

CHAPTER 70

PROBABILISTIC STRUCTURE OF RANDOM WAVE GROUPS

Ke Yu *

Abstract

This paper deals with the statistical properties of wave groups in a stationary ergodic normal process. For a narrowband Gaussian process, a method based on Kimura's theory is developed to estimate the characteristics of wave groups directly from the wave spectrum. For a non-narrowband Gaussian process with an arbitrary bandwidth, a new model is established to predict the formation of the wave groups by means of zero-upcrossing method. Thus the probabilistic structure of the wave groups in a Gaussian process with an arbitrary bandwidth can be determined. Using this model, the mean run length of the wave groups above any amplitude and the probability distribution of run length at any level can be obtained. On the other hand, a representative wave period of the wave group is suggested to describe the time intervals between two successive maxima. The computational data shows that the bandwidth parameter has a significant effect on the statistical properties of wave groups.

1 Introduction

A wave group can be conveniently defined as several successive waves which exceed a given amplitude level. The waves in the group have similar wave height and wave period. Wave groupiness is an important feature of stochastic wave processes. It has been proven that wave groups have significant effects on many ocean engineering problems, such as the resonance of offshore structures and the overtopping above coastal structures. Because of the importance of the groupiness phenomenon in naval, ocean and communication engineering, extensive studies have been made on this property of stochastic waves in the past twenty years, for example, Kimura (1980) and Funke *et al.* (1979). Using the Markov chain concept in expressing the relationship between two maximum points, Kimura (1980) derived the probability distributions about the runs of high waves, the runs of low waves, and the runs of resonant wave periods respectively. Funke *et al.* (1979) developed a different method, SIWEH, to describe a wave process. A parameter GF was defined to express the groupiness degree in a random wave process. Though SIWEH may have the advantages in describing non-Gaussian processes, the author appreciates Kimura's theory

*Research engineer, Department of River and Harbor Engineering, Nanjing Hydraulic Research Institute, Nanjing 210024, P.R.China.

Current address: Center for Applied Coastal Research, Department of Civil Engineering, University of Delaware, Newark, DE 19716, U.S.A.

more for the wave group problem because it provides the information about the probabilistic structure of wave groups which is of concern by engineers and researchers.

However, there are some weaknesses in Kimura's theory. One is the adoption of the narrowband assumption. It is observed that most wave processes in the sea environment have spectrum bandwidth parameter values varying from 0.5 to 0.9, hence the application of Kimura's theory in this case may overestimate the correlation among successive maximum points. The other (Battjes *et al.*, 1984) is the inconvenience in deriving the statistical characteristics of wave groups. It is also worth mentioning that in previous studies, the wave period corresponding to a high wave run is neglected whereas the importance of this parameter is evident.

Therefore, the present study emphasizes the problem in assuming that the process is a stationary ergodic Gaussian process with an arbitrary bandwidth and that the maximum series is subject to the Markov chain condition. The author expects to derive the probabilistic structure of wave groups which include the probability distribution of wave runs above any given amplitude level and to provide some description about the representative wave amplitude and wave period at any given amplitude level.

2 Formation of the Probabilistic Structure of Wave Groups

2.1 Wave run and its probability distribution

Following Kimura (1980), we can consider a stationary ergodic Gaussian process $x_1(t)$ with a zero mean and another random process $x_2(t)$ which is essentially identical to $x_1(t)$ but has a time shift λ prior to $x_1(t)$, i.e.,

$$x_2(t) = x_1(t + \lambda). \quad (1)$$

Assuming the successive maxima of $x_1(t)$ and $x_2(t)$ are subject to the Markov chain condition, the maxima of $x_1(t)$ and $x_2(t)$ are written as: $A_1 = \{x_1(t), |x_1(t) \geq 0, \dot{x}_1 = 0, \ddot{x}_1 < 0\}$, $A_2 = \{x_2(t), |x_2(t) \geq 0, \dot{x}_2 = 0, \ddot{x}_2 < 0\}$. If the time interval λ is defined as the expected wave period between two successive maxima, the joint probability density function of two maxima $f(A_1, A_2)$ becomes the probability density function of two successive maxima. The probability density function of A_1 (or A_2) is defined as $f(A_1)$ and can be derived from $f(A_1, A_2)$.

Two states of wave height are defined. One state is $S_0 = \{A_i, |A_i < A_L, i = 1, 2\}$, the second is $S_1 = \{A_i, |A_i \geq A_L, i = 1, 2\}$. Hence, all the maxima of $x_1(t)$ and $x_2(t)$ can be classified into these two states according to their amplitude values and the given amplitude level A_L . The two states combined form a state space $\Omega = \{S_0, S_1\}$.

For this two state Markov chain, the one-step transition probability matrix is given by

$$P^{[1]} = P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad (2)$$

where

$$P_{11} = Prob\{A_2 \in S_0, |A_1 \in S_0\} = \frac{\int_0^{A_L} \int_0^{A_L} f(A_1, A_2) dA_1 dA_2}{\int_0^{A_L} f(A_1) dA_1}, \tag{3}$$

$$P_{12} = Prob\{A_2 \in S_1, |A_1 \in S_0\} = \frac{\int_{A_L}^{\infty} \int_0^{A_L} f(A_1, A_2) dA_1 dA_2}{\int_0^{A_L} f(A_1) dA_1}, \tag{4}$$

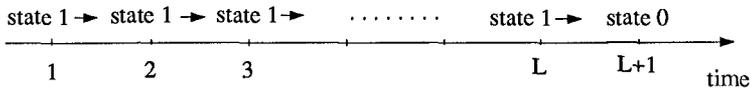
$$P_{21} = Prob\{A_2 \in S_0, |A_1 \in S_1\} = \frac{\int_0^{A_L} \int_{A_L}^{\infty} f(A_1, A_2) dA_1 dA_2}{\int_{A_L}^{\infty} f(A_1) dA_1}, \tag{5}$$

$$P_{22} = Prob\{A_2 \in S_1, |A_1 \in S_1\} = \frac{\int_{A_L}^{\infty} \int_{A_L}^{\infty} f(A_1, A_2) dA_1 dA_2}{\int_{A_L}^{\infty} f(A_1) dA_1}. \tag{6}$$

Because of the homogeneous feature and the Markov property of maximum point process, the N-step transition probability matrix can be easily obtained as follows

$$P^{[N]} = P^N. \tag{7}$$

The N-step transition probability matrix describes the probabilities of N-step transition through all possible paths. Now we consider a wave run which has a run length L. The transition path of this run must be the pattern as shown in the following figure .



The initial state probability distribution is taken as $P_0 = [0 \quad 1]$. The row matrix P_0 means that the wave run starts when a wave exceeds the amplitude level A_L initially. The run ends when the Lth wave exceeds A_L at last.

The probability matrix after L-step transition can be expressed as

$$P_L = P_0 P^L = [0 \quad 1] \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}^L. \tag{8}$$

From the above equation, the cumulative probability of a wave run which follows the mentioned pattern can be found as below

$$P(L) = P_{22}^{L-1} (1 - P_{22}), \tag{9}$$

where L is the length of the run . The mean run length of wave groups above some amplitude level can also be derived from equation (9) as follows

$$\bar{L} = \frac{1}{(1 - P_{22})}. \tag{10}$$

By a similar method, the probability distribution of a run of low waves and the distribution of the total wave run can be obtained. For conciseness, these results are not included in this paper.

2.2 Representative amplitude and period of wave groups

When an amplitude level is given the wave groups are defined. The global statistical properties of wave groups can be determined by using the method described above. However, in order to describe a wave group in detail, a wave amplitude A^* and a wave period T^* can be defined to represent the wave run.

A^* could be defined as the expected amplitude value above the level A_L

$$A^* = \frac{\int_{A_L}^{\infty} A_1 f(A_1) dA_1}{\int_{A_L}^{\infty} f(A_1) dA_1} \tag{11}$$

and T^* could be defined as the expected wave period under the condition of $A = A^*$

$$T^* = \int_0^{\infty} T f(T, |A^*) dT, \tag{12}$$

where $f(T, |A^*)$ can be derived from the joint probability density function of wave amplitude and wave period.

In addition to A^* and T^* , the characteristic extreme value of the maxima in a wave group with run length L can be calculated based on extreme value theory. For example, the probability density function of extreme values in a wave group can be expressed as

$$g(\xi) = L \{ f(\xi) [1 - P(\xi)]^{L-1} \} \tag{13}$$

and the required characteristic values of extreme value distribution can be evaluated from equation (13).

3 Wave Groups in a Narrowband Gaussian Process

3.1 The joint probability density function of two maxima

Suppose a narrowband Gaussian process $x(t)$ has a zero mean and a spectrum $S_X(\omega)$. It is a well known fact that $x(t)$ can be written as (Ochi, 1982)

$$x(t) = X_C(t) \cos(\omega_0 t) - X_S(t) \sin(\omega_0 t), \tag{14}$$

in which X_C and X_S are two orthogonal components which vary slowly with time and ω_0 is the central frequency. It can be proven that X_C and X_S are also the Gaussian processes which have the exact same mean and variance with $x(t)$. The auto-spectrum and co-spectrum of X_C and X_S can be expressed in terms of $S_X(\omega)$ as

$$S_{X_C}(\omega) = S_{X_S}(\omega) = S_X(\omega - \omega_0) + S_X(\omega + \omega_0), \tag{15}$$

$$S_{X_C X_S}(\omega) = -S_{X_S X_C}(\omega) = i [S_X(\omega - \omega_0) - S_X(\omega + \omega_0)]. \tag{16}$$

An inverse Fourier transfer is applied to get the auto-correlation or correlation functions of X_C and X_S as follows

$$\begin{aligned} R_{X_C}(\lambda) &= R_{X_S}(\lambda) = \int_{-\infty}^{\infty} S_{X_C}(\omega) \exp\{i\omega\lambda\} d\omega \\ &= 2 \int_0^{\infty} S_X(\omega) \cos[(\omega - \omega_0)\lambda] d\omega, \end{aligned} \tag{17}$$

$$R_{X_C X_S}(\lambda) = -R_{X_S X_C}(\lambda) = \int_{-\infty}^{\infty} S_{X_C X_S}(\omega) \exp\{i\omega\lambda\} d\omega \quad (18)$$

$$= 2 \int_0^{\infty} S_X(\omega) \sin[(\omega - \omega_0)\lambda] d\omega,$$

where $\omega_0 = \omega_p$ (peak frequency) and λ is defined as the expected period between two successive maxima estimated by following equation

$$\lambda = 2\pi \sqrt{\frac{m_0}{m_2}}, \quad (19)$$

where

$$m_i = \int_{-\infty}^{\infty} \omega^i S_X(\omega) d\omega, \quad i = 0, 2.$$

Alternatively, equations (17) and (18) can be expressed in terms of the one-side spectrum $S(\omega)$

$$R_{X_C}(\lambda) = R_{X_S}(\lambda) = \int_0^{\infty} S(\omega) \cos[(\omega - \omega_0)\lambda] d\omega, \quad (20)$$

$$R_{X_C X_S}(\lambda) = -R_{X_S X_C}(\lambda) = \int_0^{\infty} S(\omega) \sin[(\omega - \omega_0)\lambda] d\omega, \quad (21)$$

where $S(\omega) = 2S_X(\omega)$.

If the spectrum is symmetric about ω_0 , then $R_{X_C X_S}(\lambda) = R_{X_S X_C}(\lambda) = 0$. It can be inferred (Middleton, 1960) that the two dimensional amplitude probability density function is given by

$$f(A_1, A_2) = \frac{A_1 A_2}{B} I_0 \left\{ \frac{\sqrt{R_{X_C}^2 + R_{X_C X_S}^2}}{B} A_1 A_2 \right\} \exp \left\{ -\frac{m_0(A_1^2 + A_2^2)}{2B} \right\}, \quad (22)$$

where $B = m_0^2 - [R_{X_C}^2 + R_{X_C X_S}^2]$ and $I_0(x)$ is the zeroth order Bessel function of the first kind.

After introducing two non-dimensional amplitudes, ξ and η , defined as

$$\xi = \frac{A_1}{\sqrt{m_0}}, \quad \eta = \frac{A_2}{\sqrt{m_0}}, \quad (23)$$

equation (22) can be rewritten as

$$f(\xi, \eta) = \frac{\xi \eta}{1 - \rho^2} I_0 \left\{ \frac{\rho \xi \eta}{1 - \rho^2} \right\} \exp \left\{ -\frac{\xi^2 + \eta^2}{2(1 - \rho^2)} \right\}. \quad (24)$$

The correlation coefficient ρ can be determined by the following formula

$$\rho = \frac{\sqrt{R_{X_C}^2(\lambda) + R_{X_C X_S}^2(\lambda)}}{m_0}. \quad (25)$$

The one dimensional amplitude probability density function can be derived from equation (22) as

$$f(A_1) = \frac{A_1}{m_0} \exp \left\{ -\frac{A_1^2}{2 m_0} \right\} \tag{26}$$

and the dimensionless form of $f(A_1)$ is

$$f(\xi) = \xi \exp \left\{ -\frac{\xi^2}{2} \right\}. \tag{27}$$

3.2 Representative amplitude and period of wave groups in a narrowband Gaussian process

Applying the maximum distribution into equation (11), the representative amplitude of a wave group above any given level ξ_L can be obtained as

$$\xi^* = \xi_L + \sqrt{2\pi} \exp\{\xi_L^2/2\} [1 - \Phi(\xi_L)], \tag{28}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp \left\{ -\frac{u^2}{2} \right\} du.$$

As for the representative wave period, the joint probability distribution of wave amplitude and period suggested by Longuet-Higgins (1975) is used to get the probability distribution of wave period under the condition of $\xi = \xi^*$

$$f(\xi, \tau) = \frac{1}{1 - \Phi(-\xi/\nu)} \frac{1}{\sqrt{2\pi}} \xi^2 \exp \left\{ -\frac{\xi^2(1 + \tau^2)}{2} \right\}, \tag{29}$$

$$0 \leq \xi < \infty, \quad -\frac{1}{\nu} < \tau < \infty$$

where

$$\xi = \frac{A}{\sqrt{m_0}}, \quad \tau = \frac{T - \bar{T}}{\nu T}, \quad \nu = \sqrt{\frac{m_0 m_2}{m_1^2} - 1}, \quad \bar{T} = 2\pi \sqrt{\frac{m_0}{m_2}}.$$

$\Phi(x)$ is the error function and ν is another bandwidth parameter which is determined by the spectral moments.

The conditional probability density function of the wave period is

$$f(\tau, |\xi^*) = \frac{f(\xi, \eta)}{f(\xi)} = \frac{1}{1 - \Phi(-\xi^*/\nu)} \frac{1}{\sqrt{2\pi}} \xi^* \exp \left\{ -\frac{1}{2} (\xi^* \tau)^2 \right\} \tag{30}$$

and the representative wave period can be calculated by equation (12)

$$\tau^* = \frac{1}{\sqrt{2\pi} \xi^* [1 - \Phi(-\xi^*/\nu)]} \exp \left\{ -\frac{1}{2} (\xi^* \tau)^2 \right\}. \tag{31}$$

Applying the two dimensional density function of two maxima presented by equation (22) and one dimensional density function expressed by equation (26) into equation (6), the probability P_{22} under the condition of a given amplitude level ξ^* can be determined. The P_{22} value can also be used to determine the probability distribution of wave runs which are above the level ξ_L . Equations (28) and (31) give the analytic expressions of representative wave amplitude and wave period of wave runs above a given amplitude level.

4 Wave Groups in a Non-narrowband Gaussian Process

4.1 The two dimensional maximum distribution

Given a stationary Gaussian process $x_1(t)$ with a zero mean and an arbitrary spectral bandwidth and another process $x_2(t)$ defined in equation (1), the time shift λ can be determined by the bandwidth parameter ϵ and the moments of the spectrum

$$\lambda = 4\pi \frac{\sqrt{1-\epsilon^2}}{1+\sqrt{1-\epsilon^2}} \sqrt{\frac{m_0}{m_2}}, \quad (32)$$

$$\epsilon = \sqrt{1 - \frac{m_2^2}{m_0 m_4}}. \quad (33)$$

It can be proven that the two processes $x_1(t)$ and $x_2(t)$ and their derivatives $\dot{x}_1(t)$, $\ddot{x}_1(t)$, $\dot{x}_2(t)$, $\ddot{x}_2(t)$ are subject to a six dimensional normal distribution (Ochi, 1979). Their joint probability density function can be written as

$$f(X) = \frac{1}{(2\pi)^3 |\Sigma|^{1/2}} \exp \left\{ -\frac{1}{2} X \Sigma^{-1} X^T \right\}, \quad (34)$$

in which the row matrix X is

$$X = [x_1(t) \quad \dot{x}_1(t) \quad \ddot{x}_1(t) \quad x_2(t) \quad \dot{x}_2(t) \quad \ddot{x}_2(t)].$$

The covariance matrix can be expressed in terms of the spectral density function

$$\Sigma = \begin{bmatrix} m_0 & 0 & -m_2 & m_{0C} & -m_{1S} & -m_{2C} \\ 0 & m_2 & 0 & m_{1S} & m_{2C} & -m_{2S} \\ -m_2 & 0 & m_4 & -m_{2C} & m_{3S} & m_{4C} \\ m_{0C} & m_{1S} & -m_{2C} & m_0 & 0 & -m_2 \\ -m_{1S} & m_{2C} & m_{3S} & 0 & m_2 & 0 \\ -m_{2C} & -m_{2S} & m_{4C} & -m_2 & 0 & m_4 \end{bmatrix},$$

where

$$m_i = \int_0^\infty \omega^i S(\omega) d\omega, \quad i = 0, 2, 4,$$

$$m_{iC} = \int_0^\infty \omega^i S(\omega) \cos(\omega\lambda) d\omega, \quad i = 0, 2, 4,$$

$$m_{iS} = \int_0^\infty \omega^i S(\omega) \sin(\omega\lambda) d\omega, \quad i = 1, 2, 3.$$

As mentioned before, the positive maxima of $x_1(t)$ and $x_2(t)$ must satisfy the conditions, $x_i(t) \geq 0$, $\dot{x}_i = 0$, $\ddot{x}_i < 0$, $i = 1, 2$. The expected number of maxima

which are above the respective amplitude levels A_1 and A_2 per unit time can be evaluated by the following equation

$$\bar{N}_{A_1, A_2} = \int_{A_1}^{\infty} \int_{A_2}^{\infty} \int_{-\infty}^0 \int_{-\infty}^0 |\dot{x}_1| |\dot{x}_2| f(x_1, 0, \ddot{x}_1, x_2, 0, \ddot{x}_2) dx_1 dx_2 d\ddot{x}_1 d\ddot{x}_2. \quad (35)$$

Letting $A_1 = A_2 = 0$ in equation (35), equation (35) can be rewritten as

$$\bar{N}_{0,0} = \int_0^{\infty} \int_0^{\infty} \int_{-\infty}^0 \int_{-\infty}^0 |\dot{x}_1| |\dot{x}_2| f(x_1, 0, \ddot{x}_1, x_2, 0, \ddot{x}_2) dx_1 dx_2 d\ddot{x}_1 d\ddot{x}_2 \quad (36)$$

and it can be used to estimate the expected number of maxima in $x_1(t)$ and $x_2(t)$ per unit time. The probability that the maxima in $x_1(t)$ and $x_2(t)$ exceed respective amplitude level A_1 and A_2 can be considered approximately as the ratio of these two expected numbers. Hence the joint probability density function of two maxima can be derived from the ratio $\bar{N}_{A_1, A_2} / \bar{N}_{0,0}$

$$\begin{aligned} f(A_1, A_2) &= \frac{\partial^2}{\partial A_1 \partial A_2} \left(1 - \frac{\bar{N}_{A_1, A_2}}{\bar{N}_{0,0}} \right) \\ &= \frac{\int_{-\infty}^0 \int_{-\infty}^0 |\dot{x}_1| |\dot{x}_2| f(A_1, 0, \ddot{x}_1, A_2, 0, \ddot{x}_2) d\ddot{x}_1 d\ddot{x}_2}{\int_0^{\infty} \int_0^{\infty} \int_{-\infty}^0 \int_{-\infty}^0 |\dot{x}_1| |\dot{x}_2| f(x_1, 0, \ddot{x}_1, x_2, 0, \ddot{x}_2) dx_1 dx_2 d\ddot{x}_1 d\ddot{x}_2}. \end{aligned} \quad (37)$$

Now, define two new Gaussian processes which are derived from $x_1(t)$ and $x_2(t)$

$$x'_1(t) = \frac{1}{\sqrt{m_0}} x_1(t) \quad , \quad x'_2(t) = \frac{1}{\sqrt{m_0}} x_2(t) \quad (38)$$

and two dimensionless maxima as before

$$\xi = \frac{A_1}{\sqrt{m_0}} \quad , \quad \eta = \frac{A_2}{\sqrt{m_0}}.$$

The two dimensional distribution of two dimensionless maxima ξ and η can be obtained simply by replacing $x'_1, x'_2, \dot{x}'_1, \dot{x}'_2, \xi$ and η for the proper terms in equation (34) and (37). Note the covariance matrix Σ becomes Σ' and satisfies the following relation

$$\Sigma' = \frac{1}{m_0} \Sigma. \quad (39)$$

The probability density function of the maxima in process $x_1(t)$ has the form

$$\begin{aligned} f(A_1) &= \frac{2/\sqrt{m_0}}{1 + \sqrt{1 - \epsilon^2}} \left[\frac{\epsilon}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2\epsilon^2} \left(\frac{A_1}{\sqrt{m_0}} \right)^2 \right\} + \sqrt{1 - \epsilon^2} \left(\frac{A_1}{\sqrt{m_0}} \right) \right. \\ &\quad \left. \times \exp \left\{ -\frac{1}{2} \left(\frac{A_1}{\sqrt{m_0}} \right)^2 \right\} \left\{ 1 - \Phi \left(-\frac{\sqrt{1 - \epsilon^2}}{\epsilon} \frac{A_1}{\sqrt{m_0}} \right) \right\} \right], \quad (40) \\ &\quad 0 \leq A_1 < \infty \end{aligned}$$

and the non-dimensional form of equation (40) is as follows

$$\begin{aligned} f(\xi) &= \frac{2}{1 + \sqrt{1 - \epsilon^2}} \left[\frac{\epsilon}{\sqrt{2\pi}} \exp \left\{ -\frac{\xi^2}{2\epsilon^2} \right\} + \sqrt{1 - \epsilon^2} \xi \right. \\ &\quad \left. \times \exp \left\{ -\frac{\xi^2}{2} \right\} \left\{ 1 - \Phi \left(-\frac{\sqrt{1 - \epsilon^2}}{\epsilon} \xi \right) \right\} \right]. \end{aligned} \quad (41)$$

4.2 The representative amplitude and period of wave groups in a non-narrowband Gaussian process

The representative dimensionless amplitude ξ^* can be estimated by equations (41) and (11).

For a non-narrowband Gaussian process, the joint probability density function of the maxima and the time intervals between two successive maxima can be used to calculate the representative wave period (Arhan *et al.*, 1976)

$$f(\xi, \tau) = \frac{2}{\sqrt{2\pi}} \frac{\alpha^3}{\epsilon(1-\epsilon^2)} \frac{\xi^2}{\tau^5} \exp \left\{ -\frac{\xi^2}{2\epsilon^2\tau^4} [(\tau^2 - \alpha^2)^2 + \alpha^4\beta^2] \right\}, \quad (42)$$

in which $\xi = A_1/\sqrt{m_0}$, $\tau = T/\lambda$, $\alpha = 0.5(1 + \sqrt{1 - \epsilon^2})$, $\beta = \epsilon/\sqrt{1 - \epsilon^2}$.

The conditional probability density function of τ can be expressed as

$$f(\tau, |\xi) = \frac{f(\xi, \tau)}{\int_0^\infty f(\xi, \tau) d\tau} \quad (43)$$

and the representative wave period τ^* can be written as

$$\tau^* = \int_0^\infty \tau f(\tau, |\xi^*) d\tau. \quad (44)$$

Note that τ^* is not a wave period according to the exact definition of a wave period by means of zero-upcrossing. It is just a time interval between two successive maxima. However, when the wave amplitude is large enough, τ^* can be considered as a good approximation of wave period.

5 Computational Results and Discussions

In order to find the effect of the spectral bandwidth on the probability structure of wave groups, different ϵ values should be applied in this computation. But the common wave spectra used in analysis and experiment, such as the JONSWAP spectrum and the Bretschneider spectrum, only have limited range of ϵ values. Hence some typical wave records used in Ochi's study are used here again. The ϵ values of these spectra vary from 0.46 to 0.8. A JONSWAP spectrum with $\alpha = 0.05$, $f_p = 0.1 Hz$, $\gamma = 7.0$ and $\epsilon = 0.685$ is applied to the computation. Table 1 displays the features of the spectra which include the bandwidth parameter ϵ and the correlation coefficient ρ calculated by equations (33) and (25).

Table 1. Wave records and wave spectrum used in the computation

wave spectrum	spectral bandwidth parameter ϵ	correlation coefficient ρ	peak frequency $\omega(2\pi * Hz)$
WS1	0.468	0.458	0.875
WS2	0.594	0.299	0.563
WS3	0.685	0.676	0.626
WS4	0.806	0.267	0.413

It is clear that the numerical integration in equation (37) is the major work in this computation. Here, 2.401×10^7 discretizing points are used in calculating the quadruple integration in order to guarantee accuracy.

Figure 1 shows the relationship of P_{22} and ξ_L . For each spectrum, two $\epsilon - \xi_L$ curves based on the non-narrowband and the narrowband theory respectively, are presented to compare the difference between two theories. It can be seen that the narrowband theory will give larger a P_{22} value when ϵ is increased for fixed ξ_L . In this case it seems only the value of ϵ will determine the $P_{22} - \xi_L$ relation under the non-narrowband assumption. When $\epsilon = 0.46$, the two theories give almost the same results for small amplitude levels. For the wave spectra which have very small ϵ values, the strong singularity of the co-variance matrix makes its determinant value approach zero and leads the elements in the inverse matrix Σ^{-1} to infinity. It will induce the non-existence of the joint probability density function expressed by equation (37). In this situation, the narrowband assumption could be applied to simplify the problem. We can also infer that when the value of ϵ approaches zero, it is the correlation coefficient ρ , not the bandwidth parameter ϵ , that will determine the $P_{22} - \xi_L$ relation if ϵ is very small. This estimation can be demonstrated by the one dimensional analytic distribution of wave amplitude in a non-narrowbanded Gaussian process which has a small bandwidth. From equation (41) it can be seen that the wave amplitude distribution with ϵ value in the region $0 < \epsilon < 0.5$ is almost the same with the Rayleigh distribution expressed by equation (27).

From equation (30) it can be seen that only the correlation coefficient ρ affects the joint distribution and finally determines the probability distribution of wave runs if the amplitude level remains constant. Employing equations (22) and (26) into equation (6), the probability distribution of wave runs can be obtained from equation (9). Figure 2 displays the relationship between the probability of wave run and its run-length with different ρ values.

Figure 2 also shows two probability distribution of wave groups based on narrowband theory and non-narrowband theory respectively. In the figure, the amplitude level $\xi_L = 1.245$ represents the mean amplitude in a standard narrowbanded normal process. In figure 3, the level $\xi_L = 2.005$ is just the significant amplitude in a narrowbanded normal process. By applying the two P_{22} values corresponding to a given amplitude level into equation (9), two probability distributions of the high wave runs above the amplitude level can be easily determined.

Applying equations (42) and (43) into equations (11) and (12), the dimensionless representative wave amplitude and wave period in a non-narrowband Gaussian process can be determined. Figure 4 displays the relation between the representative wave amplitude and the representative period with different ϵ values. The curve noted as $\epsilon = 0$ represents the $\xi^* - \xi_L$ relation expressed by equation(28). Figure 5 shows the relationship of τ^* and ξ with different ϵ values. Within the narrowband theory, the representative wave period τ^* calculated by equation (31) is zero when ξ^* takes its minimum value of 1.245. In other words, the expected wave period is considered to be the representative wave period in this case.

It is worthy to mention here that the present analysis and computation is in fact the extension of Kimura's theory to a more general case. We know from the previous analysis that two parameters, the bandwidth parameter, ϵ , and the correlation coef-

ficient, ρ , decide the statistical properties of wave groups. For an ideal narrowband Gaussian process, the effect of the bandwidth parameter on wave groups reduces to zero and only the correlation coefficient determines the properties. For a non-narrowband Gaussian process with a large ϵ value, the computational data shows the opposite result. The bandwidth parameter plays a major role in determining the probabilistic features of the wave groups. But for a small ϵ value we do not know which parameter is more important. We expect to see that as ϵ increases, the bandwidth parameter will gradually replace the correlation coefficient and dominate the statistical properties of wave groups. To see these effects more clearly, further numerical analysis is necessary with variance on the two parameters. For example, wave spectra having the same ϵ value but different ρ values or having the same ρ value but different ϵ values.

Another tricky problem arising in this study is the validity of the present theory when it is applied to a random process which tends to be white noise, i.e., the bandwidth parameter and the correlation coefficient approach to one and zero respectively. In this case the assumption that the successive maxima in the process subject to the Markov chain condition becomes questionable, since the successive maxima tend to be independent variables. To determine how far we can go with this model, more wave records and field data are necessary. It is definitely interesting and promising work to analyze and compare the experimental and field data about the random wave groups with the existing models. This will lead to a deeper understanding to the groupiness phenomenon in the real sea environment.

6 Conclusions

The following conclusions can be made from the above analysis:

1. When a stationary Gaussian process has a small bandwidth parameter, i.e., $\epsilon < 0.5$, the narrowband assumption can be applied and the statistical properties of wave groups derived from the narrowband theory can be considered as a good approximation.
2. If the spectrum of a stationary Gaussian process is not assumed narrowbanded, i.e., $\epsilon \geq 0.5$, the application of the narrowband assumption will overestimate the correlation between two successive maxima.
3. For a narrowband Gaussian process the correlation coefficient ρ will determine the statistical properties of wave groups. For a non-narrowband Gaussian process with large ϵ value, it seems that the spectral bandwidth parameter will finally determine the probabilistic structure of wave groups.
4. Based on the formulas presented in this paper, the probabilistic structure of wave groups in a stationary ergodic Gaussian random process can be obtained directly from its wave spectrum.

Acknowledgments

The author thanks Mr. Weidon Su for his meaningful insight which helped to start this work. In addition, the author expresses his appreciation to Dr. Jian Ye and Mr. Bingcan Din for their assistance in the numerical computation which was beneficial

to the development of this paper. The author also thanks Mr. Michael Poff and Mr. Daniel Cox who read this paper carefully and made many suggestions. Finally, the author's deep appreciation goes to Prof. Ib A. Svendsen, the head of Civil Engineering Department at University of Delaware, and Prof. Robert A. Dalrymple, the director of Center for Applied Coastal Research at University of Delaware, who provided a considerable financial aid for this trip. Their encouragement and support made this representation possible.

Reference

- Arhan, M. K., Cavanié, A., and Ezraty, R. (1976). "Etude théorique et expérimentale de la relation hauteur-période des vagues de tempete," I.F.P. 24191. Centre National pour l'Exploitation des Oceans.
- Battjes, J. A. and Van Vledder, G. Ph., (1984). "Verification of Kimura's Theory for Wave Groups Statistics", *Proc. 19th Int. Conf. Coast. Engrng.*, ASCE.
- Funke, E. R. and Mansard, E. P. D., (1979). "On the Synthesis of Realistic Sea States in a Laboratory Flume", *Hydraulics Laboratory Report LTR-HR-66*, N.R.C., Ottawa, Canada.
- Kimura, A., (1980). "Statistical Properties of Random Wave Groups", *Proc. 17th Int. Conf. Coast. Engrng.*, ASCE.
- Longuet-Higgins, M. S. (1975). "On the Joint Distribution of the Periods and Amplitudes of Sea Waves", *JGR, J. of Geophys. Res.*, 80, No.18.
- Middleton, D., (1960). "An Introduction to Statistical Communication Theory", McGraw-Hill Book Company Inc.
- Ochi, M. K., (1979). "Extreme Values of Waves and Ship Responses Subject to Markov Chain Condition", *Journal of Ship Research*, Vol. 23, No. 3, Sept.
- Ochi, M. K., (1982). "Stochastic Analysis and Probabilistic Prediction of Random Seas", *Advances in Hydroscience*, Vol. 13-1982, Academic Press.

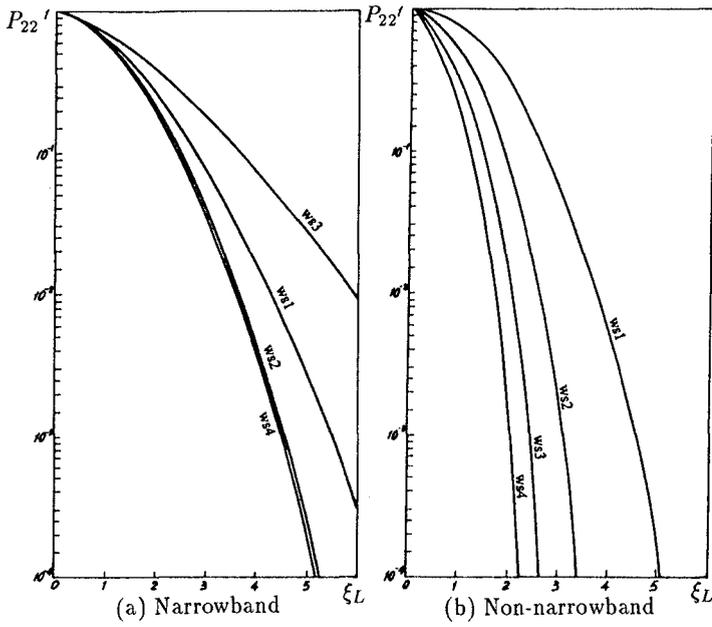


Fig. 1 Change of P_{22} value with amplitude level ξ_L

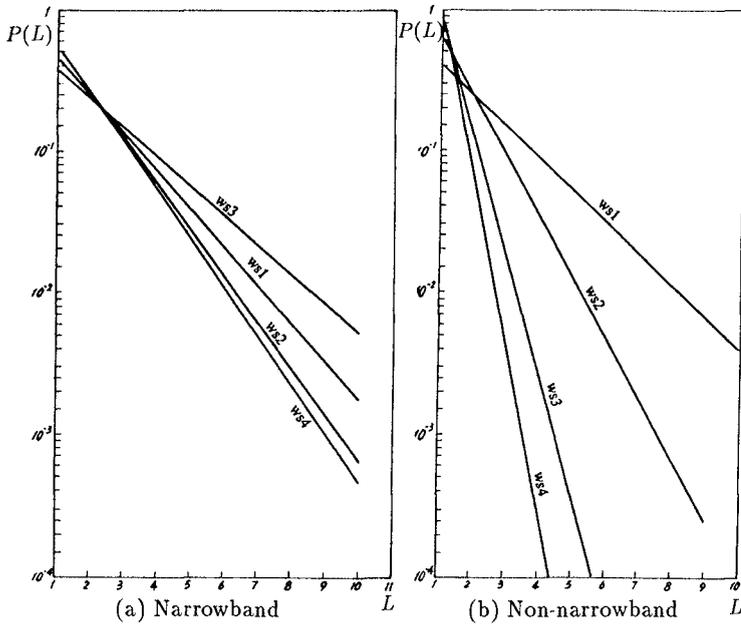


Fig. 2 Probability distribution of high wave runs ($\xi_L=1.245$)

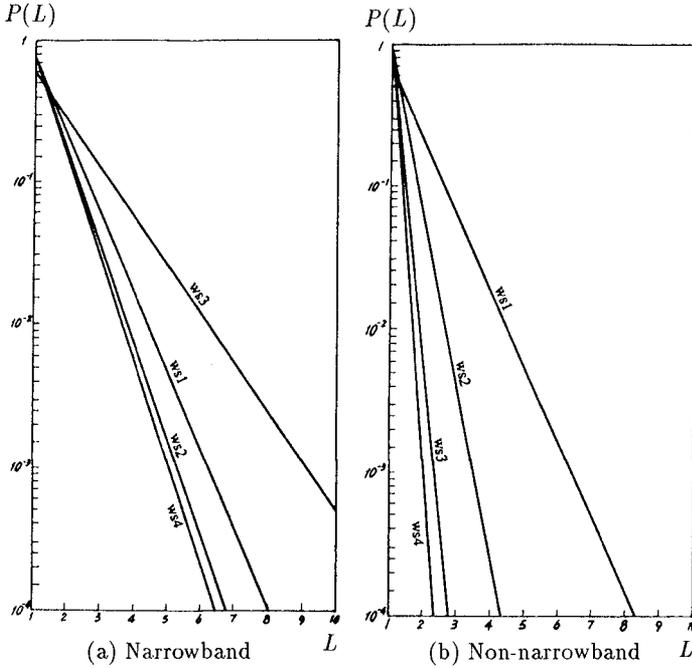


Fig. 3 Probability distribution of high wave runs ($\xi_L=2.005$)

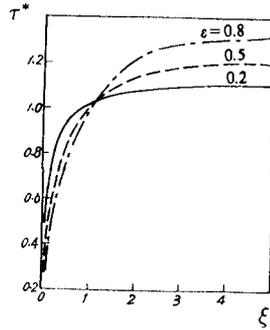
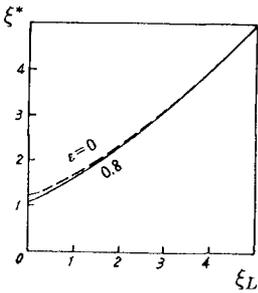


Fig. 4 Relationship between ξ^* and ξ_L Fig. 5 Relationship between τ^* and ξ