CHAPTER 13

THE INTERACTION OF DEEP-WATER GRAVITY WAVES AND A CURVED SHEARING CURRENT

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1. ABSTRACT:

The interaction of steady deep-water gravity waves with a pre-existing large-scale curved current has been investigated. In order to investigate the influence of the curvature of the current on the wave field, the current field was represented by a section of an annular current with a particular non-dimensional radius R.

As a first approximation the interaction of a family of linear axi-symmetrical waves and the current was investigated. Exact linear ray solutions were obtained which, in the limit when $R \to \infty$, reduce to the analytical straight current solutions of Longuet-Higgins and Stewart (1961).

2. INTRODUCTION:

The emphasis of this paper is on the linear theory of the interaction of deep-water waves, generated on still water, and pre-existing large scale currents. Longuet-Higgins and Stewart (1960, 1961) were the first to give an accurate description of linear wave-current interactions and introduced the concept of radiation stress. Further contributions in our understanding of the interaction of linear waves and large scale currents came from, among others, Whitham(1962), Bretherton and Garrett(1968) and Peregrine (1976), who examined a number of different situations.

In almost all of the above studies the analysis was confined to two special situations of steady currents, namely (i) straight currents, varying with distance along the stream or (ii) straight

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currents varying across the stream. Exact linear solutions for these situations were derived by Longuet-Higgins and Stewart (1960, 1961).

The purpose of this paper is to extend the linear theory of the interactions of waves with a large scale current to more general current situations. Here we extend the range of known linear solutions by considering the simplest formulation of interaction with a curved current, namely steady axi-symmetrical waves on an axi-symmetrical annular current. This restriction simplifies the mathematics, but even so, solutions have been found for a wide range of cases.

3. MATHEMATICAL FORMULATION

Consider, in polar co-ordinates, an annular current of the form

$$\underline{U} = U_\theta(r)\, \underline{e}_\theta \,, \tag{3.1}$$

where r and θ are the polar coordinates and $\underline{e}_\theta$ is a unit vector in the θ direction. Equation (3.1) describes an axi-symmetric current with arbitrary velocity profile which is only a function of the radial distance.

For the annular current (3.1) the basic equations of wave kinematics are given by:

DISPERSION:

$$\sigma^2 = gk \text{ or } c^2 = g/k, \quad k = |\underline{k}| \tag{3.2}$$

where σ is the intrinsic frequency of the waves and k is the corresponding wavenumber. The celerity of the waves is denoted by c.

DOPPLER SHIFT:

The apparent frequency, i e the frequency of waves passing a fixed point, is

$$\omega = \sigma(\underline{k}) + \underline{k}.\underline{U}$$
$$= \sigma + k \sin \alpha \, U_\theta$$
$$= k[c + U_\theta \sin \alpha], \quad (3.3)$$

where α is the angle between the wavenumber $\underline{k}$ and the unit vector $\underline{e}_r$ in the radial direction (see figure 1).

ANGULAR WAVENUMBER:

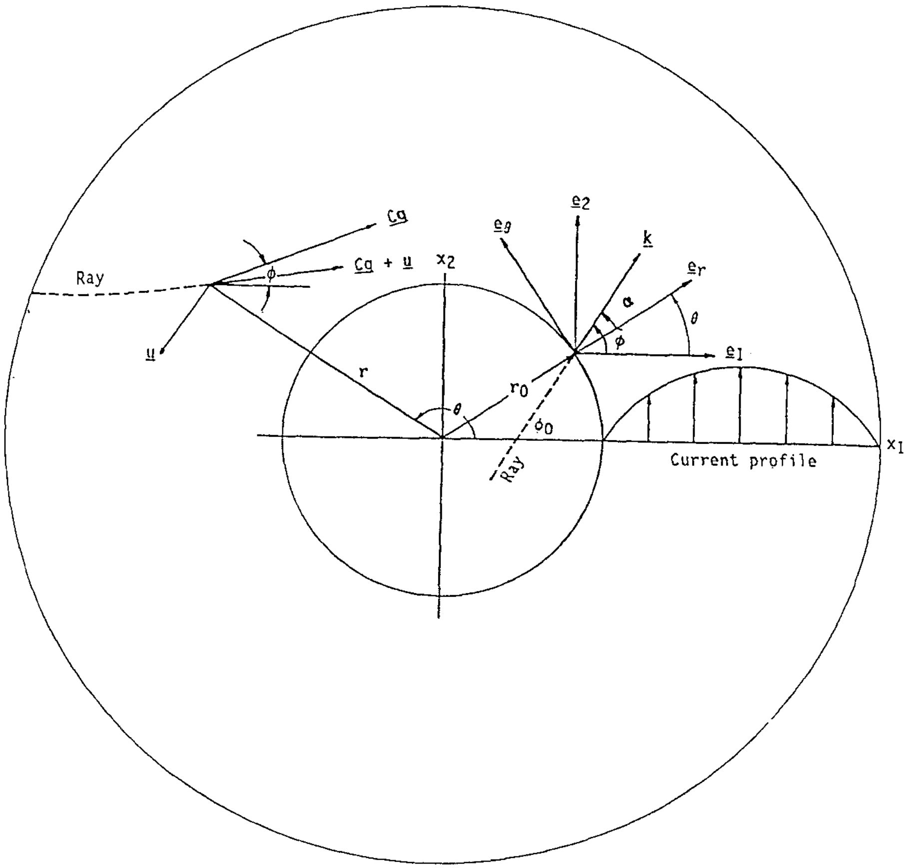
$$m = rk \sin \alpha, \quad (3.4)$$

with m a constant.

Non-dimensional variables may now be introduced to simplify the solutions. Consider a wave ray initially outside the influence of the current and denote, when $U_\theta = 0$, the wavenumber k by k_0 and the wave celerity c by c_0 A scaled wavenumber $K = k/k_0$, as well as a scaled celerity $C = c/c_0$, may then be introduced. Also let $V = U_\theta/c_0$ denote the scaled current velocity.

In order to depict the outside (or inside) radius of the annulus at the point of entry of the ray, let the radius of the current at this point be r_0. Assume further that $U_\theta = 0$ just outside (or inside) the annulus, i e where $r \to r_0$. This suggests a logical choice for the non-dimensional radius is $R = r/r_0$.

From the symmetry of the current it is clear that, without loss of generality, a polar angle $\theta_0 = 0$ can be selected for the point of entry r_0. Since $\alpha = \phi - \theta$, where ϕ is the angle between $\underline{k}$ and $\underline{e}_1$, $\alpha_0 = (\phi - \theta)_0$, and the angle between $\underline{k}$ and the x-axis at r_0 is then ϕ_0 (see figure 1).



igure 1 : Definition diagram of rays interacting with an annular current.

Equations (3.3) and (3.4), by using (3.2), then become:

$$C = \frac{R \operatorname{cosec} \phi_0}{R \operatorname{cosec} \phi_0 - V} \tag{3.5}$$

$$K = \left[\frac{R \operatorname{cosec} \phi_0 - V}{R \operatorname{cosec} \phi_0}\right]^2 \tag{3.6}$$

$$\sin \alpha = \frac{R \operatorname{cosec} \phi_0}{[R \operatorname{cosec} \phi_0 - V]^2} \tag{3.7}$$

so that, for a current field given by V(R), the wave properties may be found from (3.5), (3.6) and (3.7).

We now also introduce the description that for rays initially outside the annulus, which then penetrate the circular current at $R = 1$, so that $R < 1$ within the annulus where $V \neq 0$, the term convex current (to the direction of wave approach) will be used. Conversely, concave currents have waves that originate inside the annulus before they penetrate the annulus at $R = 1$. R will then become greater than unity within the annulus where $V \neq 0$. Figure 2 is a schematic representation of convex and concave currents as defined above.

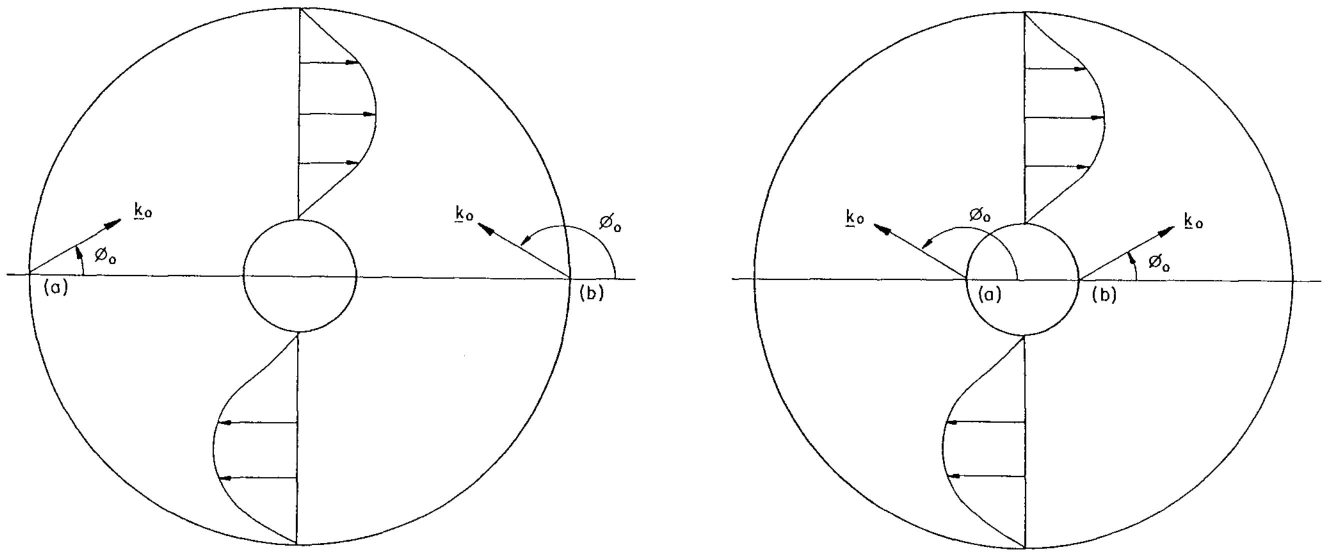
4. SPECIAL SOLUTIONS

It is clear that the right hand side of equation (3.7) can have a magnitude greater than one for a range of V values This defines upper and lower limits to V for which solutions exist.

The critical velocities bounding the region without waves are:

$$V = R \operatorname{cosec} \phi_0 \pm (R \operatorname{cosec} \phi_0)^{\frac{1}{2}} \tag{4.1}$$

At these critical velocities $\alpha = 90^0$ so that the waves travel parallel to the current. In practise the rays are tangent to a caustic curve, concentric with the eddy, and reflection of the rays result. It is important to note that with this model the wave motion along rays is entirely reversible. Note also that "reflection" in this paper imply $\alpha = 90^0$ and $V \neq 0$. The



<u>Convex currents</u>

(a) $R = 1$; $\phi_0 < 90^o$

(b) $R = 1$; $\phi_0 > 90^o$

Inside annulus : $R < 1$.

<u>Concave currents</u>

(a) $R = 1$; $\phi_0 > 90^o$

(b) $R = 1$; $\phi_0 < 90^o$

Inside annulus : $R > 1$.

Figure 2 : Definition diagram for concave and convex currents.

corresponding circular caustic curve is thus at a fixed radius from the origin within the current. For axi-symmetrical wave fields caustics can, however, also occur in the absence of "reflection". This is when the wave rays cross within the core of the annulus and although $\alpha = 90^0$, $V = 0$.

For currents opposing the direction of wave approach two other refraction configurations, at other α angles, may also be identified. "Blocking" is when the component of the group velocity in the direction of $\underline{e}_\theta$, $Cg \sin \alpha$, becomes equal to $|U_\theta|$

$$Cg \sin \alpha + U_\theta = 0 \tag{4.2}$$

and the waves are blocked in the $\underline{e}_\theta$ direction. As in the case of "reflection" a fixed "blocking" radius from the origin may also be identified.

The second, or "stopping", configuration results when the local group velocity of the waves becomes equal and opposite to the convection velocity of the current:

$$Cg + U_\theta \sin \alpha = 0 \tag{4.3}$$

The crests of the waves are refracted to be parallel to the ray direction and the waves are stopped in the $\underline{k}$ direction. As before, a fixed "stopping" radius from the origin may be identified.

By using equation (3.7) the resultant α values may be contoured in the $(R \operatorname{cosec} \phi_0, V)$ plane. Superposition of the current profile, as a function of the radius and the initial angle, $V(R \operatorname{cosec} \phi_0)$, then provides an easy mechanism to study the variation of the waves. For equation (3.7) the contours of figure 3 are obtained. The intersection of the various contours with the $R \operatorname{cosec} \phi_0$ axis then indicate the $\operatorname{cosec} \phi_0$ values of the initial ϕ_0 entry angles, that is where $R = 1$ and $V = 0$. As before, since for concave (convex) currents $R > 1$ $(R < 1)$, the abscissa values in figure 3 will be increasing (decreasing) from the initial $\operatorname{cosec} \phi_0$ value when rays penetrate the annulus from the concave (convex) side.

The $\alpha = 90^0$ contour in figure 3 is of particular interest since it represents the linear caustic curve where reflection of the wave rays take place. Other important contours in figure 3 are those

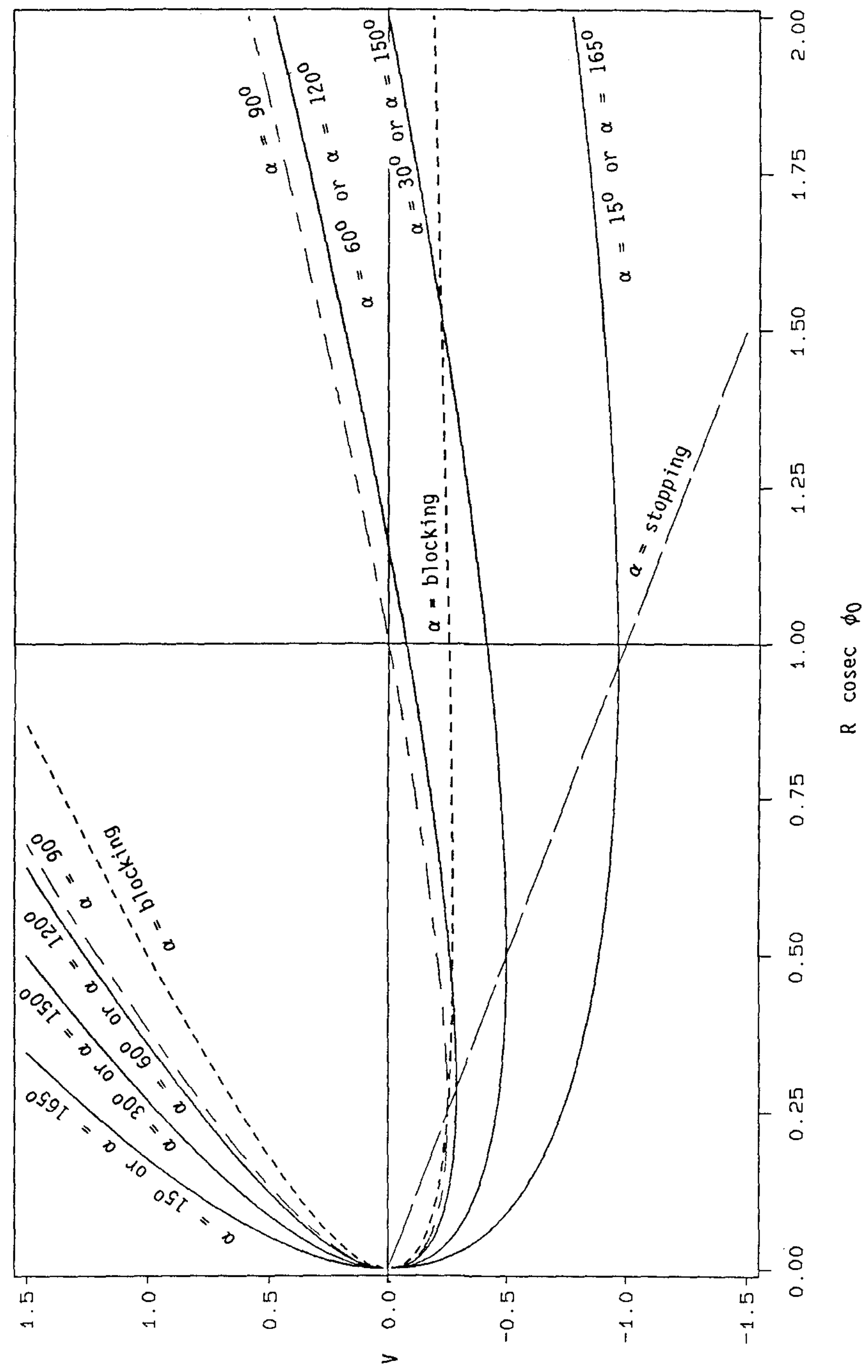


Figure 3 : Contours of α on the (R cosec ϕ_0, V) plane.

which indicate where the rays are blocked or stopped by the current. These contours were obtained from equations (4.2) and (4.3). It is clear that only the lower branch of the blocking contour in figure 3 is relevant. Figure 3 also shows that for waves initially on still water, for both concave and convex currents, the blocking condition will always occur before the stopping condition can be satisfied. Furthermore, in practical application the linear stopping velocity condition will only be satisfied for waves of relatively short period. This is due to the relatively large opposing current values needed for (4.3) to apply.

5. NUMERICAL RAY SOLUTIONS

Waves interacting with a shearing current can, in general, exhibit four different types of behaviour. That is, the waves can (i) penetrate the current, (ii) be reflected by the current, (iii) become blocked by the current or (iv) are stopped by the current. Whereas all four these types of behaviour can be expected from waves opposed to the flow direction of the current, waves that propagate in the same direction as the current can not be blocked or stopped by the current.

The information contained in figure 3 is very useful since it allows us to illuminate the different features of straight, concave and convex shearing currents. For example, for given initial angle of incidence, ϕ_0, different annular current distributions of the form (3.1) may be superimposed on the (R cosec ϕ_0, V) plane and the variation of the waves followed graphically. Various numerical ray simulations of an axi-symmetrical wave field interacting with an annular current of the form (3.1) are shown in figures 5 - 11. The point of entry of the rays in each of these figures is marked by "E" while the maximum current velocity within the annulus is indicated by a dashed line. The wave crests are also shown in some of these figures. For each of these figures the corresponding parabolic current profile is shown in figure 4. In particular instances where the waves are reflected by the current the relevant part of the current profile is indicated by a bold line. Also, the position on the ray where the waves are blocked, reflected and stopped by the current are shown by the filled circles marked "B", "F" and "S" in figures 5 - 11.

Figure 5 shows a family of rays penetrating the following concave annular current marked (a) in figure 4. The initial angle between the ray and the x-axis, ϕ_0, was taken as 45^o, corresponding to R cosec ϕ_0 = 1.41. The maximum value of the parameter V occurs at the dashed centerline-radius of the annulus and for this example V_{max} = 0.28.

Figure 6 is an example of two rays interacting with the opposing convex current marked (b) in figure 4. The initial angle $\phi_0 = 120^o$

so that R cosec ϕ_0 = 1.15. Only the bold part of the profile is relevant since the waves are reflected by the current. The filled circles in figure 6 correspond to the positions where the waves are blocked before being reflected by the current. Since the wave motion along the rays are reversible, the blocking contour is crossed twice (figure 4) before the waves exit the annulus.

It is interesting to note that on a straight opposing current, such as the current marked (c) in figure 4, reflection of the waves is not possible. This is also shown in figure 7. As before, the filled circles indicate the positions where the waves are blocked and stopped by the current. The waves are only stopped at relatively large values of V; in this example V_{max} = -0.96.

Figure 8 is another example of waves interacting with an opposing convex current. Here the dimensionless current velocity, V, is such that the waves are both blocked and stopped before they reflect. The initial angle of incidence is, similar to that of figure 6, taken as $\phi_0 = 120^o$ and the relevant current profile is marked (d) in figure 4. The value of V_{max} = -0.96.

The current profile marked (e) in figure 4 was used to generate the rays in figure 9. This profile is similar to that used in figure 8, except that here the waves approach the opposing current from the concave side.

Waves may also be trapped by an annular current. Waves generated on still water, before interacting with a concave current, may undergo multiple reflections within a certain radius and thus become trapped inside the annulus. Figure 10 is a trapped ray solution corresponding to the current profile marked (f) in figure 4. The angle of initial incidence $\phi_0 = 45^o$ while V_{max} = 0.64.

For this annular configuration, and for waves initially inside the annulus, while propagating in the same direction as the current, it can also be seen from figure 4 that profiles that reach up to the upper branch of the caustic line $\alpha = 90^o$ may have trapped waves on them for chosen initial conditions. Figure 11 is an example of such a single trapped ray. The bold part of the current profile marked (g) in figure 4 corresponds to the ray solution shown in figure 11. It is clear that relatively large V values are needed to trap the waves. In this example V_{max} = 1.4.

6. CONCLUSIONS

Exact linear solutions for the interaction of steady, axi-symmetric deep-water gravity waves and an axi-symmetric annular current have been derived. Two important non-dimensional parameters, namely a current velocity parameter, $V = U_\theta/c_0$, and a radius-angle parameter, R cosec ϕ_0, where $R = r/r_0$, were

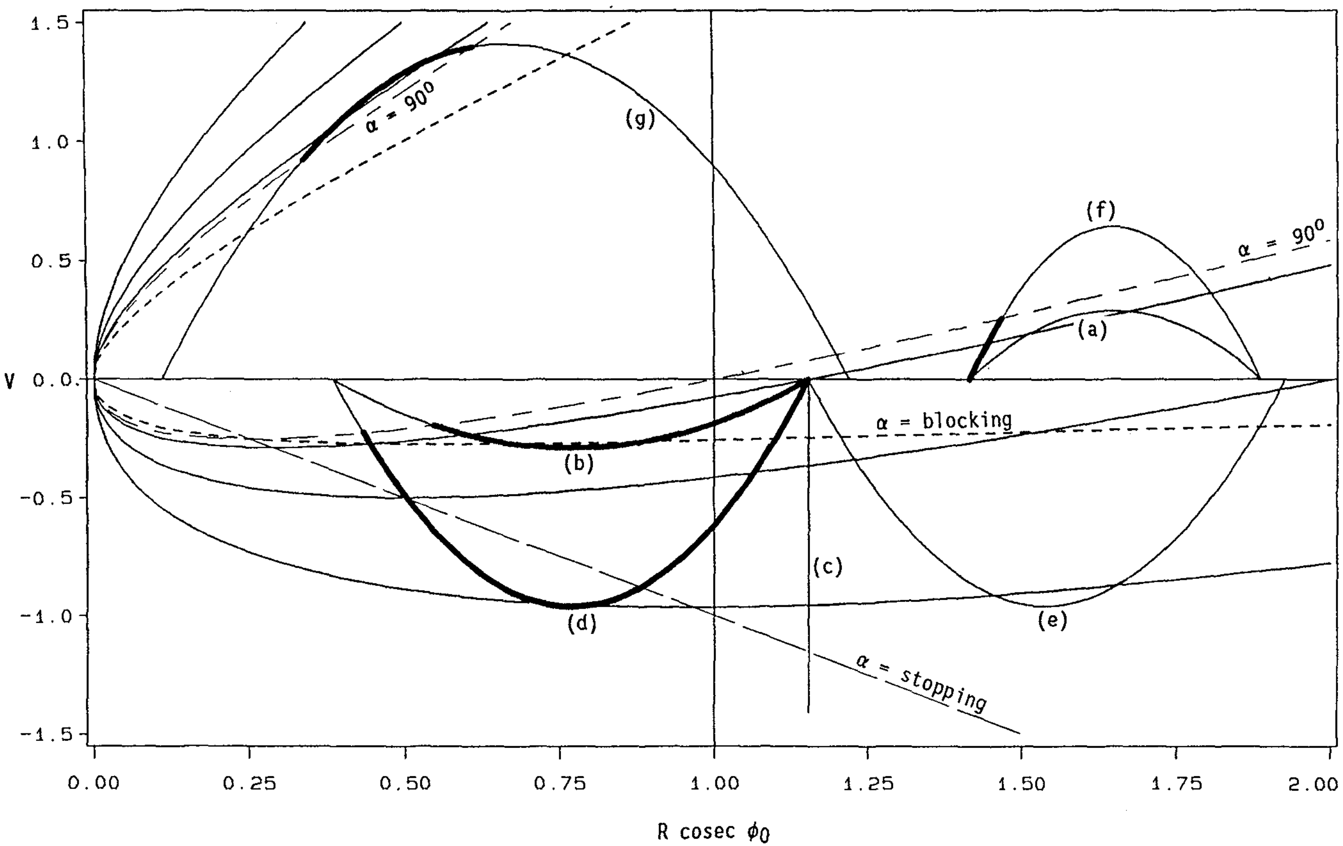


Figure 4 : The annular current configurations used to generate the wave ray examples of figures 5 - 11.

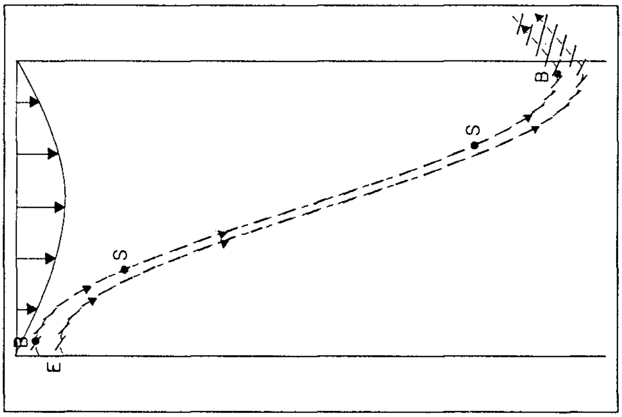


Figure 7 :

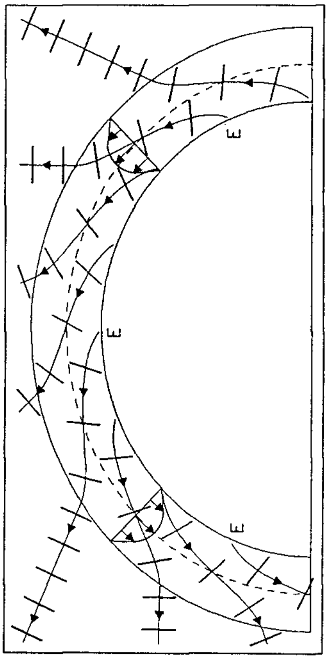


Figure 5 :

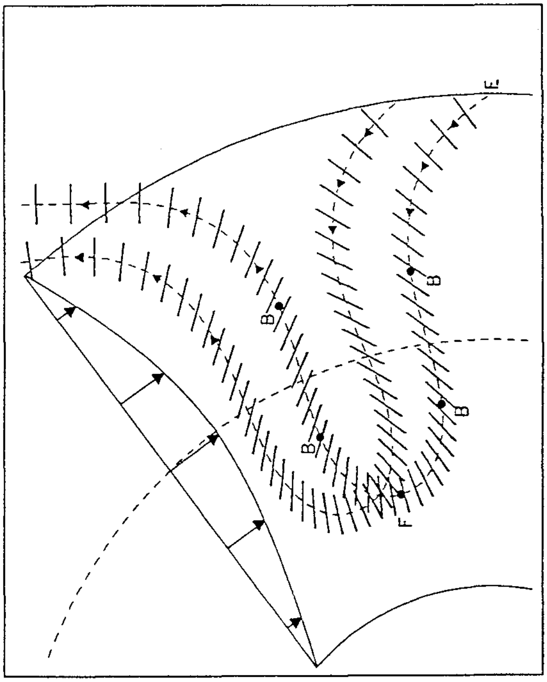


Figure 6 :

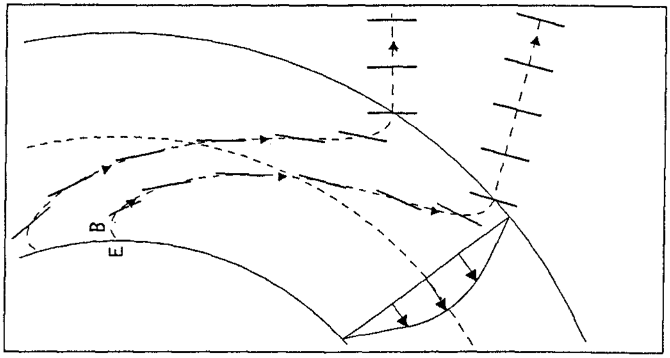


Figure 9 :

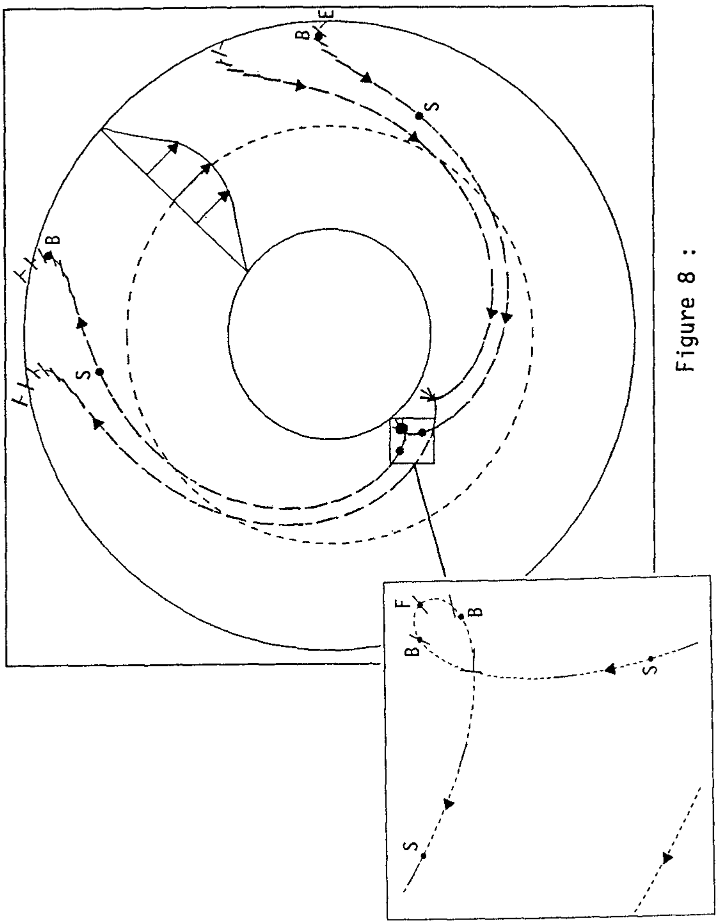


Figure 8 :

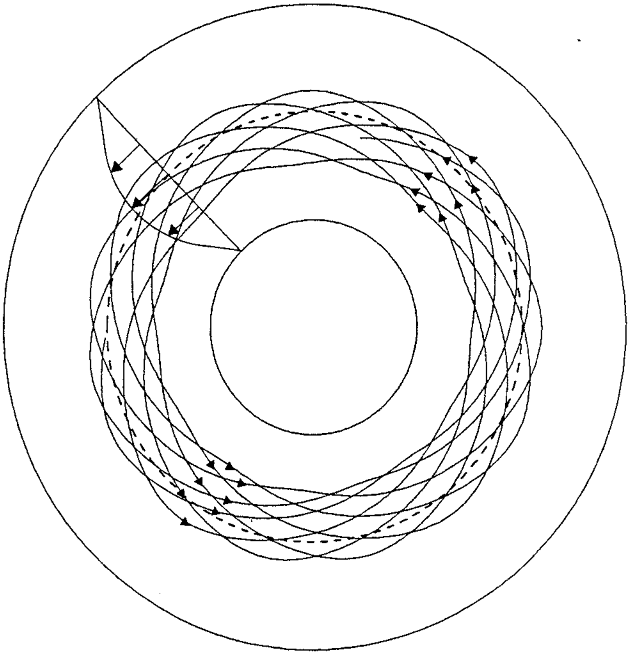


Figure 11 :

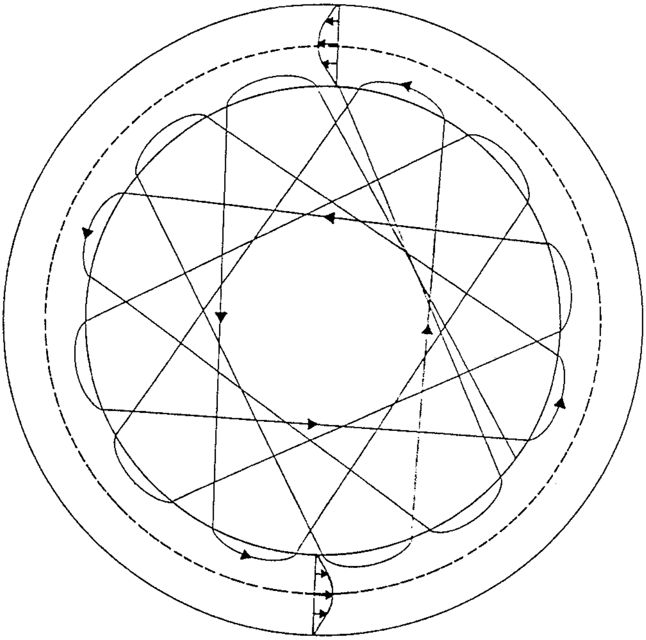


Figure 10 :

identified. For R > 1 concave currents were considered while convex currents have R < 1.

Wave reflections, as well as blocking and stopping of the waves by the current, were investigated. Both positive convex currents and positive concave currents admit reflections, but reflections are only possible for negative convex currents. Reflections may also occur on opposing convex currents before the waves are blocked. On negative concave currents the linear waves may also be stopped by the current. However, very large opposing current velocities are required to do so. Furthermore, reflections on an adverse convex current will occur more frequently than the "stopping" velocity criterion can be satisfied. This is so since large negative values of V are needed to stop the waves.

Wave rays may also be trapped within the boundaries of the current. Waves that are generated on still water inside the annulus, and which penetrate the annulus, while travelling in the same direction as the concave current, may undergo multiple reflections and remain trapped within a certain reflection radius of the current. Only waves generated within the boundaries of the annulus can be trapped so as to remain within the annulus. The theory presented in this paper limits the waves, and therefore also the current distributions, to cases where R cosec ϕ_0 > 1. Figure 4 then shows that it is not possible to construct an adverse current configuration which can trap waves.

7. REFERENCES

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