CHAPTER 22

Time-dependent mild-slope equations for random waves

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Abstract

Linear and nonlinear governing equations are derived to calculate the time evolution of random waves subject to refraction and diffraction.

In the linear theory, the frequency-dependent coefficients in the mild-slope equation (Berkhoff, 1972) are approximated by a rational function of the frequency, and then a time-dependent and frequency-independent expression of the mild-slope equation is derived. The resulting equation is applicable to simulate the transformation of random waves in the nearshore zone. Results of numerical calculation agree well with experimental results for random wave shoaling in the offshore zone.

A set of nonlinear governing equations is also derived to simulate the nonlinear wave transformation. The velocity potential for the wave motion is expressed as a series in terms of a given set of vertical distribution functions. Then, the Lagrangian is integrated vertically and the variational principle is applied to yield a set of nonlinear, time-dependent, two-dimensional governing equations for the nonlinear random wave tranformation. Comparison between the results of numerical calculation and flume experiment shows good agreement for the random wave shoaling near the breaking point and for wave disintegration due to a submerged breakwater.

1 Introduction

The mild-slope equation derived by Berkhoff (1972) has widely been used in the numerical calculation of refraction and diffraction of regular waves. However, the randomness of sea waves has a significant effect on the wave transformation especially due to refraction and diffraction. In this paper, linear and nonlinear governing equations are derived to calculate the time evolution of random waves subject to refraction and diffraction. In the linear equation, a term for the energy dissipation due to breaking is added to simulate the random wave field in the near shore zone. Results of numerical calculations are compared with those of laboratory experiments in wave flumes.

2 Linear Theory

2.1 Derivation

2.1.1 rational approximation

The mild-slope equation derived by Berkhoff (1972) is written as

$$\nabla(cc_g\nabla\hat{\eta}) + k^2cc_g\hat{\eta} = 0 \tag{1}$$

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where $\hat{\eta}$ is the complex amplitude of the water surface elevation, c the wave celerity, c_g the group velocity and k the wave number, and ∇ denotes the differential operator in the horizontal two directions. To simplify the equation, the transformation by Radder (1979) is employed:

$$\hat{\phi} = \hat{\eta}/\sqrt{cc_g} \tag{2}$$

Then, within the accuracy up to the first order in the bottom slope, the resultant equation becomes a Helmholtz equation:

$$\nabla^2\hat{\phi} + k^2\hat{\phi} = 0 \tag{3}$$

The time-dependent quantity, ϕ, corresponding to $\hat{\phi}$ is expressed as

$$\phi = \hat{\phi}e^{-i\omega t} \tag{4}$$

where ω is the angylar frequency and t the time. For the random wave analysis, ϕ is composed of an infinite number of component waves and the angular frequency differs from component to component; however, a unique value must be chosen to express ϕ of random waves. Thus a slowly varying amplitude, $\tilde{\phi}$, is defined from ϕ as

$$\phi = \tilde{\phi}e^{-i\bar{\omega}t} \tag{5}$$

where $\bar{\omega}$ is a certain representative angular frequency such as the average frequency. Comparison between Eqs. (4) and (5) gives

$$\tilde{\phi} = \hat{\phi}e^{-i\omega' t} \tag{6}$$

where ω' is the deviation from the representative frequency and defined as

$$\omega' = \omega - \bar{\omega} \tag{7}$$

Equation (6) implies that $\tilde{\phi}$ is a slowly varying function of time.

Since the Helmholtz equation (3) is independent of time, the governing equation for $\tilde{\phi}$ has the same form. When an energy dissipation term which is expressed in terms of the energy dissipation coefficient, f_D, is added, the equation is expressed as

$$\nabla^2\tilde{\phi} + k^2(1 + if_D)\tilde{\phi} = 0 \tag{8}$$

Equation (8) cannot be used to calculate $\tilde{\phi}$ of random waves directly since the coefficients included vary with the frequency. Linear approximation (Kubo *et al.*, 1992) and parabolic approximation (Kotake *et al.*, 1992) to the coefficients were employed in the previous studies. To improve the accuracy of approximation, a rational function is used in the present study.

Consider the following approximation to Eq. (8):

$$\nabla^2\tilde{\phi} - ia_1\nabla^2(\frac{\partial\tilde{\phi}}{\partial t}) + (b_0 + ic_0)\tilde{\phi} + i(b_1 + ic_1)\frac{\partial\tilde{\phi}}{\partial t} - b_2\frac{\partial^2\tilde{\phi}}{\partial t^2} = 0 \tag{9}$$

where the coefficients a_1, b_0, b_1, b_2, c_0 and c_1 are constants and independent of the frequency. Theoretical consideration on stability condition requires that the highest orders of approximation for b and c should be second and first, respectively. The order

for a is lower than that for b so that the ADI method may be available in the numerical calculation.

For monochromatic progressive waves expressed as

$$\tilde{\phi} = a e^{i(\hat{k}x\cos\theta + \hat{k}y\sin\theta - \omega' t)} \tag{10}$$

Eq. (9) becomes

$$-\hat{k}^2 + a_1\hat{k}^2\omega' + (b_0 + ic_0) + (b_1 + ic_1)\omega' + b_2\omega'^2 = 0 \tag{11}$$

from which the approximated dispersion relation is obtained as

$$\hat{k}^2 = (b_0 + b_1\omega' + b_2\omega'^2)/(1 - a_1\omega') \tag{12}$$

The values of the coefficients should be determined so that the error in the approximation (12) may become minimum without causing numerical instability.

2.1.2 determination of coefficients

Equation (11) can be solved for ω' as

$$\omega' = \{-(a_1\hat{k}^2 + b_1 + ic_1) \pm \sqrt{(a_1\hat{k}^2 + b_1 + ic_1)^2 - 4b_2(-\hat{k}^2 + b_0 + ic_0)}\}/(2b_2) \tag{13}$$

To avoid numerical divergence, $Im\{\omega'\} \leq 0$. This requires that the magnitude of the imaginary part for $\sqrt{\ }$ should not exceed c_1. Let the real and imaginary parts in the $\sqrt{\ }$ be denoted by X and Y, respectively, then the condition is written as

$$X \geq 0 \qquad (c_1 = 0) \tag{14}$$

$$X \geq (Y/2c_1)^2 - c_1^2 \qquad (c_1 > 0) \tag{15}$$

The above condition should be satisfied for an arbitrary $\hat{k}$, which yields

$$b_1^2 - 4b_0b_2 \geq 0, \quad c_0 = 0 \qquad (c_1 = 0) \tag{16}$$

$$(\frac{c_0}{c_1})^2 - (\frac{b_1}{b_2})(\frac{c_0}{c_1}) + (\frac{b_0}{b_2}) \leq 0 \qquad (c_1 > 0) \tag{17}$$

Within the above restrictions, we take the equal sign for the sake of convenience. Then,

$$b_1 = 2\sqrt{b_0b_2} \tag{18}$$

$$c_1 = (2b_2/b_1)c_0 \tag{19}$$

By considering the above two equations, independent parameters are a_1, b_0, b_2 and c_0.

When we determine the values of these parameters, we compensate for the error included in the finite difference form of the equation. For waves progressive in the x-direction, the central finite difference expressions for each term in Eq. (9) are related with the corresponding derivatives as

$$\left.\begin{aligned} &\left.\frac{\partial^2\tilde{\phi}}{\partial x^2}\right|_{\text{F.D.}} = \alpha_2\beta_0\frac{\partial^2\tilde{\phi}}{\partial x^2}, \quad \left.\frac{\partial^2}{\partial x^2}(\frac{\partial\tilde{\phi}}{\partial t})\right|_{\text{F.D.}} = \alpha_2\beta_1\frac{\partial^2}{\partial x^2}(\frac{\partial\tilde{\phi}}{\partial t}), \quad \left.\tilde{\phi}\right|_{\text{F.D.}} = \beta_0\tilde{\phi} \\ &\left.\frac{\partial\tilde{\phi}}{\partial t}\right|_{\text{F.D.}} = \beta_1\frac{\partial\tilde{\phi}}{\partial t}, \qquad \left.\frac{\partial^2\tilde{\phi}}{\partial t^2}\right|_{\text{F.D.}} = \beta_2\frac{\partial^2\tilde{\phi}}{\partial t^2} \end{aligned}\right\} \tag{20}$$

where $|_{\mathrm{F.D.}}$ denotes the finite difference expressions and

$$\left.\begin{aligned} &\beta_0 = (2/3)\cos\omega'\Delta t + (1/3), \qquad \beta_1 = \{(\sin\omega'\Delta t)/(\omega'\Delta t)\}^2 \\ &\beta_2 = \left\{\left(\sin\frac{\omega'\Delta t}{2}\right)/\left(\frac{\omega'\Delta t}{2}\right)\right\}^2, \quad \alpha_2 = \left\{\left(\sin\frac{\hat{k}\Delta x}{2}\right)/\left(\frac{\hat{k}\Delta x}{2}\right)\right\}^2 \end{aligned}\right\} \tag{21}$$

are correction factors. Then, instead of Eq. (11), the finite difference equation for Eq. (9) implies

$$-\alpha_2\hat{k}^2\beta_0 + a_1\alpha_2\hat{k}^2\beta_1\omega' + b_0\beta_0 + b_1\beta_1\omega' + b_2\beta_2\omega'^2 = 0 \tag{22}$$

for $c_0 = c_1 = 0$. Three independent parameters can be determined from three sets of ω' and k which satisfy the dispersion realation exactly:

$$-b_2^*\alpha_2 k_l^2\beta_0 + a_1^*\alpha_2 k_l^2\beta_1\omega_l' + \xi^2\beta_0 + 2\xi\beta_1\omega_l' + \beta_2\omega_l'^2 = 0 \quad (l = 1,2,3) \tag{23}$$

where

$$b_2^* = 1/b_2, \quad \xi = \sqrt{b_0/b_2}, \quad a_1^* = a_1/b_2 \tag{24}$$

Since Equations expressed by (23) are linear in b_2^* and a_1^*, these parameters can be eliminated to yield a parabolic equation in terms of ξ. After solving for ξ, we can determine b_2, b_0, a_1 and b_1 by Eqs. (18) and (24).

2.1.3 breaking wave model

Breaking wave model used is the same as Isobe (1987). First, the relative wave amplitude is defined by

$$\gamma = |\eta|/h \tag{25}$$

The critical relative amplitude, γ_{b}, for breaking of an individual wave in the random wave train is given as

$$\gamma_{\mathrm{b}} = 0.8 \times \gamma_{\mathrm{b}}' \tag{26}$$

$$\gamma_{\mathrm{b}}' = 0.53 - 0.3\exp(-3\sqrt{h/\bar{L}_{\mathrm{o}}}) + 5\tan^{3/2}\beta\,\exp\{-45(\sqrt{h/\bar{L}_{\mathrm{o}}} - 0.1)^2\} \tag{27}$$

After breaking, the energy dissipation coefficient is introduced as follows:

$$f_{\mathrm{D}} = \frac{5}{2}\tan\beta\sqrt{\frac{1}{\bar{k}_{\mathrm{o}}h}}\sqrt{\frac{\gamma - \gamma_{\mathrm{r}}}{\gamma_{\mathrm{s}} - \gamma_{\mathrm{r}}}} \tag{28}$$

$$\gamma_{\mathrm{s}} = 0.4 \times (0.57 + 5.3\tan\beta) \tag{29}$$

$$\gamma_{\mathrm{r}} = 0.135 \tag{30}$$

From f_{D} determined at the representative frequency, $c_0 = \bar{k}^2 f_{\mathrm{D}}$ and c_1 is calculated by Eq. (19). Thus all the coefficients in Eq. (9) are determined and $\tilde{\phi}$ can be calculated.

2.1.4 water surface elevation

In Eq. (2) which determines the water surface elevation from the calculated $\tilde{\phi}$, the coefficient $\sqrt{cc_{\mathrm{g}}}$ is also a function of frequency but can accurately be approximated by a second-order polynomial function. Therefore, Eq. (2) is approximated by

$$\tilde{\eta} = d_0\tilde{\phi} + d_1\frac{\partial\tilde{\phi}}{\partial t} + d_2\frac{\partial^2\tilde{\phi}}{\partial t^2} \tag{31}$$

where the constants, d_0, d_1 and d_2, are determined from the values of $\sqrt{cc_{\mathrm{g}}}$ at three different frequencies:

$$d_0 + d_1\beta_1\omega_l' + d_2\beta_2\omega_l'^2 = (1/\sqrt{cc_{\mathrm{g}}})_l \quad (l = 1,2,3) \tag{32}$$

2.2 Error evaluation

Since $1/\sqrt{cc_g}$ is constant for low frequency and proportional to the frequency for high frequency, the second-order approximation has a high accuracy. On the other hand, since k^2 is proportional to ω^2 and ω^4 for low and high frequency, respectively, even the rational approximation may not have a sufficient accuracy.

Figure 1 shows the interval from $\omega_{\min}$ to $\omega_{\max}$ within which the relative error of k^2 is less than 1%. The three frequencies for determining the coefficients are denoted

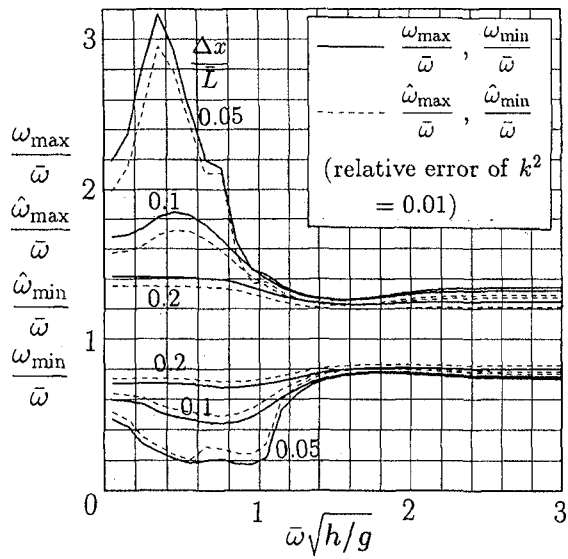

Figure 1: Frequency intervals within which the maximum relative error of k^2 is 1%

by $\hat{\omega}_{\min}$, $\bar{\omega}$ and $\hat{\omega}_{\max}$, and therefore ω_1', ω_2' and ω_3' in Eq. (23) are $\hat{\omega}_{\min} - \bar{\omega}$, 0 and $\hat{\omega}_{\max} - \bar{\omega}$, respectively. The horizontal axis is the nondimensionalized representative angular frequency $\bar{\omega}\sqrt{h/g}$ (h: the water depth; g: the gravitational acceleration). Lines are drawn for various relative grid size $\Delta x/\bar{L}$ ($\bar{L}$: the wavelength at $\bar{\omega}$). Figure 2 shows the same interval for various relative errors. From these figures, the interval becomes narrowest at about $\bar{\omega}\sqrt{h/g} = 1.4$. Finally, Fig. 3 shows the narrowest interval as a function of the relative error. The relative grid size, $\Delta x/\bar{L}$, is assumed to be 0.1 but does not have much influence.

From Fig. 3, if the relative error of k^2 is permitted up to 10%, most of the energy in random waves will be included in the interval and therefore the transformation of random waves can be analyzed by Eq. (9). This may usually be the case because the wave energy at the frequency far different from the representative frequency is usually small. However, for random waves with a very wide banded spectrum and a small relative error of k^2, the frequency interval have to be divided into several sections and the results of calculation for each section are superimposed.

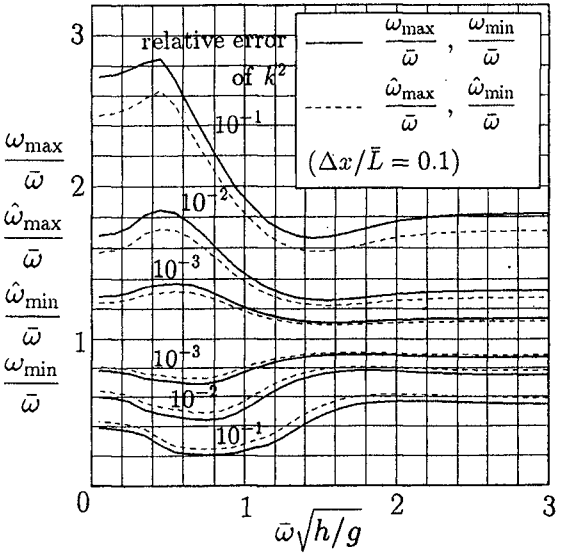

Figure 2: Frequency intervals for various maximum relative errors of k^2

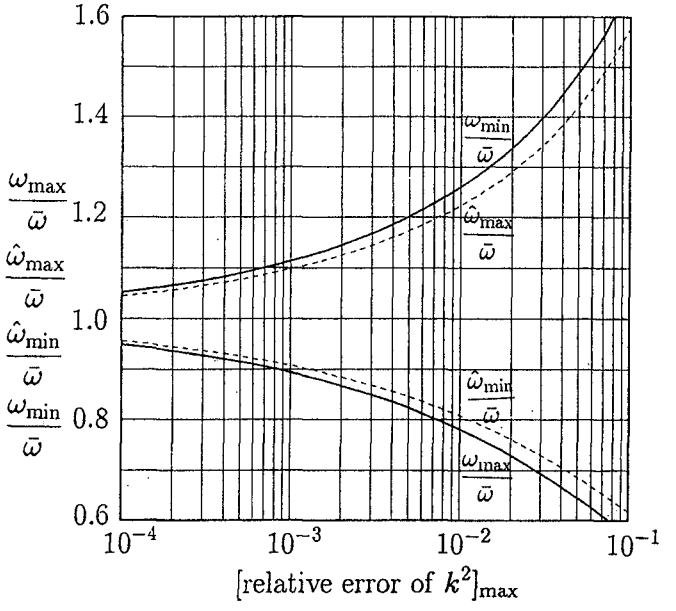

Figure 3: Frequency intervals in terms of the allowable relative error of k^2

2.3 Sample calculations

Once the representative frequency and the grid size are fixed, the gird size to wavelength ratio may become large for high frequency. Figure 4 examines the effect of grid size for analyzing wave shoaling on a uniformly sloping bottom. The angular frequency, ω, of the waves analyzed is $0.8\bar{\omega}$ for which the error of k^2 becomes almost maximum. In the upper figure, the agreement with the analytical solution is very good. Even in the lower figure for which the relative grid size is as large as 0.321, the agreement is not bad, which may be acceptable in analyzing far side frequency band component.

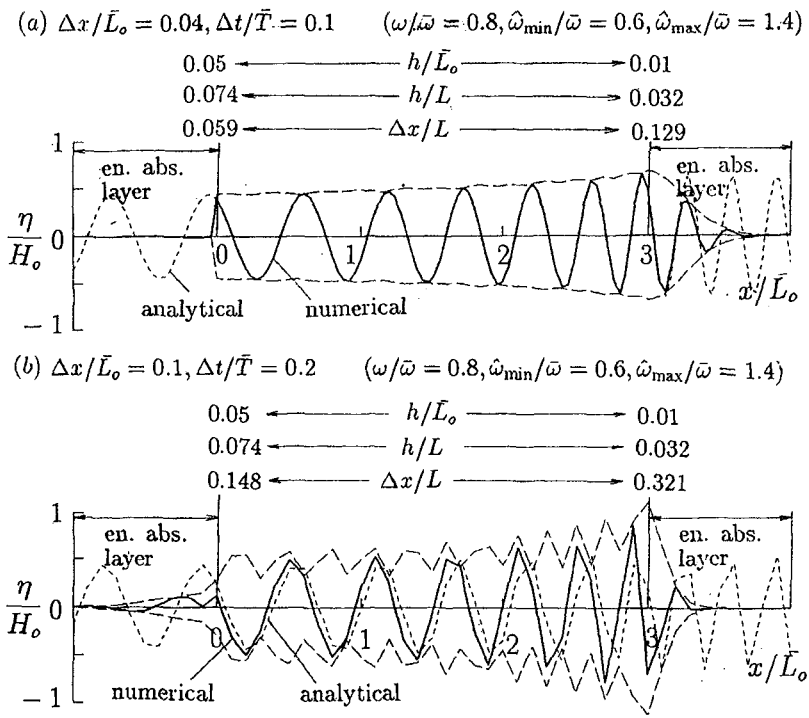

Figure 4: Effect of grid size and time interval on the accuracy

Figure 5 compares calculated and measured water surface elevation due to shoaling random waves. The incident wave profile which is shown in the upper figure was measured on a horizontal bottom with water depth of 40cm. From the point, a horizontal bottom with 0.4m in length, a 1/10 slope with 1m, and a 1/30 slope are installed. The onshoreward measuring point is located 2.6m from the beginning of the 1/30 slope and the water depth there is 21cm. The frequency interval was divided into four sections in the numerical calculation. The agreement is seen to be very good. However, near the breaker zone where nonlinearity of waves is strong, steepening of wave crests can not be reproduced by the present linear theory, even though energies of wave groups are fairly well reproduced. This implies that the present linear theory can be used to

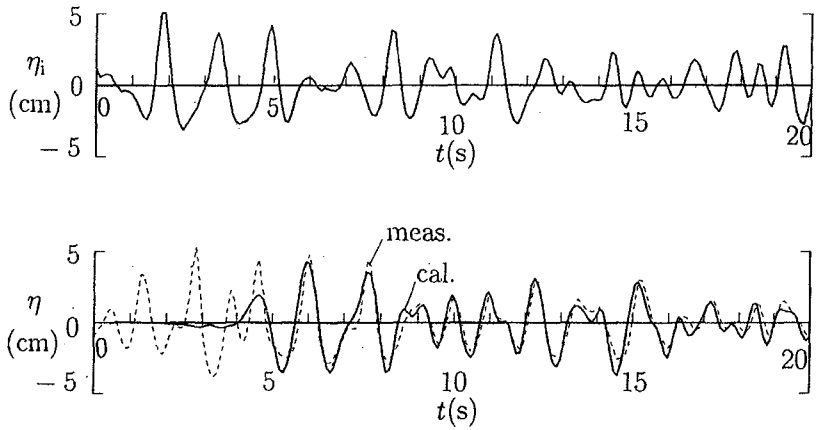

Figure 5: Comparison between calculated and measured water surface elevation in shoaling water

predict the distribution of integral properties such as the wave energy and radiation stress. To predict the wave profile in the nearshore zone, nonlinear theory must be employed.

3 Nonlinear Theory

3.1 Derivation

3.1.1 definition of Lagrangian

A Lagrangian L which is euivalent to the basic equation and boundary conditions for water surface waves is given as follows (Luke, 1967):

$$L[\phi,\eta] = \int_{t_1}^{t_2}\iint_A\int_{-h}^{\eta}\left\{\frac{\partial\phi}{\partial t}+\frac{1}{2}(\nabla\phi)^2+\frac{1}{2}\left(\frac{\partial\phi}{\partial z}\right)^2+gz\right\}dz\,dA\,dt \tag{33}$$

where unknown functions are the velocity potential ϕ and the water surface elevation η, and t_1 and t_2 denote the beginning and end of time, A the area of concern in (x,y) plane, h the water depth, g the gravitational acceleration, $\nabla = (\frac{\partial}{\partial x},\frac{\partial}{\partial y})$ the differential operator in the horizontal directions, $(x,y)=\mathbf{x}$ the horizontal coordinates, z the vertical coordinates, and t the time.

The variation of L due to small variations of ϕ and η is obtained from Eq. (33):

$$\begin{aligned}\delta L = &-\int_{t_1}^{t_2}\iint_A\int_{-h}^{\eta}\left(\nabla^2\phi+\frac{\partial^2\phi}{\partial z^2}\right)\delta\phi\,dz\,dA\,dt\\ &-\int_{t_1}^{t_2}\iint_A\left[\left\{\frac{\partial\phi}{\partial t}+\frac{1}{2}(\nabla\phi)^2+\frac{1}{2}\left(\frac{\partial\phi}{\partial z}\right)^2+g\eta\right\}\delta\eta\Big|_{z=\eta}\right.\\ &\left.+\left\{\frac{\partial\eta}{\partial t}+\nabla\eta\nabla\phi-\frac{\partial\phi}{\partial z}\right\}\delta\phi\Big|_{z=\eta}+\left\{\nabla h\nabla\phi+\frac{\partial\phi}{\partial z}\right\}\delta\phi\Big|_{z=-h}\right]dA\,dt\end{aligned}$$

$$+\int_{t_1}^{t_2}\oint_C\int_{-h}^{\eta}\frac{\partial\phi}{\partial n}\,\delta\phi\,dz\,ds\,dt+\iint_A\int_{-h}^{\eta}\left[\delta\phi\right]_{t_1}^{t_2}dz\,dA \tag{34}$$

where C denotes the boundary of A. To terminate L for small variations of ϕ and η in an arbitrary point, all the integrands in the above equation must vanish. The Laplace equation for ϕ can be obtained from the first integral, and the dynamic and kinematic free surface boundary conditions and the bottom boundary condition, respectively, from the first, second and third terms in the second integral. Therefore the application of the variational principle to L results in the basic equation and boundary conditions for water surface waves. The third and fourth integrals are, respectively, related with the lateral and initial conditions which are given in each specific problem.

3.1.2 vertical distribution functions

Wave equations such as the mild-slope equation and Boussinesq equation are two-dimensional equations which are obtained by integrating vertically the govering three-dimensional equations. For the integration, vertical distribution functions are introduced theoretically or a priori. Massel (1993) derived an extended mild-slope equation by introducing a vertical distribution function of hyperbolic cosine type and integrating the governing equation. A clear and generalized consept of this procedure was proposed by Nadaoka and Nakagawa (1993) and Nadaoka *et al.* (1994) in deriving a strongly-nonlinear, strongly-dispersive wave equation by applying the Galerkin method to the Euler equations of motion. Nochino (1994) used another set of vertical distribution functions to derive a nonlinear dispersive equation. The present theory also introduces vertical distribution functions and integrate the Lagrangian to yield a nonlinear mild-slope equation.

First, the three-dimensional dependent variable, ϕ, is expanded into a series in terms of a certain set of vertical distribution functions, $Z_\alpha(z)$, given a priori:

$$\phi(\mathbf{x},z,t)=\sum_{\alpha=1}^{N}Z_\alpha(z;h(\mathbf{x}))\,f_\alpha(\mathbf{x},t)\equiv Z_\alpha\,f_\alpha \tag{35}$$

where the function, Z_α, may change according to the water depth h, and f_α is the coefficient for Z_α and a function of x and t but not of z. The summation convention will be applied hereafter.

Then, after substituting Eq. (35) into Eq. (33), the integration is carried out in the vertical direction:

$$L[f_\alpha,\eta]=\int_{t_1}^{t_2}\iint_A\xi(f_\alpha,\frac{\partial f_\alpha}{\partial t},\eta,\frac{\partial\eta}{\partial t})\,dA\,dt \tag{36}$$

$$\begin{aligned}\xi(f_\alpha,\frac{\partial f_\alpha}{\partial t},\eta,\frac{\partial\eta}{\partial t})=&\frac{g}{2}(\eta^2-h^2)+\tilde{Z}_\beta\frac{\partial f_\beta}{\partial t}+\frac{1}{2}A_{\gamma\beta}\nabla f_\gamma\nabla f_\beta+\frac{1}{2}B_{\gamma\beta}f_\gamma f_\beta\\&+C_{\gamma\beta}f_\gamma\nabla f_\beta\nabla h+\frac{1}{2}D_{\gamma\beta}f_\gamma f_\beta(\nabla h)^2\end{aligned} \tag{37}$$

where

$$\tilde{Z}_\alpha=\int_{-h}^{\eta}Z_\alpha\,dz \tag{38}$$

$$A_{\alpha\beta} = \int_{-h}^{\eta} Z_\alpha Z_\beta \, dz \tag{39}$$

$$B_{\alpha\beta} = \int_{-h}^{\eta} \frac{\partial Z_\alpha}{\partial z} \frac{\partial Z_\beta}{\partial z} \, dz \tag{40}$$

$$C_{\alpha\beta} = \int_{-h}^{\eta} \frac{\partial Z_\alpha}{\partial h} Z_\beta \, dz \tag{41}$$

$$D_{\alpha\beta} = \int_{-h}^{\eta} \frac{\partial Z_\alpha}{\partial h} \frac{\partial Z_\beta}{\partial h} \, dz \tag{42}$$

The above coefficients, $\tilde{Z}_\alpha$, $A_{\alpha\beta}$, $B_{\alpha\beta}$, $C_{\alpha\beta}$ and $D_{\alpha\beta}$, are obtained from given vertical distribution functions and then the Lagrangian is expressed by Eq. (36) as an integral in the horizontal two-dimensional plane.

3.1.3 variational principle

Application of the variational principle to Eq. (36) in terms of η and f_α yields Euler equations which are expressed in general forms:

$$\frac{\partial \xi}{\partial f_\alpha} = \frac{\partial}{\partial t}\left[\frac{\partial \xi}{\partial(\partial f_\alpha/\partial t)}\right] + \nabla\left[\frac{\partial \xi}{\partial(\nabla f_\alpha)}\right] \tag{43}$$

$$\frac{\partial \xi}{\partial \eta} = \frac{\partial}{\partial t}\left[\frac{\partial \xi}{\partial(\partial \eta/\partial t)}\right] + \nabla\left[\frac{\partial \xi}{\partial(\nabla \eta)}\right] \tag{44}$$

Substituting Eq. (37) into Eqs. (43) and (44), a set of nonlinear partial differential equations is obtained for analyzing nonlinear water wave transformation:

$$Z_\alpha^\eta \frac{\partial \eta}{\partial t} + \nabla(A_{\alpha\beta}\nabla f_\beta) - B_{\alpha\beta} f_\beta + \nabla(C_{\beta\alpha} f_\beta \nabla h) - C_{\alpha\beta}\nabla f_\beta \nabla h - D_{\alpha\beta} f_\beta (\nabla h)^2 = 0 \tag{45}$$

$$g\eta + Z_\beta^\eta \frac{\partial f_\beta}{\partial t} + \frac{1}{2} Z_\gamma^\eta Z_\beta^\eta \nabla f_\gamma \nabla f_\beta + \frac{1}{2}\frac{\partial Z_\gamma^\eta}{\partial z}\frac{\partial Z_\beta^\eta}{\partial z} f_\gamma f_\beta + \frac{\partial Z_\gamma^\eta}{\partial h} Z_\beta^\eta f_\gamma \nabla f_\beta \nabla h + \frac{1}{2}\frac{\partial Z_\gamma^\eta}{\partial h}\frac{\partial Z_\beta^\eta}{\partial h} f_\gamma f_\beta (\nabla h)^2 = 0 \tag{46}$$

where

$$Z_\alpha^\eta = Z_\alpha|_{z=\eta} \tag{47}$$

The above equations includes terms up to the second order in the bottom slope; however, vertical distribution functions given will usually be consistent only with a horizontal or mild-slope bottom. Therefore, on assuming that the bottom slope is mild, the terms of the second order are neglected to yield a set of nonlinear mild-slope equations:

$$Z_\alpha^\eta \frac{\partial \eta}{\partial t} + \nabla(A_{\alpha\beta}\nabla f_\beta) - B_{\alpha\beta} f_\beta + (C_{\beta\alpha} - C_{\alpha\beta})\nabla f_\beta \nabla h + \frac{\partial Z_\beta^\eta}{\partial h} Z_\alpha^\eta f_\beta \nabla \eta \nabla h = 0 \tag{48}$$

$$g\eta + Z_\beta^\eta \frac{\partial f_\beta}{\partial t} + \frac{1}{2} Z_\gamma^\eta Z_\beta^\eta \nabla f_\gamma \nabla f_\beta + \frac{1}{2}\frac{\partial Z_\gamma^\eta}{\partial z}\frac{\partial Z_\beta^\eta}{\partial z} f_\gamma f_\beta + \frac{\partial Z_\gamma^\eta}{\partial h} Z_\beta^\eta f_\gamma \nabla f_\beta \nabla h = 0 \tag{49}$$

The total number of equations is $(N + 1)$ since Eqs. (48) and (49) contain 1 and N components, respectively. On the other hand, the total number of unknowns, η and f_α ($\alpha = 1$ to N), is $(N + 1)$. Therefore, with appropriate boundary conditions, the equations can be solved numerically. Then, the velocity is obtained through the velocity potential expressed by Eq. (35).

3.2 Sample vertical distribution functions

A set of vertical distribution functions should be selected so that the velocity potential may accurately be expressed by Eq. (35) with a small number of terms. As understood from the small amplitude wave theory, hyperbolic cosine functions may be effective for deep to intermediate water, whereas polynomial functions for very shallow water. Here, for the sake of simplicity, polynomial functions are chosen and analytical expressions for the coefficients are shown.

As inferred from shallow water wave theory, we select a set of even-order polynomial functions:

$$Z_\alpha = \left(\frac{h+z}{h}\right)^{2(\alpha-1)} \tag{50}$$

Then, Eqs. (47) and (39) to (41) give

$$Z_\alpha^\eta = \zeta^{2\alpha_1} \tag{51}$$

$$A_{\alpha\beta} = h\frac{\zeta^{2(\alpha_1+\beta_1)+1}}{2(\alpha_1+\beta_1)+1} \tag{52}$$

$$B_{\alpha\beta} = \frac{4\alpha_1\beta_1}{h}\frac{\zeta^{2(\alpha_1+\beta_1)-1}}{2(\alpha_1+\beta_1)-1} \tag{53}$$

$$C_{\alpha\beta} = -2\alpha\left[\frac{\zeta^{2(\alpha_1+\beta_1)+1}}{2(\alpha_1+\beta_1)-1} - \frac{\zeta^{2(\alpha_1+\beta_1)}}{2(\alpha_1+\beta_1)}\right] \tag{54}$$

where

$$\zeta = (h+z)/h \tag{55}$$

$$\alpha_1 = \alpha - 1 \tag{56}$$

$$\beta_1 = \beta - 1 \tag{57}$$

To check the effectiveness of the polynomial functions, the dispersion relation of the linearized equation is examined for a horizontal bottom. In the linear theory, the coefficients expressed by Eqs. (51) to (53) are evaluated at $z = 0$ instead of $z = \zeta$. By denoting the quantities at $z = 0$ by superscript ^{o}, the following equation can be obtained by eliminating η from the linearized forms of Eqs. (48) and (49) on a horizontal bottom:

$$-\frac{1}{g}Z_\alpha^o Z_\beta^o\frac{\partial^2 f_\beta}{\partial t^2} + \nabla(A_{\alpha\beta}^o\nabla f_\beta) - B_{\alpha\beta}^o f_\beta = 0 \tag{58}$$

where from Eqs. (51) to (53)

$$Z_\alpha^o = 1 \tag{59}$$

$$A_{\alpha\beta}^o = h/\{2(\alpha_1+\beta_1)+1\} \tag{60}$$

$$B^{o}_{\alpha\beta} = 4\alpha_1\beta_1/[h\{2(\alpha_1+\beta_1)-1\}] \tag{61}$$

For progressive waves,

$$f_\alpha = a_\alpha e^{i(\hat{k}-\omega t)} \tag{62}$$

By substituting the above expression into Eq. (58), the following homogenious equations are obtained:

$$\sum_{\beta=1}^{N}\left(\frac{\omega^2}{g}-\frac{1}{h}\frac{4\alpha_1\beta_1}{2(\alpha_1+\beta_1)-1}\right)a_\beta = \hat{k}^2\sum_{\beta=1}^{N}\frac{h}{2(\alpha_1+\beta_1)+1}a_\beta \tag{63}$$

For a given ω, $\hat{k}^2$ is obtained as an eigenvalue which gives a non-trivial solution to the above equations. At least up to $N = 4$, it was confirmed numerically that only one eigenvalue is positive and the others are negative. A positive value of $\hat{k}^2$, *i.e.*, a real value of $\hat{k}$, corresponds to progressive waves, and a negative value to evernescent waves. The relationship between the frequency and wave celerity of the progressive waves is shown in Fig. 6 for various N. As can be seen, agreement with the linear wave theory

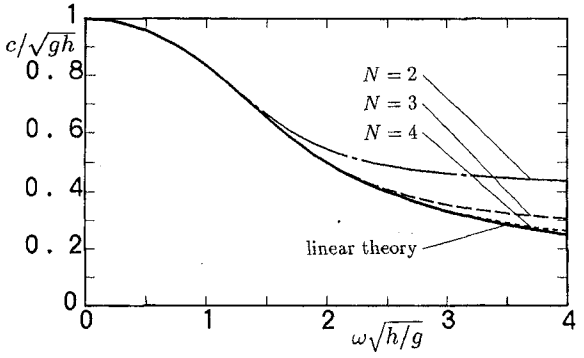

Figure 6: Dispersion relation for vertical distribution functions of even-order polynomials

is good for shallow to deep water even with small N.

3.3 Sample calculations

Figure 7 compares calculated and measured water surface elevation η and bottom velocity u_b in shoaling water. Even for Case 2-2 in which nonlinearity is strong at the measuring point, agreement is good for the bottom velocity as well as the water surface elevation.

Figure 8 compares calculated and measured water surface elevation around a submerged breakwater. Nonlinearity and dispersion are significant on the breakwater and on the horizontal bottom, respectively. Agreement is good even with a small number of N ($N = 3$).

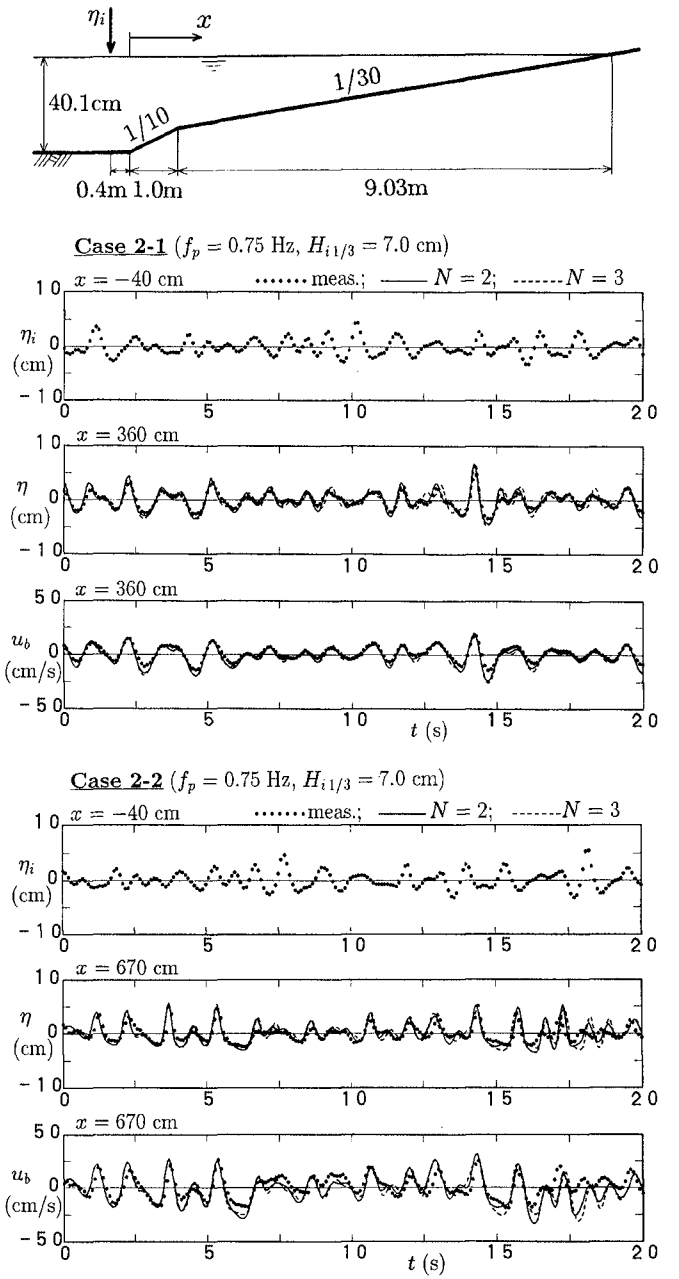

Figure 7: Comparison between calculated and measured water surface elevation and bottom velocity in shoaling water

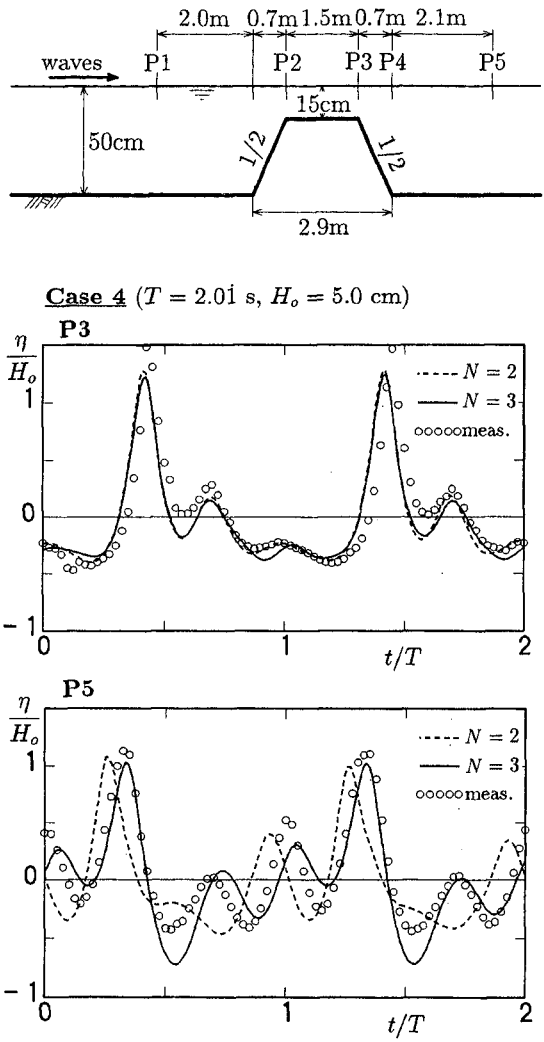

Figure 8: Comparison between calculated and measured water surface elevation on and behind the submerged breakwater

4 Conclusions

A time-dependent mild-slope equation for random waves is derived from the mild-slope equation by approximating rational function to the frequency-dependent coefficients. This equation allows to simulate the time evolution of short-crested random waves in the nearshore area. Agreement between calculated and measured water surface elevation in the offshore zone is good because wave nonlinearity is not essential.

A nonlinear mild-slope equation is derived by expanding the velocity potential into a series in terms of vertical distribution functions and then applying the variational principle to a Lagrangian. Comparison between calculated and measured quantities confirms the validity of the theory even for a strongly nonlinear and dispersive wave field.

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