = \frac{\partial u_f}{\partial t} + \frac{\partial}{\partial z}\left(\nu_t \frac{\partial u}{\partial z}\right) \tag{4.3}$$

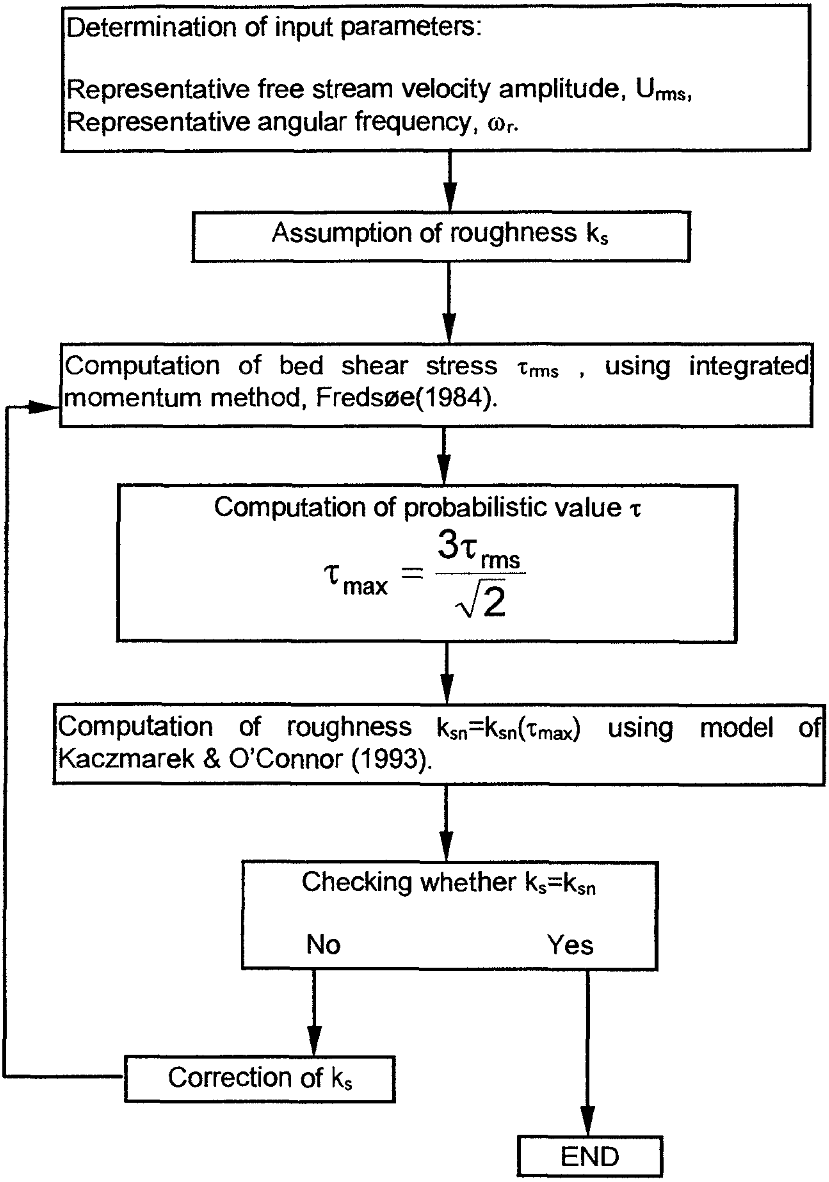

Figure 6: Modified iteration scheme for spectral waves.

Turbulent Energy, k:

$$\frac{\partial k}{\partial t} = \frac{\partial}{\partial z}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial z}\right) + \nu_t\left(\frac{\partial u}{\partial z}\right)^2 - \varepsilon \tag{4.4}$$

Dissipation Rate, ε:

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial}{\partial z}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial z}\right) + c_{1\varepsilon}\frac{\varepsilon}{k}\nu_t\left(\frac{\partial u}{\partial z}\right)^2 - c_{2\varepsilon}\frac{\varepsilon^2}{k} \tag{4.5}$$

Turbulent Eddy Viscosity, ν_t:

$$\nu_t = c_1\frac{k^2}{\varepsilon} \tag{4.6}$$

The upper boundary condition for the k-ε model is given by :-

$$u_o(t) = \sqrt{\frac{2M_{oc}}{N}}\sum_{n=1}^{N}\left(\frac{\omega_n}{\sinh(k_n d)}\right)\cos(-\omega_n t + \delta_n) \tag{4.7}$$

Results from the model appear to indicate that the shear stress time series is not necessarily Rayleigh in its distribution. A typical model value for R was 2.6.

5. Results

5.1 Spectral bed roughness

The ability of the present iteration procedure, shown in Figure 6 to evaluate moveable bed roughness, k_s, under spectral waves was checked for a sandy bed: $s = \rho_s/\rho = 2.66$ $\varphi = 24.4°$ with different grain size and various wave conditions. The results of the computations plotted in Figure 7 are for both irregular and regular waves.

In an attempt to explain the reduction of spectral wave friction factors the present theoretical approach was compared with Madsen *et al.* (1990) laboratory data. The results are shown in Figure 5 with the previous results for a monochromatic wave. The parameters are defined as before except that for spectral waves as the skin Shields parameter is given by:

$$\theta' = \frac{u_{fr}'^2}{(s-1)gd} = \frac{\tau'_{rms}}{\rho(s-1)gd} \tag{5.1}$$

The calculation of the friction factors were carried out in two steps. First, the values of the bed roughness k_s were obtained using the modified iterative scheme (Figure 6) with Fredsøe's (1984) model used to determine the bed shear stress, τ_{rms}. Then the friction factors were calculated on the basis of adjusted the semi-empirical formula of Jonsson

& Carlsen (1976) , as for monochromatic waves, in order to include the contribution of vortex formation in the lee of the roughness crests on the shear stress. Here, Jonsson & Carlsen's (1976) formulae were proposed for the calculations of both the friction factors and the dimensionless skin shear stresses.

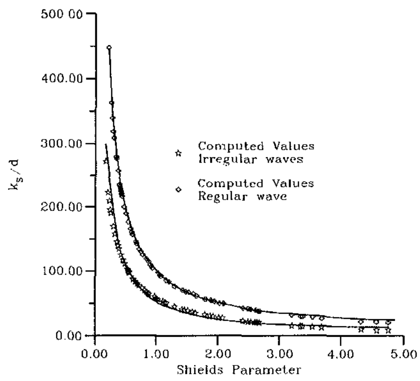

Figure 7: Results of sheet flow model for regular and random waves.

Similarly as for monochromatic waves, the calculations were performed for two different sediments (0.2mm and 0.12mm diameter quartz sands). Again the agreement between theoretical and experimental results appears quite satisfactory.

6. Conclusions

The sheet flow model appears to produce reasonable results for the conditions tested. However further testing is required.

The use of the model for a range of flow conditions and grain sizes produces a trend of large bed roughnesses at low flow regimes. According to this trend it is suggested that the sheet flow model provides a simple method, or rather an analogy, for the investigation of rippled bed conditions.

Using τ_{rms} to represent mono-frequency waves and τ_{max} to represent spectral waves produces a reasonable agreement with laboratory data.

The simple model results for k_s / d may be of use in preliminary engineering estimates although further testing is required. The present findings can be summarized for both the plane and rippled bed by the equation:

$$\tau_{rms1,2} = F_{1,2}\left[U_{rms}, T_p, k_s = f\left(\frac{3\tau_{rms1}}{\sqrt{2}}, s, d\right)\right] \tag{6.1}$$

The subscripts 1 and 2 refer to the plane and rippled bed respectively.
The function f is described by the proposed iterative procedure and may be represented by the approximating formula:

$$\log\left[\frac{k_s}{d}\right] = -1.05 \log\left[\theta_{rms1}\right] + 4.00 \tag{6.2}$$

where the Shields parameter is calculated using Fredsøe's (1984) model.
The above approximation differs from that given for monochromatic waves due to the largest waves causing a reduction in the roughness parameter.
To calculate the function F in the case of a plane bed, Fredsøe's (1984) model is recommended ($F_1 \rightarrow \tau_{rms1}$). For the rippled bed case, the empirical formula of Kamphuis (1975) or semi-empirical formula of Jonsson & Carlsen (1976) have been used ($F_2 \rightarrow \tau_{rms2}$) in order to include the effects of the vortices formed in the lee of the roughness crest on the turbulent mixing.
Based on experimental data, it was found that the representative period equals the peak period. It appears as though the proposed method of predicting bed roughness in spectral waves by using ideas derived for sheet-flow modelling and a representative design wave is capable of providing realistic values for effective bed roughness height. Further work is in progress on the application of the model to additional North Sea data.

7. References

Brevik, I. (1981). 'Oscillatory rough turbulent boundary layers.' J. Waterways, Port, Coastal and Ocean Eng. Div., ASCE, Vol. 107, No.WW3, pp175-188.

Fredsøe, J. (1984). 'The turbulent boundary layer in combined wave and current motion.' J. Hydraulic Eng., ASCE, Vol. 110, No. HY8, pp1103-1120.

Horikawa, K., Watanabe, A. and Katori, S. (1982). 'Sediment transport under sheet flow condition.' Proc. 18th Int. Conf. on Coastal Eng., ASCE, Cape Town, South Africa, pp1335-1352.

Jonsson, I.G. and Carlsen, N.A. (1976). 'Experimental and theoretical investigations in an oscillating turbulent boundary layer.' J. Hydr. Res., Vol. 14, No. 1, pp45-59.

Kaczmarek, L.M. and O'Connor, B.A. (1993a). 'A new theoretical approach for predictive evaluation of wavy roughness on a moveable-flat bed.' Part I, Report No. CE/14/93, Department of Civil Engineering, University of Liverpool, 31pp.

Kaczmarek, L.M. and O'Connor, B.A. (1993b). 'A new theoretical approach for predictive evaluation of wavy roughness on a moveable-rippled bed.' Part II, Report No. CE/15/93, Department of Civil Engineering, University of Liverpool, 29pp.

Kamphuis, J.W. (1975). 'Friction factor under oscillatory waves.' J. Waterways, Port, Coastal and Ocean Eng. Div., ASCE, Vol. 101, No. WW2, pp135-144.

Kajiura, K. (1968). 'A model of the bottom boundary layer in water waves.' Bull. Earthq. Res. Inst., Univ. Tokyo, Vol. 46, pp75-123.

Madsen, O.S., Mathison, P.P. and Rosengaus, M.M. (1990). 'Moveable bed friction factors for spectral waves.' Proc. 22nd Int. Conf. on Coastal Eng., ASCE, pp420-429.

Nielsen, P. (1992). 'Coastal bottom boundary layers and sediment transport'. Advanced Series on Ocean Engineering, Vol. 4, World Scientific, Singapore, 324pp.

O'Connor, B.A., Harris, J.M., Kim, H., Wong, Y.K., Oebius, H.U. and Williams, J.J. (1992). 'Bed boundary layers.' Proc. 23rd Int. Conf. on Coastal Eng., ASCE, pp2307-2320.

Savage, S.B. (1984). 'The mechanics of rapid granular flows.' Advances in Applied Mechanics, Vol. 24, pp289-367.

Sayed, M. and Savage, S.B. (1983). 'Rapid gravity flow of cohesionless granular materials down inclined chutes.' J. Applied Mathematics and Physics (ZAMP), Vol.34, pp84-100.