CHAPTER 55

Interrelation of Cnoidal Wave Theories

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Abstract

The aim of this paper is to analytically clarify the interrelation among all of the existing cnoidal wave theories of the third order approximation. The result is that the Chappelear theory expressed with two expansion parameters is found to include the other theories. This means that they are able to be derived from the Chappelear theory by changing expansion parameters in the expressions of wave characteristics and integral properties and identifying the definition for wave celerity to be used in the theory. The expressions for the trajectory of a water particle, mass transport velocity and integral properties of waves are newly derived using these theories, and the wave characteristics and integral properties are examined based on the numerical computation and comparison with the experiments. As a result, use of the theory expressed with a single expansion parameter is recommended from a practical viewpoint. The figures describing a limiting range of wave parameters for the application of the above-mentioned theory are also provided by comparing wave characteristics and integral properties of the theory with those of Fenton's Fourier approximation wave theory.

1. Introduction

Higher order solutions of cnoidal waves have been derived by Laitone(1960, 1965; the second order approximation, the second definition for wave celerity), Chappelear(1962; the third order approximation, the first definition), Fenton(1979; the ninth order approximation, the first definition), Tsuchiya & Yasuda(1985; the third

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order approximation, the third definition) and Isobe
(1988; the third order approximation, the second defini-
tion). These solutions are different from each other in
the definitions used for the derivation of wave celerity
and the expansion parameters for the expression of wave
characteristics.

Yamaguchi & Tsuchiya (1974) mathematically proved
that the Laitone theory can be derived from the Chappe-
lear theory within the second order approximation when
the expressions of wave characteristics in the Chappe-
lear theory are rewritten with the parameter in the
Laitone theory under the condition that the same defini-
tion is applied for the derivation of wave celerity in
each theory.

Now there are at least 4 kinds of third order cnoi-
dal wave theories derived with different expansion
parameters and definitions for wave celerity.

The aim of this paper is to mathematically clarify
the interrelation among these third order solutions and
to examine the wave characteristics and integral proper-
ties, and limiting conditions for practical application
based on numerical computation and the comparison with
experiments.

2. Theoretical Considerations

(1) Definitions for wave celerity
    Usually, either of the first and second definitions
proposed by Stokes has been used for the determination
of wave celerity, although Fenton(1990) recommends use of
the term "approximation" in place of the term "defini-
tion". The first and second definitions for wave celeri-
ty are given respectively as

$$\overline{u^{\mathrm{I}}} = 0 \qquad \text{or} \qquad c^{\mathrm{I}} = \overline{\tilde{u}} \tag{1}$$

$$\overline{\int_{-D}^{\eta} u^{\mathrm{II}} dz} = 0 \quad \text{or} \quad c^{\mathrm{II}} = \overline{\int_{-D}^{\eta} \rho \tilde{u} dz / \rho D} \tag{2}$$

where c is the wave celerity, u the horizontal water
particle velocity in a fixed frame of reference, $\tilde{u}$ the
horizontal water particle velocity in a moving frame of
reference, D the water depth, η the surface displacement,
z the ordinate taken vertically upward on the still water
level, '——' the time-mean, and superscripts 'l' and 'Il'
indicate the first and second definitions respectively.
The first definition means that time-averaged horizontal
water particle velocity at a level under the trough in a
frame of reference moving with a constant speed vanishes,
and the second definition means that time-mean of depth-

integrated momentum in a frame of reference moving with a constant speed vanishes.

Tsuchiya & Yasuda have defined the wave celerity as gradient of the characteristics in x-t space. In this case, the Bernoulli constant in wave field of the steady state C is equivalent to that in the coordinate system moving with wave celerity in still water.

$$C = gD + (c^{III})^2/2 \tag{3}$$

where g is the acceleration of gravity. This is tentatively termed the third definition for wave celerity, in which case superscript 'Ill' is used as the indication.

(2) The Chappelear theory

Chappelear derived a cnoidal wave theory with two expansion parameters L_0 and L_3. The Bernoulli constant and horizontal water particle velocity in a moving frame of reference, wave celerity and horizontal water particle velocity in a fixed frame of reference based on the first definition are expressed respectively as

$$
\begin{aligned}
C = \{3/2 + L_0(1+x^2) + 3L_3 + (9/10)L_0^2(2+3x^2+2x^4) \\
+ 6L_0L_3(1+x^2) + (3/2)L_3^2 + (3/35)L_0^3(45+97x^2+97x^4+45x^6) \\
+ 9L_0^2L_3(2+3x^2+2x^4) + 15L_0L_3^2(1+x^2)\}gD
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
\bar{u}/\sqrt{gD} = 1 + L_3 + L_0x^2\mathrm{sn}^2\gamma\chi + \{L_0^2(1+x^2) + 5L_0L_3\}x^2\mathrm{sn}^2\gamma\chi \\
+ L_0^2x^4\mathrm{sn}^4\gamma\chi - (y/D)^2(3/4)L_0^2\{x^2 - 2(1+x^2)x^2\mathrm{sn}^2\gamma\chi + 3x^4\mathrm{sn}^4\gamma\chi\} \\
+ \{(1/5)L_0^3(7+19x^2+7x^4) + 9L_0^2L_3(1+x^2) + 10L_0L_3^2\}x^2\mathrm{sn}^2\gamma\chi \\
+ \{(9/5)L_0^3(1+x^2) + 9L_0^2L_3\}x^4\mathrm{sn}^4\gamma\chi + (6/5)L_0^3x^6\mathrm{sn}^6\gamma\chi \\
- (1/4)(y/D)^2[3L_0^3x^2(1+x^2) + 15L_0^2L_3x^2 + 6\{L_0^3(-1+x^2-x^4) \\
- 5L_0^2L_3(1+x^2)\}x^2\mathrm{sn}^2\gamma\chi + 15\{-L_0^3(1+x^2) + 3L_0^2L_3\}x^4\mathrm{sn}^4\gamma\chi \\
+ 30L_0^3x^6\mathrm{sn}^6\gamma\chi] + (3/16)(y/D)^4L_0^3\{-x^2(1+x^2) \\
+ (2+13x^2+2x^4)x^2\mathrm{sn}^2\gamma\chi - 15(1+x^2)x^4\mathrm{sn}^4\gamma\chi + 15x^6\mathrm{sn}^6\gamma\chi\}
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
c^I/\sqrt{gD} = 1 + L_3 + L_0(1-e) + 5L_0L_3(1-e) + (1/3)L_0^2\{5 \\
+ 4x^2 - 5(1+x^2)e\} + 10L_0L_3^2(1-e) + (1/25)L_0^3\{81 + 146x^2 \\
+ 58x^4 - (81+169x^2+81x^4)e\} + 3L_0^2L_3\{5+4x^2-5(1+x^2)e\}
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
u^I/\sqrt{gD} = L_0(1-e-x^2\mathrm{sn}^2\psi) + 5L_0L_3(1-e-x^2\mathrm{sn}^2\psi) \\
+ (1/3)L_0^2\{5+4x^2-5(1+x^2)e-3(1+x^2)x^2\mathrm{sn}^2\psi-3x^4\mathrm{sn}^4\psi\} \\
+ (y/D)^2(3/4)L_0^2\{x^2 - 2(1+x^2)x^2\mathrm{sn}^2\psi + 3x^4\mathrm{sn}^4\psi\} \\
+ 10L_0L_3^2(1-e-x^2\mathrm{sn}^2\psi) + 3L_0^2L_3\{5+4x^2-5(1+x^2)e \\
-3(1+x^2)x^2\mathrm{sn}^2\psi-3x^4\mathrm{sn}^4\psi\} + (1/25)L_0^3\{81+146x^2+58x^4 \\
-(81+169x^2+81x^4)e-5(7+19x^2+7x^4)x^2\mathrm{sn}^2\psi
\end{aligned}
$$

$$-45(1+x^2)x^4\text{sn}^4\phi-30x^6\text{sn}^6\phi\}+(3/4)(y/D)^2[L_0{}^3\{(1+x^2)x^2$$
$$-2(1-x^2+x^4)x^2\text{sn}^2\phi-5(1+x^2)x^4\text{sn}^4\phi+10x^6\text{sn}^6\phi\}$$
$$+5L_0{}^2L_3\{x^2-2(1+x^2)x^2\text{sn}^2\phi+3x^4\text{sn}^4\phi\}]$$
$$+(3/16)(y/D)^4L_0{}^3\{(1+x^2)x^2-(2+13x^2+2x^4)x^2\text{sn}^2\phi$$
$$+15(1+x^2)x^4\text{sn}^4\phi-15x^6\text{sn}^6\phi\} \tag{7}$$

where x is the argument of elliptic function, e=E/K, K and E are the complete elliptic integrals of the first and second kinds, sn, cn and dn the Jacobian elliptic functions, y=D+z, $\gamma=\sqrt{3L_0}/2D$ and $\phi=\gamma(x-ct)$.

The relation between two expansion parameters is written as

$$2L_3+L_0(x^2+e)+L_0{}^2\{(1/5)(-1+6x^2+9x^4)+2(1+x^2)e\}$$
$$+6L_0L_3(x^2+e)+L_3{}^2+L_0{}^3\{(1/175)(-102+223x^2+944x^4+675x^6)$$
$$+(1/25)(111+214x^2+111x^4)e\}+2L_0{}^2L_3\{-1+6x^2+9x^4$$
$$+10(1+x^2)e\}+15L_0L_3{}^2(x^2+e)=0 \tag{8}$$

and the relation between wave height H and expansion parameters is expressed as

$$H/D=x^2L_0\{1+(1/4)L_0(10+7x^2)+6L_3+(1/40)L_0{}^2(251+369x^2$$
$$+151x^4)+(5/2)L_0L_3(10+7x^2)+15L_3{}^2\} \tag{9}$$

Yamaguchi et al.(1990) confirmed that the Chappelear theory is correct except for some printing errors, and newly calculated the trajectory of a water particle and integral properties such as energy flux.

(3) Relation with the Tsuchiya & Yasuda theory
Tsuchiya & Yasuda derived a cnoidal wave theory with a single parameter L_0.

If the parameter L_3 in the Chappelear theory is expanded into the power series form of L_0, L_3 is expressed as

$$L_3=-(1/2)(x^2+e)L_0+(1/40)\{4-24x^2+19x^4+10(-4+7x^2)e$$
$$+55e^2\}L_0{}^2-(1/2800)\{-816+3884x^2-5048x^4+1375x^6+(8316$$
$$-21616x^2+10941x^4)e+2625(-8+9x^2)e^2+14875e^3\}L_0{}^3 \tag{10}$$

and the relation between wave height H/D and L_0 becomes

$$H/D=x^2L_0+(1/4)x^2(10-5x^2-12e)L_0{}^2+(1/8)x^2\{55-55x^2$$
$$+13x^4+74(-2+x^2)e+96e^2\}L_0{}^3 \tag{11}$$

Substitution of L_3 relation into the wave characteristics in the Chappelear theory yields the following expressions.

$$c^{III}/\sqrt{gD} = 1-(1/2)(-2+x^2+3e)L_0+(1/40)\{64-64x^2+19x^4$$
$$+90(-2+x^2)e+135e^2\}L_0^2-(1/2800)\{-10448+15672x^2-7974x^4$$
$$+1375x^6+21(2288-2288x^2+563x^4)e+36225(-2+x^2)e^2+36225e^3\}L_0^3 \qquad (12)$$

$$u^{III}/\sqrt{gD} = (1-e-x^2\mathrm{sn}^2\psi)L_0+[(1/2)\{3-2x^2-(7-x^2)e+4e^2$$
$$-(2-3x^2-5e)x^2\mathrm{sn}^2\psi-2x^4\mathrm{sn}^4\psi\}+(3/4)(y/D)^2\{x^2-2(1+x^2)x^2\mathrm{sn}^2\psi$$
$$+3x^4\mathrm{sn}^4\psi\}]L_0^2+\{(1/200)[688-842x^2+209x^4-(2838-1888x^2$$
$$+63x^4)e+75(49-12x^2)e^2-1525e^3+5\{-76+148x^2-71x^4$$
$$+10(38-37x^2)e-375e^2\}x^2\mathrm{sn}^2\psi-180(2-3x^2-5e)x^4\mathrm{sn}^4\psi$$
$$-240x^6\mathrm{sn}^6\psi]+(3/8)(y/D)^2[x^2(2-3x^2-5e)+2\{-2+7x^2+3x^4$$
$$+5(1+x^2)e\}x^2\mathrm{sn}^2\psi-5(2+5x^2+3e)x^4\mathrm{sn}^4\psi+20x^6\mathrm{sn}^6\psi]$$
$$-(3/16)(y/D)^4\{-x^2(1+x^2)+(2+13x^2+2x^4)x^2\mathrm{sn}^2\psi-15(1+x^2)x^4\mathrm{sn}^4\psi$$
$$+15x^6\mathrm{sn}^6\psi\}]L_0^3 \qquad (13)$$

These expressions coincide perfectly with those in the Tsuchiya & Yasuda theory. Relation between L_0 and the expansion parameter in the Tsuchiya & Yasuda theory λ is given as

$$\lambda = x^2 L_0 \qquad (14)$$

(4) Relation with the Isobe theory
  Isobe gave a third order solution of cnoidal waves expressed with relative wave height H/D.

  If the expansion parameters L_0 and L_3 in the Chappelear theory are expanded into the power series of H/D, L_0 and L_3 are written as

$$L_0 = (1/x^2)(H/D)-(1/4x^4)(10-5x^2-12e)(H/D)^2$$
$$+(1/8x^6)\{3(15-15x^2+4x^4)-46(2-x^2)e+48e^2\}(H/D)^3 \qquad (15)$$

$$L_3 = -(1/2x^2)(x^2+e)(H/D)+(1/40x^4)\{2(2+13x^2-3x^4)$$
$$+5(2-3x^2)e-5e^2\}(H/D)^2+(1/2800x^6)\{-584-2659x^2+2073x^4$$
$$-150x^6+(-511+4011x^2-861x^4)e+1050(1-x^2)e^2-175e^3\}(H/D)^3 \qquad (16)$$

and if these L_0 and L_3 relations are used, the wave characteristics in the Chappelear theory are transformed as

$$c^{II}/\sqrt{gD} = 1+(1/2x^2)(2-x^2-3e)(H/D)+(1/40x^4)\{2(-8+8x^2$$
$$-3x^4)+5(2-x^2)e+15e^2\}(H/D)^2+(1/2800x^6)\{1558-2337x^2$$
$$+1079x^4-150x^6-(2653-2653x^2+203x^4)e+350(2-x^2)e^2$$
$$+175e^3\}(H/D)^3 \qquad (17)$$

$$u^{11}/\sqrt{gD} = (1/x^2)(1-e-x^2sn^2\psi)(H/D)+(1/4x^4)[-2-x^2+x^2e$$
$$+2e^2+(6+x^2-2e)x^2sn^2\psi-4x^4sn^4\psi+3(y/D)^2\{x^2-2(1+x^2)x^2sn^2\psi$$
$$+3x^4sn^4\psi\}](H/D)^2+(1/x^6)\ [(1/200)[\{153+23x^2-71x^4+(-153-97x^2$$
$$+47x^4)e+25(-1+2x^2)e^2+25e^3\}+5\{-101-27x^2+19x^4+10(10+x^2)e$$
$$-15e^2\}x^2sn^2\psi+20(32+2x^2-15e)x^4sn^4\psi-240x^6sn^6\psi]$$
$$+(3/8)(y/D)^2[(-8+2x^2+7e)x^2+2\{2(4+6x^2-x^4)-7(1+x^2)e\}x^2sn^2\psi$$
$$+(-40-10x^2+21e)x^4sn^4\psi+20x^6sn^6\psi]+(3/16)(y/D)^4\{(1+x^2)x^2$$
$$-(2+13x^2+2x^4)x^2sn^2\psi+15(1+x^2)x^4sn^4\psi-15x^6sn^6\psi\}\} \ (H/D)^3 \qquad (18)$$

These expressions are exactly the same as the Isobe theory.

(5) Relation with the Fenton theory

Fenton derived a cnoidal wave theory expressed with relative wave height H/h_t, in which h_t is the water depth under the trough,

If L_0 and L_3 are expanded into the power series of H/h_t and h_t/D relation is taken into account, L_0, L_3 and h_t/D are expressed as a function of H/h_t.

$$L_0 = (1/x^2)(H/h_t)+(1/4x^4)(-6+x^2+8e)(H/h_t)^2+(1/8x^6)\{3(3+x^2)$$
$$-8(2+x^2)e+8e^2\}(H/h_t)^3 \qquad (19)$$

$$L_3 = -(1/2x^2)(x^2+e)(H/h_t)+(1/40x^4)\{2(2+3x^2+7x^4)-5(2-5x^2)e$$
$$+15e^2\}(H/h_t)^2-(1/2800x^6)\{24+279x^2+307x^4+710x^6+7(53-153x^2$$
$$+253x^4)e-1050(1-2x^2)e^2+875e^3\}(H/h_t)^3 \qquad (20)$$

$$h_t/D = 1+(1/x^2)(1-x^2-e)(H/h_t)+(1/4x^4)\{2(1-3x^2+2x^4)$$
$$-(6-7x^2)e+4e^2\}(H/h_t)^2-(1/400x^6)\{-66+399x^2-733x^4+400x^6$$
$$+2(233-683x^2+458x^4)e-200(4-5x^2)e^2+400e^3\}(H/h_t)^3 \qquad (21)$$

Substitution of L_0, L_3, and h_t/D relations into the wave characteristics of the Chappelear theory yields the following expressions, which agree exactly with those in the Fenton theory again.

$$c^1/\sqrt{gh_t} = 1+(1/2x^2)(1-2e)(H/h_t)+(1/120x^4)\{-(13+2x^2+3x^4)$$
$$+10(4+x^2)e\}(H/h_t)^2+(1/16800x^6)\{-2888+5697x^2-2689x^4+780x^6$$
$$+4(392-1442x^2+917x^4)e\}(H/h_t)^3 \qquad (22)$$

$$\tilde{u}/\sqrt{gh_t} = 1+(1/2x^2)(-1+2x^2-2x^2\,\mathrm{cn}^2\gamma\chi)(H/h_t)+(1/40x^4)\{3(3-8x^2$$
$$+3x^4)+10(4-5x^2)x^2\,\mathrm{cn}^2\gamma\chi+40x^4\,\mathrm{cn}^4\gamma\chi\}(H/h_t)^2+(3/4x^2)\{1-x^2$$
$$+2(-1+2x^2)\,\mathrm{cn}^2\gamma\chi-3x^2\,\mathrm{cn}^4\gamma\chi\}(y/h_t)^2(H/h_t)^2+(1/1120x^6)\{-88+167x^2$$
$$+181x^4-200x^6+4(-112+378x^2-133x^4)x^2\,\mathrm{cn}^2\gamma\chi$$
$$+16(-119+133x^2)x^4\,\mathrm{cn}^4\gamma\chi-1344x^6\,\mathrm{cn}^6\gamma\chi\}(H/h_t)^3+(3/8x^4)\{-1+x^4$$
$$+2(1-7x^2+4x^4)\,\mathrm{cn}^2\gamma\chi+(19-29x^2)x^2\,\mathrm{cn}^4\gamma\chi+20x^4\,\mathrm{cn}^6\gamma\chi\}(y/h_t)^2(H/h_t)^3$$
$$+(3/16x^4)\{1-3x^2+2x^4+(-2+17x^2-17x^4)\,\mathrm{cn}^2\gamma\chi$$
$$+15(-1+2x^2)x^2\,\mathrm{cn}^4\gamma\chi-15x^4\,\mathrm{cn}^6\gamma\chi\}(y/h_t)^4(H/h_t)^3 \qquad (23)$$

(6) Summary of the considerations

In short, substitution of these series forms for L_0 and L_3 into the wave characteristics and integral proper-ties by the Chappelear theories based on three defini-tions for wave celerity yields the same expressions as those developed by the above-mentioned researchers in all the cases investigated. This means that all of the ex-isting third order solutions of cnoidal wave theory can be derived from the Chappelear theory by properly chang-ing expansion parameters when the definition for wave celerity is identified in each solution. Details of the wave characteristics and integral properties calculated are given in the series papers by the authors(Yamaguchi et al. 1990, 1991 and 1992).

Fig. 1 summarizes the interrelation established among the existing cnoidal wave theories.
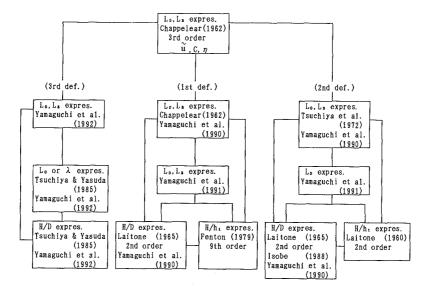

Fig. 1 Interrelation of cnoidal wave solutions.

3. Numerical Considerations

Fig.2 shows the vertical distributions of maximum horizontal water particle velocity $u_c/\sqrt{gD}$ based on the cnoidal wave theories using the first definition with two parameters, one parameter and H/D parameter indicated by c-3-1, c-3-1M and c-3-1L respectively, in which T is the wave period. The theory with H/D parameter gives a more and more unrealistic distribution in the case of larger H/D values, and the theory with L_0 parameter produces similar results to that with two parameters. But integral properties of the theory with two parameters do not show favorable behavior in the case of smaller $T\sqrt{g/D}$. Therefore, use of the cnoidal wave theory with with L_0 parameter is recommended for its practical application.
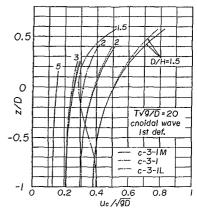

Fig. 2 Vertical distributions of maximum horizontal water particle velocity based on cnoidal wave theories with different parameters.

Examples of the comparison between computations with the experiments for wave celerity and maximum horizontal water particle velocity distribution are given in Fig. 3, which shows reasonable agreement between them. It is also
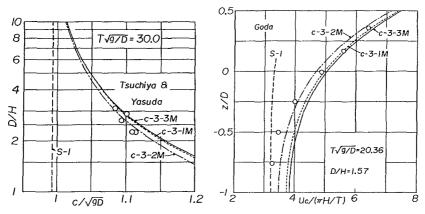

Fig. 3 Comparison between experiments and computations for wave celerity and vertical distribution of maximum horizontal water particle velocity.

seen that water particle velocity becomes larger according to the order of the theories with the second, first and third definitions.

Lagrangian velocity of a particular water particle is equal to Eulerian velocity at the position at time t

$$dx/dt = u(x, z), \quad dz/dt = w(x, z), \quad x = x_0 + \xi(t), \quad z = z_0 + \zeta(t) \tag{24}$$

where (x_0, z_0) are the initial horizontal and vertical positions of a water particle, and (ξ, ζ) the horizontal and vertical displacements of a water particle from its initial positions. Substitution of water particle velocities by cnoidal wave theory based on the first definition and use of a successive approximation method yield the trajectory of a water particle, in which case the initial conditions $x = x_0$ and $z = z_0$ at $t = 0$ are imposed. Horizontal and vertical displacements are written respectively as

$$x'/T\sqrt{gD} = x_0/T\sqrt{gD} + (1/2K)\{[ea - E(\alpha)]L_0$$
$$+ [(1/6)\{2(1-x^2)[\alpha] + 2(-4 + 2x^2 + 3e)[E(\alpha)] + (4 - 5x^2 - 15e$$
$$+ 6x^2\text{sn}^2\alpha)[ea - E(\alpha)]\} - (3/4)(y_0/D)^2x^2[\text{sn}\alpha\text{cn}\alpha\text{dn}\alpha]]L_0^2$$
$$+ [(1/600)[20(24 - 41x^2 + 17x^4)[\alpha] + 20\{-149 + 169x^2 - 34x^4$$
$$+ 5(52 - 29x^2)e - 135e^2\}[E(\alpha)] + \{-256 - 244x^2 + 369x^4$$
$$+ 150(-32 + 25x^2)e + 5625e^2\}[ea - E(\alpha)] + 200(1-x^2)x^2\text{sn}^2\alpha[\alpha]$$
$$+ 200(-4 + 2x^2 + 3e)x^2\text{sn}^2\alpha[E(\alpha)] + 100(10 - 11x^2 - 27e$$
$$+ 6x^2\text{sn}^2\alpha)x^2\text{sn}^2\alpha[ea - E(\alpha)] + 600x^2\text{sn}\alpha\text{cn}\alpha\text{dn}\alpha[ea - E(\alpha)]^2$$
$$- 8(1 + x^2)x^2[\text{sn}\alpha\text{cn}\alpha\text{dn}\alpha] + 24x^4[\text{sn}^3\alpha\text{cn}\alpha\text{dn}\alpha]]$$
$$+ (1/40)(y_0/D)^2[16(2 - 3x^2 + x^4)[\alpha] + 32(-1 + x^2 - x^4)[E(\alpha)]$$
$$- 30x^2\{1 - 2(1 + x^2)\text{sn}^2\alpha + 3x^2\text{sn}^4\alpha\}[ea - E(\alpha)]$$
$$+ 30x^4\text{sn}\alpha_0\text{cn}\alpha_0\text{dn}\alpha_0[\text{sn}^2\alpha] + (-44 + 61x^2 + 105e)x^2[\text{sn}\alpha\text{cn}\alpha\text{dn}\alpha]$$
$$- 60x^4\text{sn}^2\alpha_0[\text{sn}\alpha\text{cn}\alpha\text{dn}\alpha] - 18x^4[\text{sn}^3\alpha\text{cn}\alpha\text{dn}\alpha]]$$
$$+ (3/16)(y_0/D)^4\{-(1 + x^2)x^2[\text{sn}\alpha\text{cn}\alpha\text{dn}\alpha] + 3x^4[\text{sn}^3\alpha\text{cn}\alpha\text{dn}\alpha]\}\}L_0^3] \tag{25}$$

$$z'/T\sqrt{gD} = z_0/T\sqrt{gD} - (3/2K\sqrt{3L_0})(y_0/D)x^2\{(1/2)[\text{sn}^2\alpha]L_0^2$$
$$- (1/8)[2(-4 + 3x^2 + 7e - 2x^2\text{sn}^2\alpha_0)[\text{sn}^2\alpha] - 8\text{sn}\alpha\text{cn}\alpha\text{dn}\alpha[ea$$
$$- E(\alpha)] + (y_0/D)^2\{-2(1 + x^2)[\text{sn}^2\alpha] + 3x^2[\text{sn}^4\alpha]\}]L_0^3$$
$$+ [(1/240)[80(1 - x^2)\text{sn}\alpha\text{cn}\alpha\text{dn}\alpha[\alpha] + 80(4 - 2x^2$$
$$- 3e)\text{sn}\alpha\text{cn}\alpha\text{dn}\alpha[E(\alpha)] + 40(10 - 11x^2 - 27e + 6x^2\text{sn}^2\alpha$$
$$+ 6x^2\text{sn}^2\alpha_0)\text{sn}\alpha\text{cn}\alpha\text{dn}\alpha[ea - E(\alpha)] + 120\{1 - 2(1 + x^2)\text{sn}^2\alpha$$
$$+ 3x^2\text{sn}^4\alpha\}[ea - E(\alpha)]^2 + \{548 - 704x^2 + 213x^4 + 10(-200 + 133x^2)e$$
$$+ 1665e^2\}[\text{sn}^2\alpha] + 60x^2\text{sn}^2\alpha_0(6 - 5x^2 - 11e + 2x^2\text{sn}^2\alpha_0)[\text{sn}^2\alpha]$$
$$- 24x^2(1 + x^2)[\text{sn}^4\alpha] + 24x^4[\text{sn}^6\alpha]] + (1/16)(y_0/D)^2[8(1 + x^2$$

$$-3x^2\mathrm{sn}^2\alpha)\mathrm{sn}\alpha\mathrm{cn}\alpha\mathrm{dn}\alpha\,[ea-E(\alpha)] +2\{4-8x^2-3x^4$$

$$-7(1+x^2)e\}\,[\mathrm{sn}^2\alpha] +2x^2\mathrm{sn}^2\alpha_0(8+8x^2-3x^2\mathrm{sn}^2\alpha_0)\,[\mathrm{sn}^2\alpha]$$

$$+3x^2(7x^2+7e-6x^2\mathrm{sn}^2\alpha_0)\,[\mathrm{sn}^4\alpha]-8x^4\,[\mathrm{sn}^6\alpha]$$

$$+12x^2\mathrm{sn}\alpha_0\mathrm{cn}\alpha_0\mathrm{dn}\alpha_0\,[\mathrm{sn}\alpha\mathrm{cn}\alpha\mathrm{dn}\alpha]]+(3/160)(y_0/D)^4\{(2+13x^2$$

$$+2x^4)\,[\mathrm{sn}^2\alpha]-15x^2(1+x^2)\,[\mathrm{sn}^4\alpha]+15x^4\,[\mathrm{sn}^6\alpha]\}]L_0^4] \qquad (26)$$

where $[f(\alpha)] = f(\alpha)-f(\alpha_0)$, $\alpha = 2K(x_0-ct)/L$

$$\alpha_0 = 2Kx_0/L, \qquad\qquad y_0/D = 1+z_0/D, \qquad (27)$$

and $E(\alpha)$ is the Jacobian E function defined as

$$E(\alpha) = \int_0^\alpha \mathrm{dn}^2\alpha d\alpha \qquad (28)$$

Mass transport velocity is calculated as time-averaged residual displacement of a water particle, and the distributions based on the solutions using three definitions are given respectively as

$$U_m^1/\sqrt{gD} = (1/3)\{-1+x^2+2(2-x^2)e-3e^2\}L_0^2+[(1/30)\{-24+41x^2$$

$$-17x^4 +(149-169x^2+34x^4)e+5(-52+29x^2)e^2+135e^3\}$$

$$\underline{+(1/3)\{-1+x^2+2(2-x^2)e-3e^2\}x^2\mathrm{sn}^2\alpha_0}$$

$$+(2/5)(y_0/D)^2\{-2+3x^2-x^4+2(1-x^2+x^4)e\}]L_0^3 \qquad (29)$$

$$V_m^1/\sqrt{gD} = \underline{-(1/3\sqrt{3L_0})L_0^4(y_0/D)x^2\mathrm{sn}\alpha_0\mathrm{cn}\alpha_0\mathrm{dn}\alpha_0\{-1+x^2}$$

$$\underline{+2(2-x^2)e-3e^2\}} \qquad (30)$$

$$U_m^{II}/\sqrt{gD} = (2/15)L_0^3\{1-3(y_0/D)^2\}\{2-3x^2+x^4+2(-1+x^2-x^4)e\} \qquad (31)$$

$$V_m^{II}/\sqrt{gD} = 0 \qquad (32)$$

$$U_m^{III}/\sqrt{gD} = (1/2)\{-1+x^2+2(2-x^2)e-3e^2\}L_0^2+[(1/60)\{-66$$

$$+109x^2-43x^4 +(421-461x^2+86x^4)e+5(-146+79x^2)e^2+375e^3\}$$

$$\underline{+(1/3)\{-1+x^2+2(2-x^2)e-3e^2\}x^2\mathrm{sn}^2\alpha_0}$$

$$+(2/5)(y_0/D)^2\{-2+3x^2-x^4+2(1-x^2+x^4)e\}]L_0^3 \qquad (33)$$

$$V_m^{III}/\sqrt{gD} = \underline{-(3/2\sqrt{3L_0})L_0^4(y_0/D)x^2\mathrm{sn}\alpha_0\mathrm{cn}\alpha_0\mathrm{dn}\alpha_0\{-1+x^2}$$

$$\underline{+2(2-x^2)e-3e^2\}} \qquad (34)$$

When the initial conditions such as $x=x_0$ and $z=z_0$ at $t=0$ are used, mass transport velocities based on the third order solutions of cnoidal waves using the first and third definitions depend on the initial horizontal position of a water particle, as denoted by underline. This may be due to the deficiency of the successive approximation method as in the case of Stokes wave theory. Thus,

x_0 is set to zero.

Fig. 4 illustrates the trajectory of a water particle and vertical distribution of mass transport velocity based on cnoidal wave theories using three definitions. None of the theories describe closed orbits, and all of them result in mass transport velocity distribution.

Wave energy $\widehat{E}$ and energy flux $\widehat{F}$ nondimensionalized by those of the small amplitude wave theory are shown in Fig. 5. Increase of $T\sqrt{g/D}$ and decrease of D/H, that is to say, increase of wave nonlinearity, give rise to consistent and favorable reduction of dimensionless integral properties.
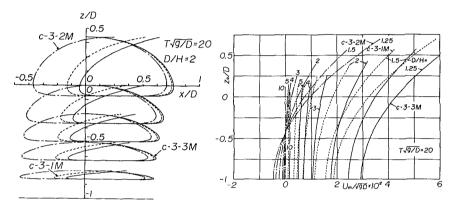

Fig. 4 Trajectory of a water particle and vertical distribution of mass transport velocity.
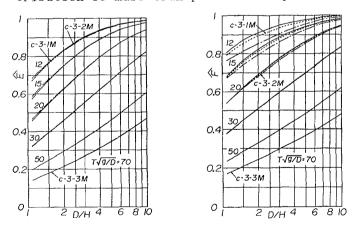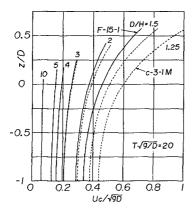

Fig. 5 Relation between integral properties and wave parameters.

4. Limiting Range for Applicability

A limiting range for the application of cnoidal wave theory with a single expansion parameter is investigated by comparing wave characteristics and integral properties of cnoidal waves with those of Fenton's Fourier approximation theory, because the Fourier approximation theory gives the almost exact solution to water wave problem on a uniform depth.

Fig. 6 shows two examples of the comparison for vertical distribution of maximum horizontal water particle velocity and dimensionless energy flux, which shows that decrease of D/H and $T\sqrt{g/D}$ gives rise to greater discrepancy between both theoretical results.
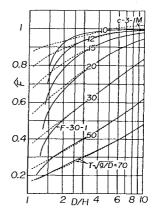

Fig. 6 Comparison between vertical distribution of maximum horizontal water particle velocity and dimensionless energy flux by cnoidal wave theory and those by Fourier approximation theory.

Fig. 7 illustrates a limiting range of wave parameters for the application of cnoidal wave theory, in which errors for wave celerity, horizontal water particle velocity and integral properties of cnoidal wave theory are less than 0.5 %, 5 % and 5 % respectively. In the figures, notations u_0, u_5 and u_b are the maximum horizontal water particle velocities at the still water level, middle point between still water level and bottom, and bottom, E_k and E_p the kinetic and potential energies, S_{11} and S_{22} the principal components of radiation stress and u_b^2 is the time mean of squared horizontal water particle velocity at the bottom. The range for individual wave characteristics and for integral property are different from each other. But roughly speaking, a limiting range of H/D for the third order solution is around 2.
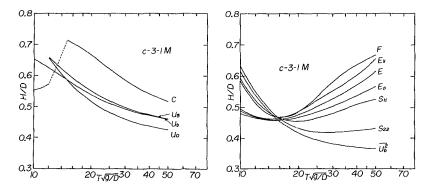
Fig. 7 Limiting range for application of the third order
solution of cnoidal wave theory.

5. Conclusions

The main results are summarized as follows;

1) The interrelation among all of the existing cnoidal
wave theories with the third order approximation is
analytically established, in which case they can be
derived from the Chappelear solution expressed with
two expansion parameters by identifying the defini-
tion for wave celerity to be used in the theory and
changing the expansion parameters in the expressions
of wave characteristics and integral properties.

2) Based on the numerical computation of the wave charac-
teristics and integral properties and the comparison
with the experiments, use of the cnoidal wave theory
expressed with a single parameter L_0 was recommended
from a practical viewpoint.

3) The figures describing a limiting range of wave pa-
rameters for the application of the cnoidal wave theo-
ry expressed with a single expansion parameter are
provided by comparing wave characteristics and inte-
gral properties of the theory with those of the
Fourier approximation theory.

6. Acknowledgements

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