CHAPTER 51

A stream function solution for waves on a strongly sheared current.

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Abstract.

A perturbation analysis is presented in which a series of small amplitude progressive gravity waves interact with a strongly sheared current. The solution, which is extended to a second order of wave steepness, shows that if the time averaged vorticity distribution varies with depth the wave motion becomes rotational. An additional wave component, expressed in terms of the first harmonic, is identified at a second order of wave steepness. This does not arise within an irrotational solution and is quite distinct from the Doppler shift associated with the surface current. Explicit solutions are given for the dispersion equation and the wave induced kinematics. These are found to be very different from the existing irrotational solutions, and suggest that the non-linear wave-current interaction terms can become very important if the current profile is strongly sheared in the vicinity of the water surface. In such cases the underlying velocity field should not be predicted by an irrotational solution based upon an "equivalent" uniform current.

1. Introduction.

The combination of waves and currents is an important feature of most marine environments. The present paper considers the fluid flow resulting from such an interaction once it has achieved a state of equilibrium. It will not consider the initial generation of waves on a strongly sheared current; nor will it consider the propagation of waves onto a strongly sheared current. The initial transfer of energy between the various components of the flow field,

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and the resulting change in the wave height, forms part of a transient problem which has already been considered by a number of authors. These include Longuet-Higgins and Stewart (1960, 1961), Bretherton and Garrett (1968), and Brink-Kjær and Jonsson (1975). A full discussion of these matters is given in the review articles by Peregrine (1976) and Jonsson (1990).

It is well known that the equilibrium conditions associated with the interaction of waves and currents are strongly dependent upon the vertical distribution of the current velocity. In many practical cases it may be assumed that the current profile is approximately uniform with depth. Important examples of this type of behaviour are the large scale ocean currents, and the majority of tidal flows. Under these conditions, the wave motion remains irrotational and, in effect, the only interaction occurs within the associated dispersion equation. At a second order of wave steepness the dispersion equation for waves propagating on a uniform current (U=Uo) is given by:-

$$c=\frac{\sigma}{k}=\left[\frac{g}{k}\tanh(kh)\right]^{1/2}+U_0 \qquad (1)$$

where c is the wave celerity, h is the water depth and g is the gravitational constant. The wave number (k) and the wave frequency (σ) are defined in the usual way so that $k=2\pi/\lambda$ and $\sigma=2\pi/T$, where λ is the wave length and T is the wave period. This solution is often referred to as a "Doppler shifted solution" since it describes the wave form propagating on the surface current.

A second example which has been widely considered is that of waves on a linear shear current, or one in which the current velocity varies linearly with depth. Tsao (1959) considered this case and showed that the wave motion, or the oscillatory component of the flow field, will remain irrotational provided the vortioity is constant throughout the water depth. In this case the stream function (ψ) can no longer be expressed in the form of a solution to Laplace's equation as would be the case in a classical Stokes' expansion (1847). The governing equation is thus expressed in the form of a Poisson equation:-

$$\nabla^2\psi=\Omega_o \qquad (2)$$

where Ω_o is the constant vorticity or the gradient of the linear shear current. Although the oscillatory motion remains irrotational, it is different from that which would be predicted in the absence of a current. If the current is assumed to be of a similar magnitude to the first order wave motion, an additional oscillatory term arises at a second order of wave steepness ($O.(a^2k^2)$). Kishida and Sobey (1988) identified this term as:-

$$\psi_{wc} = 2az\Omega_o \frac{\sinh(kh+kz)}{\sinh(kh)} \cos(kx-\sigma t) \qquad (3)$$

where a is the wave amplitude (or half the wave height, H) and (x,z) are the Cartesian co-ordinates described below.

In many practical cases neither the current velocity or the vorticity distribution are uniform. For example, in the absence of significant vertical mixing, a wind driven current decays exponentially with depth. This creates a strongly sheared current with a concentration of vorticity near the water surface. To describe an interaction of this type has hitherto required a complex numerical model similar to that proposed by Chaplin (1989). The present paper will consider this case and presents a new analytical solution which is simple to use, and which provides a first approximation to the non-linear wave-current interaction which arises in the presence of a strongly sheared current profile.

2. Theory.

The new solution will take the form of a perturbation expansion in which a series of two-dimensional monochromatic waves, propagating in water of constant depth, interact with a strongly sheared current. To simplify the non-linear boundary conditions which must be applied at the water surface, the analysis will be conducted within an orthogonal curvi-linear co-ordinate system.

The Cartesian co-ordinates shown on figure 1a translate with the phase velocity (c) to provide a steady (d/dt=0) frame of reference. These axes are mapped onto the curvi-linear co-ordinates shown on figure 1b. To achieve this a sequence of transformations must be applied so that at each order of the perturbation (see below) η=0 defines the free surface and η=-h the position of the impermeable bottom boundary. At a first order of approximation these transformations are similar to those applied by Benjamin (1959). The general form of the transformations are given by:-

$$\eta = z - \sum_{n=1}^{N} A_n \frac{\sinh(nkh+nk\eta)}{\sinh(nkh)}\cos(nk\xi) - B_n$$

$$\xi = x - \sum_{n=1}^{N} A_n \frac{\cosh(nkh+nk\eta)}{\sinh(nkh)}\sin(nk\xi) \qquad (4)$$

where the subscript defines the order of the terms, and the unknown constants (A_n, B_n), are dependent upon the water surface elevation (ζ). Since this cannot be known a priori an iterative approach is adopted in which an initial estimate of the surface profile is made, the co-ordinates are transformed, and a solution obtained as indicated below. This solution is then used in conjunction with the boundary conditions to define, where necessary, an appropriate modification of the surface elevation. In this way a unique solution can be identified which satisfies both the governing equation and the boundary conditions.

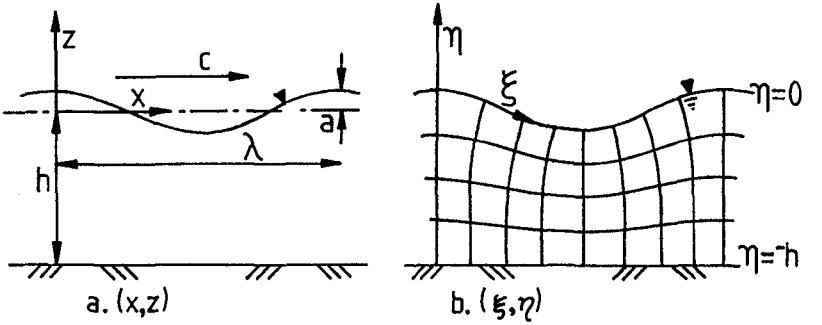

Figure 1. Co-ordinate arrangements.

To achieve this iterative approach the irrotational solution proposed by Stokes (1847) was used to provide an initial estimate of the surface elevation. If ζ is measured relative to the still water level (z=0), then the required second order expression is given by:-

$$\zeta = a\cos(\Phi) + a^2 k \cosh(kh)\frac{[2\sinh^2(kh)+3]}{4\sinh^3(kh)}\cos(2\Phi) - \frac{a^2 k}{2\sinh(2kh)} \qquad (5)$$

where Φ is the phase angle ($kx-\sigma t$). It is important to note that the surface elevation alone is being adopted as an initial estimate. No assumptions are made about the nature of the associated dispersion equation, and the flow is not assumed to be irrotational.

The transformations given above (4) are orthogonal, and the Jacobian J is defined by:

$$J=\frac{\partial(\xi,\eta)}{\partial(x,z)} \tag{6}$$

Within the (ξ,η) co-ordinates the two dimensional vorticity equation for flow in an inviscid fluid is given by:

$$\frac{\partial(\psi,J\nabla^2\psi)}{\partial(\xi,\eta)} = 0 \tag{7}$$

where the stream function (ψ) is defined so that the total velocity components in the ξ and η directions are:-

$$(U+u)_\xi=J^{1/2}\frac{\partial\psi}{\partial\eta} \quad , \quad u_\eta=-J^{1/2}\frac{\partial\psi}{\partial\xi} \tag{8}$$

In accordance with the arguments originally outlined by Stokes (1847) an exact solution of a two dimensional wave train does not exist. Consequently, a perturbation expansion must be employed in which the stream function (ψ), the surface elevation (ζ), and the Jacobian (J), are expressed in terms of a small expansion parameter (ε):

$$\begin{aligned}\psi&=\psi_0+\varepsilon\psi_1+\varepsilon^2\psi_2+\varepsilon^3\psi_3+\varepsilon^4\psi_4+\ .....\\ \zeta&=\zeta_0+\varepsilon\zeta_1+\varepsilon^2\zeta_2+\varepsilon^3\zeta_3+\varepsilon^4\zeta_4+\ .....\\ J&=J_0+\varepsilon J_1+\varepsilon^2 J_2+\varepsilon^3 J_3+\varepsilon^4 J_4+\ .....\end{aligned} \tag{9}$$

where the subscript again denotes the order of the term involved. In the present solution the wave steepness (ak) is adopted as an appropriate expansion parameter, and the co-ordinate arrangement shown on figure 1b defines the following "zero order" terms:

$$\psi_0=-c\eta \quad , \quad J_0=1 \quad , \quad \zeta_0=0 \tag{10}$$

Substituting (9) and (10) into the governing vorticity equation and collecting powers of ε gives the required expressions of the vorticity equation at successive steps in the perturbation:

$$c\frac{\partial}{\partial\xi}(\nabla^2\psi_1) = 0$$

$$c\frac{\partial}{\partial\xi}(\nabla^2\psi_2) = \frac{\partial(\nabla^2\psi_1\ ,\ \psi_1)}{\partial(\xi\ ,\ \eta)} - c\frac{\partial}{\partial\xi}(J_1\nabla^2\psi_1) \qquad (11)$$

Before discussing the boundary conditions we must define the characteristics of the current profile. Firstly, it should be noted that if the magnitude of the current velocity is close to the phase velocity (c), the vorticity equations (11) are indeterminate. Fortunately, this situation seldom arises within either a coastal or an ocean environment. Current velocities are typically much smaller than the phase velocity, and are generally more closely related to the magnitude of the first order wave motion (U=O.(ak)). This assumption is adopted within the present formulation.

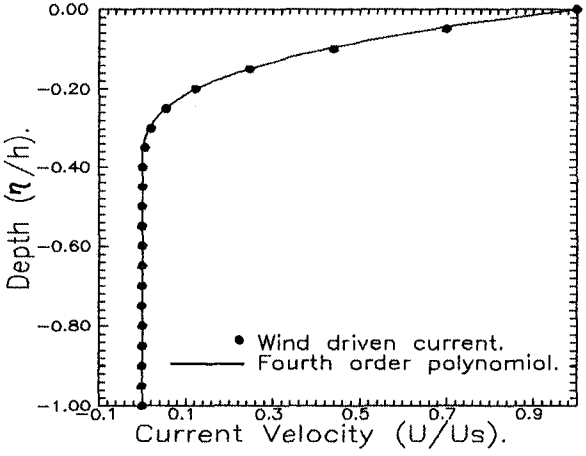

Figure 2. A strongly sheared current profile.

To ensure that an appropriate range of current profiles can be incorporated within the present solution, a polynomial representation is adopted to describe a current profile within the region $0 \geq \eta \geq -mh$ where m is a constant within the range $(0 \leq m \leq 1)$. Hence:

$$U-(P+2Q\eta+3R\eta^2+4S\eta^3)\ \delta_s(\eta+mh) \qquad (12)$$

Where δs(η+mh) is the heavyside step function defined by:

$$\delta_s(\eta+mh)=1 \quad if \quad (\eta+mh) \geq 0$$
$$\delta_s(\eta+mh)=0 \quad if \quad (\eta+mh) < 0 \tag{13}$$

Figure 2 compares the current profile defined in (12) with a wind driven profile based on the lamina flow solution proposed by Lamb (1932).

The vorticity equations (11) must be solved within the confines of the following boundary conditions:

(a) If the bottom boundary is assumed to be both horizontal and impermeable, the vertical velocity at the bed must be zero:

$$-J^{1/2}\frac{\partial \psi}{\partial \xi}=0 \quad on \quad \eta=-h \tag{14}$$

(b) Since the water surface is a streamline the kinematic condition requires the velocity normal to the surface to be zero:

$$-J^{1/2}\frac{\partial \psi}{\partial \xi}=0 \quad on \quad \eta=0 \tag{15}$$

(c) The dynamic free surface boundary condition further requires the pressure (p) acting on the water surface to be constant. In general orthogonal co-ordinates the equations of motion are given by:-

$$\frac{\partial \underline{u}}{\partial t}+\frac{1}{2}\nabla(\underline{u}^2)-\underline{u}\times\underline{\omega} = -\frac{1}{\rho}\nabla(p)+\underline{F} \tag{16}$$

where the under-bar denotes a vector quantity, ω is the vorticity distribution, and F is a body force. Taking the first component of (16) and applying the kinematic condition (15) gives the required dynamic condition:

$$\frac{\partial p}{\partial \xi} = F_\xi J^{-1/2}-\frac{1}{2}\frac{\partial J}{\partial \xi}\left[\frac{\partial \psi}{\partial \eta}\right]^2-J\frac{\partial \psi}{\partial \eta}\frac{\partial^2 \psi}{\partial \eta \partial \xi} = 0 \quad on \quad \eta=0 \tag{17}$$

where F_ξ is the resolved component of the body force per unit mass in the ξ direction. If g is the gravitational constant, and θ is the angle separating the z and η directions (figure 1), we obtain:

$$F_\xi = -g \sin(\theta) \tag{18}$$

(d) Finally the fluid motion must be continuous throughout the entire water depth. In the case of ($m<1$) the flow may be divided into two distinct regions which are characterised by the presence ($0 \geq \eta \geq -mh$), or absence ($-mh \geq \eta \geq -h$), of the current profile. If the governing equation (7) is to be consistent across this interface ($\eta = -mh$) the time averaged vorticity must be zero at the lower edge of the current profile:

$$limit_{(\eta \to -mh)}(\omega) = 0 \tag{19}$$

Furthermore, the value of the stream function and the velocity components must also be continuous across this region. If ($\eta \to -mh$) represents the limit taken in the negative η direction (from above) and ($\eta \to -mh$) represents the limit taken in the positive η direction (from below) the final boundary conditions are given by:

$$limit_{(\eta^+ \to -mh)}(\psi) = limit_{(\eta^- \to -mh)}(\psi)$$

$$limit_{(\eta^+ \to -mh)}\left(J^{1/2}\frac{\partial \psi}{\partial \eta}\right) = limit_{(\eta^- \to -mh)}\left(J^{1/2}\frac{\partial \psi}{\partial \eta}\right) \tag{20}$$

$$limit_{(\eta^+ \to -mh)}\left(-J^{1/2}\frac{\partial \psi}{\partial \xi}\right) = limit_{(\eta^- \to -mh)}\left(-J^{1/2}\frac{\partial \psi}{\partial \xi}\right)$$

3. Results and discussion.

The perturbation analysis is extended to a second order of wave steepness. Since the current velocity is assumed to be of order ak, the solution will provide a first approximation to the wave-current interaction. The resulting stream function is given by:

$$\psi_0 = -c\eta$$

$$\psi_1 = (Q\eta^2+R\eta^3+S\eta^4)\,\delta_s(\eta+mh) - P\eta\,\delta_s(-\eta-mh)$$

$$\psi_2 = a(2Q\eta+3R\eta^2+4S\eta^3+\frac{6S\eta}{k^2}+F_a)\frac{\sinh k(h+\eta)}{\sinh(kh)}\cos(k\xi)\,\delta_s(\eta+mh) \quad (21)$$

$$+a(-\frac{3R\eta}{k}-\frac{6S\eta^2}{k}+F_b)\frac{\cosh k(h+\eta)}{\sinh(kh)}\cos(k\xi)\,\delta_s(\eta+mh)$$

$$+aF_c\sinh k(h+\eta)\cos(k\xi)\,\delta_s(-\eta-mh)$$

Where the heavyside function (δ_s) is defined in (13) and the terms Fa, Fb, and Fc are constants for a given wave-current interaction (figure 3).

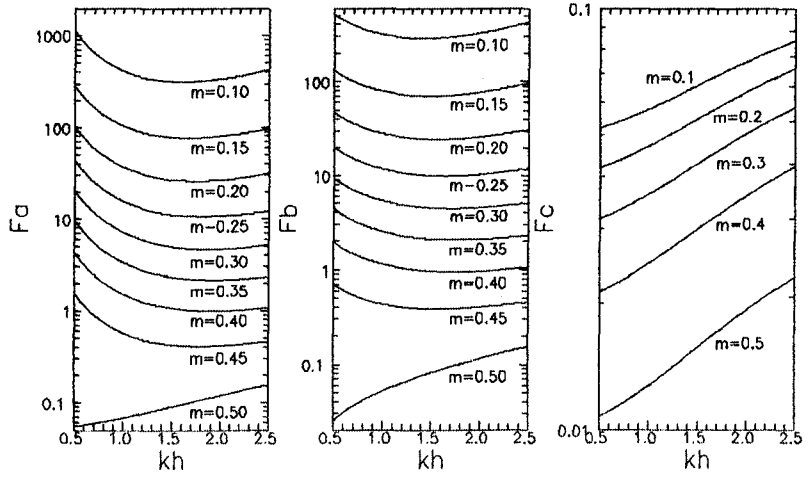

Figure 3. The constant coefficients (Fa, Fb, Fc). (Defined for a cubic shear current).

The three terms within this series solution can be identified in the following way. The zero order term (ψ_0), which arises due to the translation of the co-ordinate axes, may be used in conjunction with the Jacobian (J) to define the irrotational velocity components within a classical Stokes expansion. The first order term (ψ_1) provides a description of the current profile in a steady frame of reference, and the second order term (ψ_2) defines the additional terms associated with the wave-current interaction.

To complete the required solution a dispersion equation describing the combined wave-current motion is required. At a second order of wave steepness the phase velocity measured relative to a stationary observer is given by:-

$$c = \frac{\sigma}{k} = \left(\frac{g}{k}\tanh(kh)\right)^{\frac{1}{2}} + P - \frac{Q\tanh(kh)}{k} + \frac{3R}{2k^2} - \frac{3S\tanh(kh)}{k^3} + \frac{F_a}{2\cosh^2(kh)} \tag{22}$$

where P, Q, R and S are the constants used to define the current profile (12). Assessing these terms at the water surface (η=0) we obtain:

$$P = (U)_{\eta=0} \quad , \quad Q = \frac{1}{2!}\left(\frac{dU}{d\eta}\right)_{\eta=0}$$
$$R = \frac{1}{3!}\left(\frac{d^2U}{d\eta^2}\right)_{\eta=0} \quad , \quad S = \frac{1}{4!}\left(\frac{d^3U}{d\eta^3}\right)_{\eta=0} \tag{23}$$

The effect of the surface vorticity (ω_s) is considered in figure 4. Convention dictates that a vorticity distribution is positive if it causes an anti-clockwise rotation of the fluid particles (ω=-dU/dη). Unfortunately, this definition results in a positively sheared current (dU/dη>0) having negative vorticity. To avoid confusion the horizontal abscissa on figure 4 is expressed in terms of ($-\omega_s$) so that an increase in the positive x-direction indicates an increase in the positive shear.

Figure 4 considers a number of realistic current profiles in which a "favourable" current (U_s>0) has positive shear ($-\omega_s$>0) and an "adverse" current (U_s<0) has negative shear ($-\omega_s$<0). In all cases the vorticity distribution acts to reduce the effect of the Doppler shift associated with the surface current (U_s). For example, if a series of waves propagate onto a "favourable" current the individual waves will be "stretched" thereby producing a reduction in the wave number. If however, the current profile is sheared (dU/dη>0) then the apparent change in the wave number is much reduced. Indeed, figure 4 indicates that a highly sheared current profile can entirely negate the wave length changes associated with the Doppler shift.

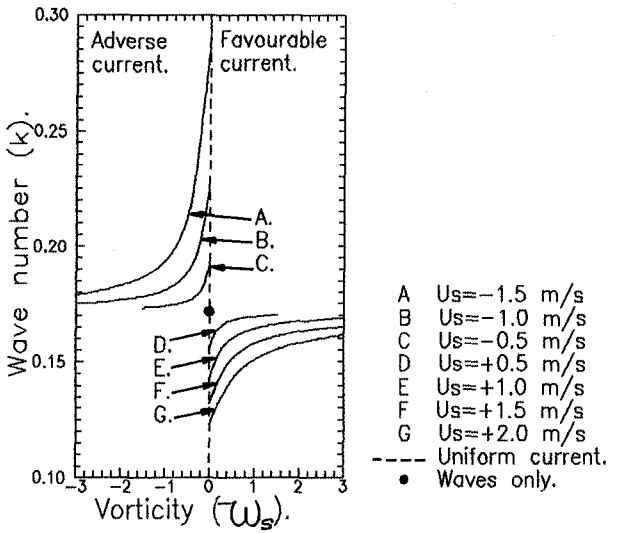

Figure 4. The wave number (k).

This reduction in the apparent Doppler shift has led some authors to suggest that the interaction with a strongly sheared current can be described by an "equivalent" uniform current ($U(\eta)=U_e$). Hedges and Lee (1992) define this "equivalent" uniform current as that which produces the same wave number (k) as the actual depth varying current for a particular wave period, wave height, and water depth. This approach is considered in figures 5a-5b which respectively concern the interaction with a positively sheared "favourable" current and a negatively sheared "adverse" current. In each case the first figure provides a description of the current profile (U), and the second figure describes the oscillatory velocity occurring beneath the wave crest (u). Four different solutions of the wave induced velocity are presented: the first is a waves only solution which neglects the effect of the current profile; the second assumes that the surface current exists uniformly with depth ($U(\eta)=U_e$); the third is based on the present solution for waves on a strongly sheared current, and the fourth is based on an "equivalent" uniform current.

Figures 5a and 5b show that the wave kinematics resulting from the interaction with a strongly sheared current are dependent upon two separate effects. The first corresponds to a change in the dispersive characteristics of the wave form given in equation (22), and the second is associated with the rotational wave components identified in equation (21). The dispersive characteristics are found to be dependent upon the magnitude of the surface current and

the time averaged vorticity distribution. In consequence the actual wave number falls between the values predicted by the waves only solution and the uniform current solution which is based upon the magnitude of the surface current (U_s).

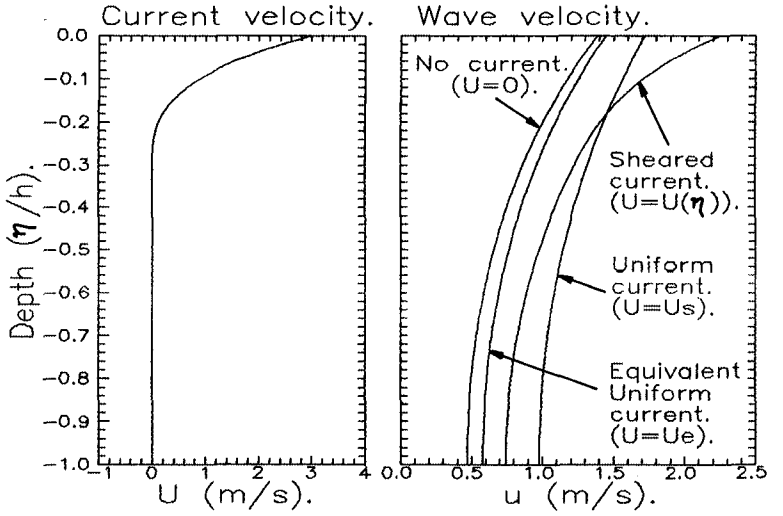

Figure 5a. Interaction with a "favourable" current. (T=5s, h=10m, H=2m).

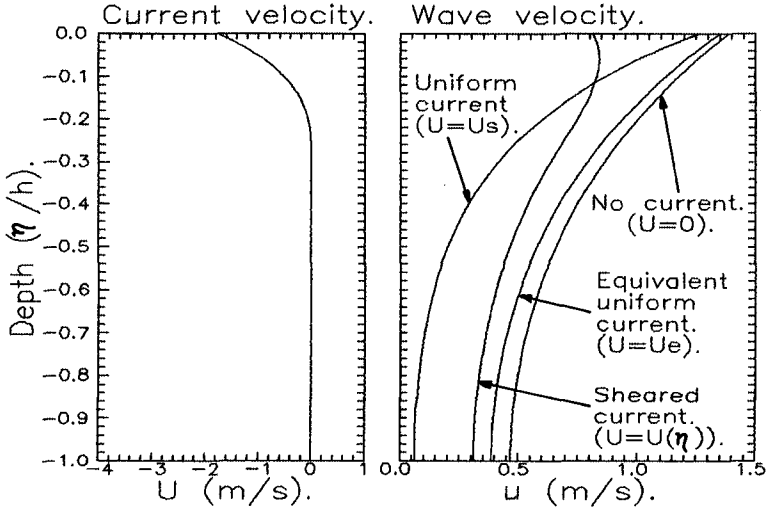

Figure 5b. Interaction with an "adverse" current. (T=5s, h=10m, H=2m).

The "equivalent" uniform current will, by definition, correctly model the dispersive characteristics of the combined wave-current motion. However, this alone is not sufficient to describe the wave induced flow field in the presence of a strongly sheared current. Figures 5a and 5b clearly indicate that the wave kinematics are directly dependent upon the vorticity distribution. In the upper region of the flow field ($0 \geq \eta \geq -mh$), where the current profile exists, figure 5a shows that a positively sheared "favourable" current produces a significant increase in the amplitude of the oscillatory velocity occurring near the water surface. In figure 5b a negatively sheared "adverse" current reduces the oscillatory velocity in this region. These changes are associated with the rotational wave component identified in equation (21). They are not related to the effective Doppler shift, and cannot therefore be predicted by an irrotational solution based on an "equivalent" uniform current.

It is interesting to note that even in the lower layers of the flow field ($-mh \geq \eta \geq -h$) where no current profile exists, and consequently the flow must be irrotational, the solution based upon an "equivalent" uniform current again provides a poor representation of the wave kinematics. This arises because of the continuity conditions (20) which must be applied at the lower edge of the current profile ($\eta = -mh$). This suggests that a wave-current interaction involving a non-uniform vorticity distribution will effect the entire wave field, even if, as in the present case, the vorticity profile is contained within a relatively narrow layer near the water surface.

4. Conclusions.

An analytical solution has been presented to describe a series of two dimensional progressive gravity waves propagating on a strongly sheared current. The solution has been extended to a second order of wave steepness and shows that the wave motion becomes rotational if the vorticity distribution varies with depth.

The dispersion equation describing the combined wave-current motion is found to contain additional first order terms which are related to the vorticity distribution. These terms have a significant effect upon the predicted wave number, and in the case of a realistic current profile lead to a reduction in the apparent Doppler shift.

The oscillatory velocities associated with the wave motion are also shown to be very different from the existing irrotational solutions. In addition to the wave number effects discussed above, a favourable current with positive

shear produces a significant increase in the amplitude of the oscillatory motion, while an adverse current with negative shear reduces the amplitude of the oscillatory motion. These changes are associated with a rotational wave component which arises at a second order of wave steepness. This additional component is directly related to the vorticity distribution and does not therefore arise within a classical Stokes' expansion. As a result, the wave motion generated in the presence of a strongly sheared current cannot be predicted by an irrotational solution, even if this includes the effect of an "equivalent" uniform current.

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