CHAPTER 7

MEAN FLUX IN THE FREE SURFACE ZONE OF WATER WAVES IN A CLOSED WAVE FLUME

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Abstract

This paper presents some results of research relating to the theoretical predictions of mass-transport velocity within the free surface zone of water waves in intermediate water depth. The theoretical results are compared with measurements made in a wave flume.

The theoretical estimate of a mean drift has allowed for a better estimation of the return flow in the wave flume. Examples of such estimation are given and graphically presented in the paper. Finally, the stability of the obtained mean velocity profiles throught the experiments is examined.

Introduction

The occurrence of a second-order mean drift is one of the more interesting, and by the same time, important non-linear features of a progressive water gravity wave. This drift, an apparent mass transport, influences such a phenomena like migration of sediments and pollutant particles in the water, and it can also result in the pilling up of water at a beach, with an associated increase in the local mean water level.

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The formulae for the mass-transport velocity and the total mean flux are usually derived in the Lagrangian frame. The total mean flux in Eulerian frame was developed by Starr (1947) for a small-amplitude wave train and by Phillips (1960) for a random wave field. These traditional approaches, in the Eulerian frame, allow us only to treat the total flux like a physical quantity "existing on a subset of zero measure," namely, *exactly* at the free surface of the wave. Within such an approach we are not able to discuss the distribution of the mean horizontal velocity in the free surface zone. We would then have the artificial situation that the mean velocity along the vertical is everywhere equal to zero except at the free surface.

Tung (1975) has shown a positive mean value of the horizontal orbital velocity of random waves in the near surface zone, but to the authors' knowledge, this has not been discussed and interpreted as a current induced by waves until the works of Cieślikiewicz and Gudmestad (1993, 1994). In those works the modified particle velocity $\overline{\mathbf{u}}$ of a wave field is introduced by following the approach of Tung (1975):

$$\overline{\mathbf{u}}(\mathbf{x}, z, t) = \begin{cases} \mathbf{u}(\mathbf{x}, z, t) & \text{for} \quad z \leq \zeta(\mathbf{x}, t) \\ \mathbf{0} & \text{for} \quad z > \zeta(\mathbf{x}, t) \end{cases} \tag{1}$$

in which $\mathbf{u}$ is unmodified water wave orbital velocity, ζ is the free surface elevation, $\mathbf{x}$ is the location vector on the horizontal plane, z-axis is directed vertically upwards and t is the time.

Random waves

Tung (1975) had derived the probability density function and the first three statistical moments of this modified velocity. The mean value of the horizontal velocity component u is given as

$$\langle \overline{u}(z) \rangle = r(z)\sigma_u(z) Z(z') \tag{2}$$

where $Z(\gamma) = (2\pi)^{-1/2} \exp(-\gamma^2/2)$, σ_ζ and σ_u are the standard deviations of ζ and u, respectively, and $z' = z/\sigma_\zeta$. Assuming that the wave is unidirectional and denoting the frequency spectrum by $S(\omega)$, the cross-correlation coefficient of u and ζ is given as

$$r(z) = \frac{1}{\sigma_u(z)\sigma_\zeta} \int_0^\infty \frac{gk}{\omega} \frac{\cosh k(z+h)}{\cosh kh} S(\omega)\, d\omega \tag{3}$$

in which g denotes the gravitational acceleration, h is the water depth, and the wavenumber k is related to the angular frequency ω by the dispersion relation.

Cieślikiewicz and Gudmestad (1993) developed the formula for total mean flux of random waves using the modified velocity (1) in the following form:

$$q = \int_0^\infty \frac{gk}{\omega \cosh kh} W(-h; \sigma_\zeta, k) S(\omega)\, d\omega \tag{4}$$

where the function W is defined as

$$\begin{aligned} W(z_a; \sigma, k) &= \int_{z_a/\sigma}^{\infty} \cosh k(z+h) Z(z)\, dz \\ &= \frac{1}{2} \exp\left[\frac{\sigma^2 k^2}{2}\right] \left\{ e^{kh} Q(z/\sigma + k\sigma) + e^{-kh} Q(z/\sigma - k\sigma) \right\} \end{aligned} \tag{5}$$

and where $Q(z) = \int_z^\infty Z(\gamma)\, d\gamma$

The approximation of that formula leads to the result obtained by Phillips (1960). Phillips' formula for total mean flux, $q = \int_{-h}^{\infty} \langle u(z) \rangle \, dz$, may be easily obtained by using the modified velocity (1) and formulae (2) and (3) (see Cieślikiewicz 1994):

$$\begin{aligned} q = \frac{1}{\sigma_\zeta} \int_{-h}^{\infty} Z(z') \left[\int_0^\infty \frac{gk}{\omega} \frac{\cosh k(z+h)}{\cosh kh} S(\omega)\, d\omega \right] dz = \\ \int_0^\infty \frac{gk}{\omega} S(\omega) \left[\int_{-h}^{\infty} \frac{\cosh k(z+h)}{\cosh kh} Z(z')\, dz' \right] d\omega \approx \\ \int_0^\infty \frac{gk}{\omega} S(\omega) \left[\int_{-h}^{\infty} Z(z')\, dz' \right] d\omega \end{aligned} \tag{6}$$

since we assume in practice that $h \gg 3\sigma_\zeta$, then for $|z| < 3\sigma_\zeta$ we obtain in the above integral an approximation $\cosh k(z+h)/\cosh kh \approx 1$. A large error in this approximation outside the region $|z| < 3\sigma_\zeta$ is nonessential since the value of $Z(z')$ is close to zero here. As we have assumed $h \gg 3\sigma_\zeta$, we have $\int_{-h}^{\infty} Z(z')\, dz' \approx \int_{-\infty}^{\infty} Z(z')\, dz' = 1$. Thus

$$q \approx \int_0^\infty \frac{gk}{\omega} S(\omega)\, d\omega \tag{7}$$

Note that for deep water waves above formula can be rewritten as $q = \int_0^\infty \omega S(\omega)\, d\omega$ which is the value of the spectral moment of the first order m_1.

Deterministic wave

In the paper of Cieślikiewicz and Gudmestad (1994), the same approach as described above for random waves, is adapted to deterministic small-amplitude waves. The result for the total mean flux $q^{(\mathrm{d})}$ in the unidirectional wave case is equal to

$$q^{(\mathrm{d})} = \frac{ga}{\omega} I_1(ak) \tag{8}$$

where $I_1(\,\cdot\,)$ is the modified Bessel function of the first order. The above formula in approximation gives a result first obtained by Starr (1947): $M^{(\mathrm{d})} = \rho q^{(\mathrm{d})} = E/C$ where E is the average energy per unit surface area and C is the phase velocity. It should be emphasised that this approximation may be easily obtained by assuming that

$$\frac{\cosh k(z+h)}{\cosh kh} \approx 1 \qquad \text{for} \quad |z| \leq a \tag{9}$$

where a is the wave amplitude. Consider a unidirectional progresive small-amplitude wave of the form

$$\zeta(x,t) = a \cos(kx - \omega t) \tag{10}$$

The associated horizontal velocity under the wave is given by

$$u(x,z,t) = \frac{gak}{\omega} \frac{\cosh k(z+h)}{\cosh kh} \cos(kx - \omega t) \qquad \text{for} \quad z \in [-h, \zeta(x,t)] \tag{11}$$

Introduce the extension of u on the z-domain $[-h, \infty)$ by the definition

$$\overline{u}(x,z,t) = \begin{cases} u(x,z,t) & \text{for} \quad z \leq \zeta(x,t) \\ 0 & \text{for} \quad z > \zeta(x,t) \end{cases} \tag{12}$$

The mean value of u over a wave period T of a deterministic wave is

$$m_{\overline{u}}^{(\mathrm{d})}(z) = < \overline{u}(x,z,t) >^{(\mathrm{d})} = \frac{1}{T} \int_{-T/2}^{T/2} \overline{u}(x,z,t)\, dt \tag{13}$$

In view of (11) and (12)

$$m_{\overline{u}}^{(\mathrm{d})}(z) = \begin{cases} \dfrac{1}{T} \displaystyle\int_{t_1(z)}^{t_2(z)} u(x,z,t)\, dt & \text{for} \quad |z| \leq a \\ 0 & \text{for} \quad |z| > a \end{cases} \tag{14}$$

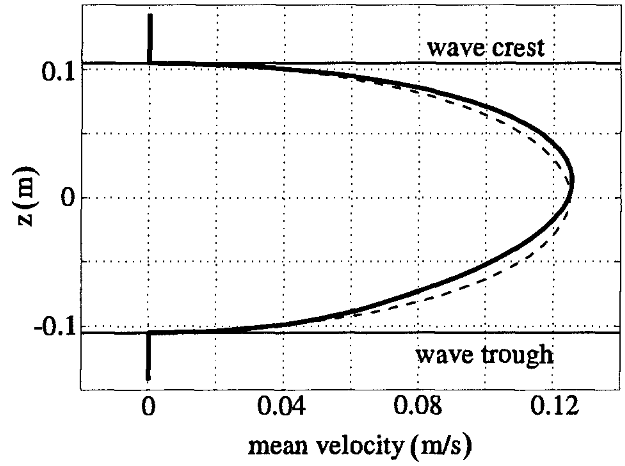

Fig. 1. Theoretical Eulerian mean velocity profile for deterministic small-amplitude wave (solid line) and its approximation (dashed line).

where t_1 and t_2 are such that

$$\begin{cases} z = \zeta(x, t_1) = \zeta(x, t_2) \\ z \le \zeta(x, t) \quad \text{for} \quad t_1 \le t \le t_2 \end{cases} \tag{15}$$

Carrying out the integration in (14) yields

$$m_{\overline{u}}^{(\mathrm{d})}(z) = \begin{cases} \dfrac{gak}{\pi\omega} \dfrac{\cosh k(z+h)}{\cosh kh} \sin\left(\arccos \dfrac{z}{a}\right) & \text{for} \quad |z| \le a \\ 0 & \text{for} \quad |z| > a \end{cases} \tag{16}$$

Taking into account the approximation (9) we obtain

$$m_{\overline{u}}^{(\mathrm{d})}(z) \approx \begin{cases} \dfrac{gak}{\pi\omega} \sin\left(\arccos \dfrac{z}{a}\right) & \text{for} \quad |z| \le a \\ 0 & \text{for} \quad |z| > a \end{cases} \tag{17}$$

The mean horizontal velocity profiles according to above expressions are presented in Fig. 1.

To obtain the total mean flux $q^{(\mathrm{d})}$ at a fixed position x (Eulerian frame) we perform the following integration

$$q^{(\mathrm{d})} = \int_{-h}^{\zeta(x,t)} m_{\overline{u}}^{(\mathrm{d})}(z)\, dz \tag{18}$$

In view of (17)

$$q^{(\mathrm{d})} \approx \int_{-a}^{a} \frac{gak}{\pi\omega} \sin\left(\arccos\frac{z}{a}\right) dz \tag{19}$$

By substituting $\theta = \arccos(z/a)$ we obtain

$$q^{(\mathrm{d})} \approx \frac{ga^2k}{\pi\omega} \int_{0}^{\pi} \sin^2\theta \, d\theta = \frac{ga^2}{2}\frac{k}{\omega} \tag{20}$$

Therefore, the flow of mass $M^{(\mathrm{d})}$ in approximation is equal to

$$M^{(\mathrm{d})} = \rho q^{(\mathrm{d})} \approx \frac{\rho g a^2}{2}\frac{k}{\omega} = \frac{E}{C} \tag{21}$$

which is the well-known result usually derived in the Lagrangian frame. The quantitative understanding of mass transport and return flow in the closed wave flume plays an important role in experimental studies of water wave kinematics. A review of recent research relating to the problem of return flow may be found in Gudmestad (1993).

Return flow

A theoretical prediction of the mean horizontal velocity (in the Eulerian frame) allows for a better estimation of the return flow in the wave flume. We suggest in the present study that the difference between the predicted and the measured mean values of mean horizontal velocity gives an estimate of the return current in the wave flume. Figs. 2, 3, and 4 show the results of such calculations applied to laboratory data. These data were collected in the Norwegian Hydrotechnical Laboratories' 33 m long, 1.02 m wide and 1.8 m deep wave channel by a two-component Laser Doppler Velocimeter (LDV). The LDV allowed wave velocity measurements from wave crest down to tank bottom but at one point in space only during one run. In order to obtain the distributions for the statistical properties of the velocity along the vertical axis it was necessary to repeat the experiment with exactly the same free surface elevation spectrum but locating the LDV station at different vertical positions. The first series I18, consisting of 12 runs is given by the significant wave height $H_S = 0.21$ m and peak period $T_p = 1.8$ s, while the second series I24 consisting of 13 runs is given by the significant wave height $H_S = 0.25$ m and peak period $T_p = 2.4$ s. The third series R15B consisting of 10 runs represens a deterministic wave given by the value of wave height $H = 0.26$ m and period $T = 1.5$ m. Digitisation of the free surface elevation and velocity time series was carried at a rate of 40 Hz and

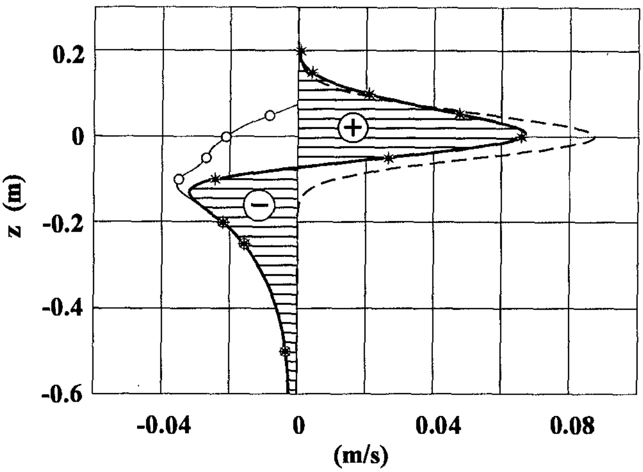

Fig. 2. Mean value of horizontal velocity and estimate of return flow: * observed mean values, - - - - theoretical mean drift, o estimated return flow. Wave case I18.

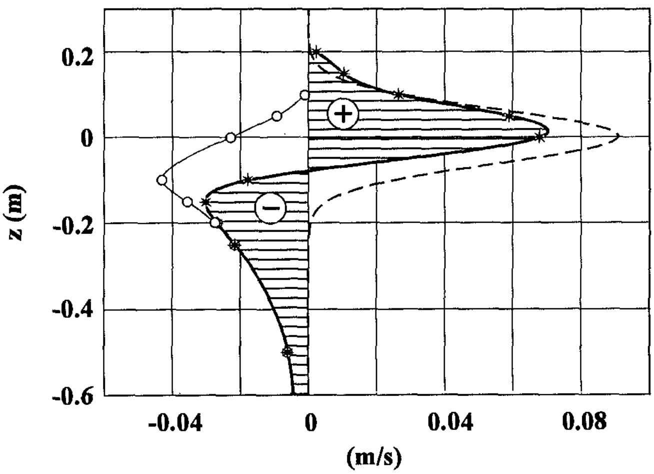

Fig. 3. Mean value of horizontal velocity and estimate of return flow: * observed mean values, - - - - theoretical mean drift, o estimated return flow. Wave case I24.

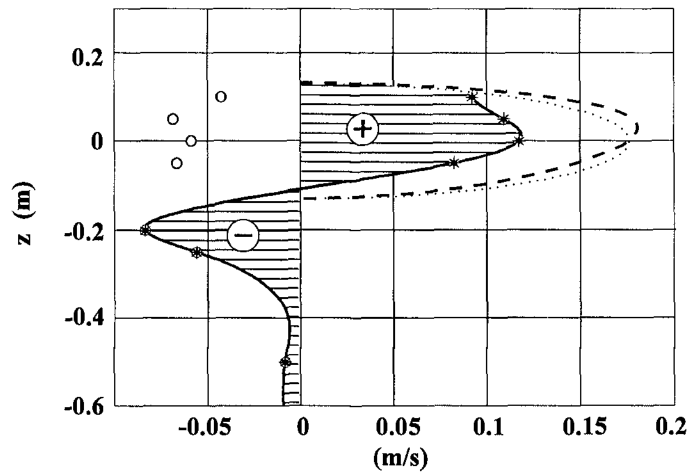

Fig. 4. Mean value of horizontal velocity and estimate of return flow for regular wave: $*$ observed mean values, – – – – theoretical mean drift and its approximation (dotted line), o estimated return flow.

samples of 32,768 measuring points were collected. The water depth was 1.3 m. The experimental arrangement is described in detail in papers by Skjelbreia *et al.* (1989, 1991).

In Figs. 2 and 3 the measured mean horizontal velocity is marked with stars for the irregular wave cases I18 and I24, respectively, from Skjelbreia's measurements. The dashed line presents the theoretical mean value of the modified (according to equation (1)) horizontal velocities. Open circles show the estimated values of the return flow.

Data for a deterministic case have been examined through analysis of data series R15B from Skjelbreia's experiments. The full line in Fig. 4 presents the measured (in the Eulerian frame) mean flow for this deterministic wave case.

The return flow profiles presented in Figs. 2, 3, and 4 were averaged over the whole velocity data collected during each run. In order to examine the stability of the mean velocity profiles obtained, each time series has been divided into four equal parts. We believe that all resulting sub-series were long enough for calculation of a statistical estimate of their mean values and standard deviations. We believe also that a comparison of those four mean values provides us with at least an indication of the stability of the mean velocity profiles. The results of these estimations are presented on Figs. 5 and 6 for the wave cases I18 and I24, respectively. It can be noted that the mean values as well as standard deviations of horizontal velocity calculated for each of four parts of time series are very much the same. Closer examination of the plots shows some similarities between I18 and I24 wave cases indicating that some trends in the mean horizontal velocity may exist. For example in both cases, for elevations below $z = -0.4\,\mathrm{m}$ the mean values in the first quarter of the experiment series have the smallest values, while, on the other hand, above that level it has the largest values.

Conclusions

In the approach of Cieślikiewicz and Gudmestad (1993) we are able, in the Eulerian frame, to discuss the distribution of the mean velocity along the vertical axis—the mean horizontal velocity is "stretched out" from the exact location at the surface onto the free surface zone. Moreover, we are able to calculate not only the total mean water flux but also the flux between two given z-elevations. The theoretical results relating to the current in the direction of the wave advance can be used for better estimation of a return current in the wave flume.

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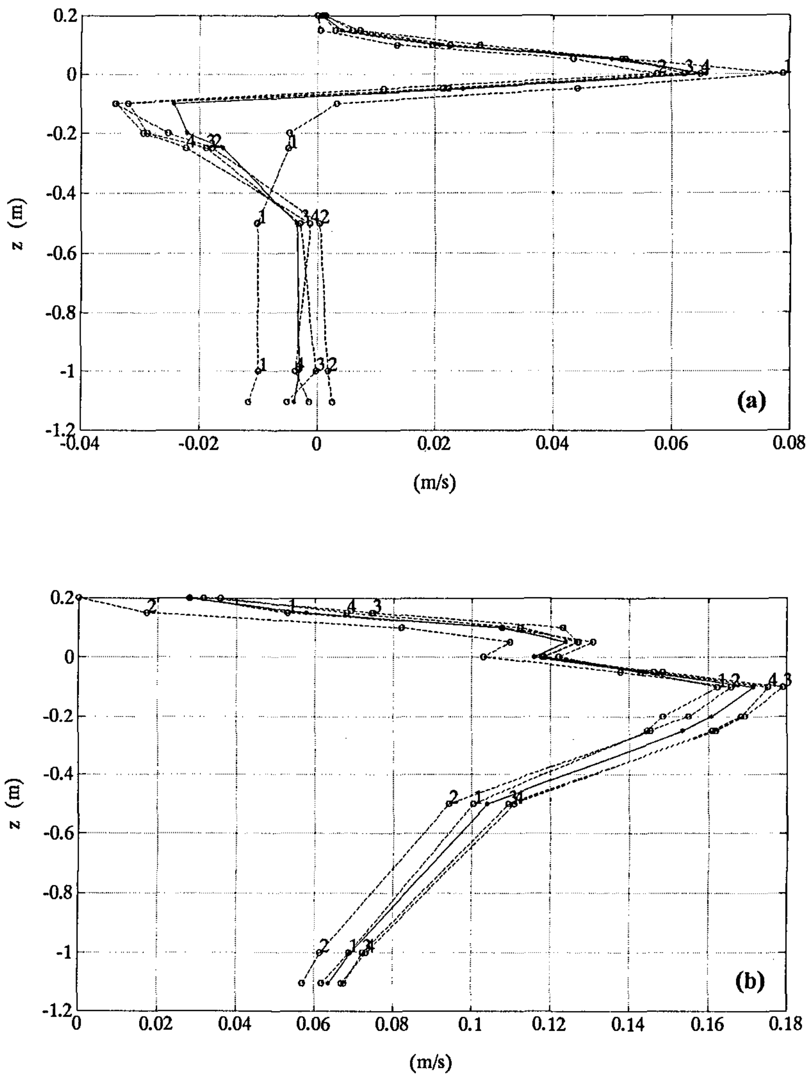

Fig. 5. Mean values (a) and standard deviations (b) of horizontal velocity calculated for each sub-series (dashed lines with circles) against the one calculated for the whole data series (solid line with stars). Wave case I18. The numbers of successive quarters of the experiment series are indicated.

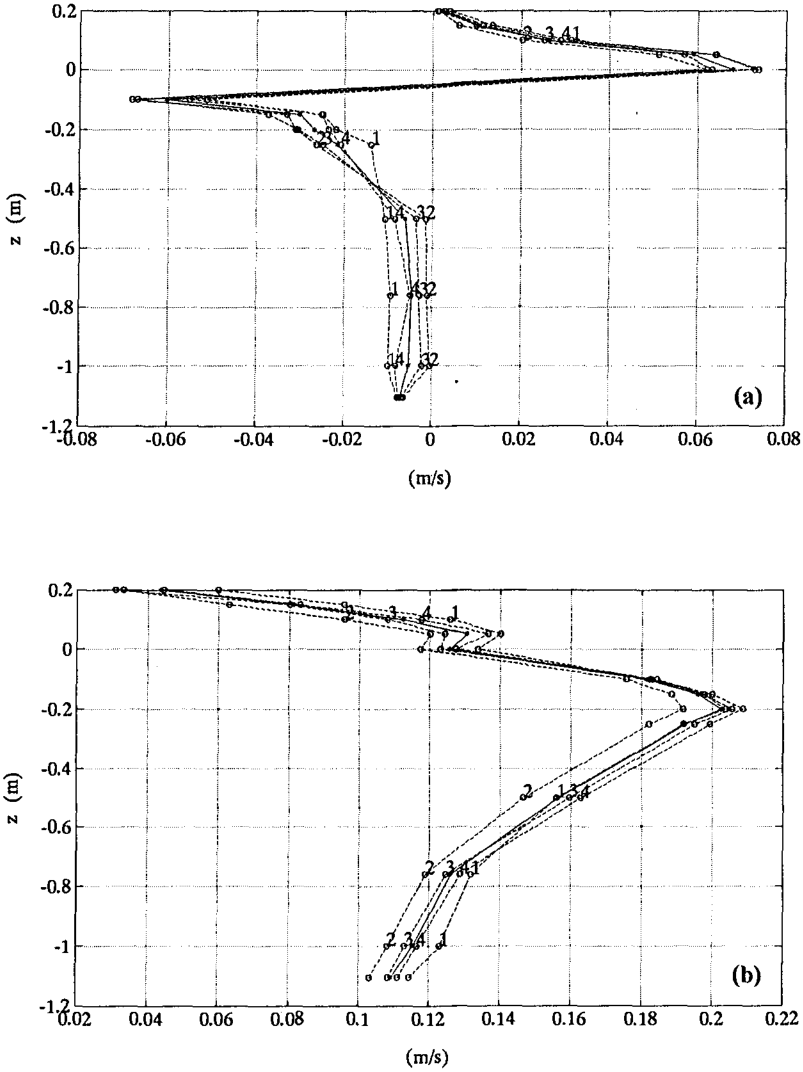

Fig. 6. Mean values (a) and standard deviations (b) of horizontal velocity calculated for each sub-series (dashed lines with circles) against the one calculated for the whole data series (solid line with stars). Wave case I24. The numbers of successive quarters of the experiment series are indicated.

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