EXTENSION OF THE BOUSSINESQ EQUATIONS TO INCLUDE WAVE PROPAGATION IN DEEPER WATER AND WAVE-SHIP INTERACTION IN SHALLOW WATER.

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1. INTRODUCTION

Mathematical short wave models based on the Boussinesq equations have been shown to be capable of reproducing the combined effects of most of the wave phenomena of interest to the coastal engineer for a relatively low cost. Today, the following phenomena can be taken into account in the most advanced numerical wave models:

- Shoaling, refraction, diffraction
- Partial reflection from breakwaters
- Irregular wave trains
- Directional wave input
- Wave-wave and wave-current interaction.

In this presentation the standard Boussinesq equations will be extended for two different purposes:

- To simulate irregular wave propagation from deep to shallow water

- To simulate wave-ship interaction in shallow water.

Section 2 will cover the first topic. The depth limitation of the standard forms of the Boussinesq equations will be discussed. It is shown that the worst form of the equations breaks down for depth to deep water wave length ratios (h/L_0) larger than 0.12 while the best forms are limited to say 0.22, corre-

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sponding to a 5% celerity error. In order to improve these limitations a new form of the Boussinesq equations is presented. With the new equations it is possible to simulate the evolution of irregular wave trains propagating from deep water to shallow water. In deep water, the new equations become effectively linear and phase celerities agree with Stokes first-order theory. In more shallow water, the new equations converge towards the standard equations which are known to provide good results for waves up to at least 75% of their breaking height. The combination of a linear wave model in deep water and a non-linear wave model in shallow water is justified by the fact that waves which are non-breaking in shallow water will be only weakly non-linear in deeper water. More details can be found in Madsen et al. (1990).

In the second part of this paper (Section 3) a new solution method for calculating wave-ship interaction in shallow water is presented. The ship motion is represented by six degrees of freedom and roll and pitch angles are assumed to be small, while there is only physical limitations on the other four displacements of the ship. The model which is still under development has the potential of solving the following types of wave-ship interaction problems:

a) Wave induced motion of ships moored in harbours or at unprotected installations, e.g. at single point moorings

b) Wave generation and wave resistance due to ships sailing in calm water or in wave fields.

A general outline of the principles used in the development is given in Section 3 while details concerning theoretical formulations and numerical solution procedures can be found in Madsen (1990).

2. IMPROVEMENT OF THE DEPTH-LIMITATION OF THE BOUSSINESQ EQUATIONS

As discussed by McCowan (1987) a variety of different forms of the Boussinesq equations exists. Firstly the dependent variables can be chosen in different ways, and typical velocity variables are the surface velocity, the bottom velocity, the depth-averaged velocity or the depth-integrated velocity. Secondly the dispersive terms can be manipulated by invoking the long wave equation as a first approximation.

A natural starting point for the derivation of the different forms of Boussinesq equations is the depth-integrated continuity equation and the Bernoulli equation for the surface velocity (see e.g. Witting, 1984). In order to close the equations, a relation between the depth-averaged velocity and the surface velocity is necessary. The classical way to do this is to apply Taylor expansion about the bottom and to express the horizontal velocity in terms of the bottom velocity (see Svendsen, 1974 or Witting, 1984). As shown by Madsen et al. (1990) three different forms of the Boussinesq equations can be derived in this way.

The first form is expressed in terms of the bottom velocity, U_b :

$$S_t + h\, U_{b_x} - \frac{1}{6} h^3\, U_{b_{xxx}} = 0$$

$$U_{b_t} + gS_x - \frac{1}{2} h^2\, U_{b_{xxt}} = 0 \qquad (1)$$

where S is the surface elevation and h the still water depth.

The second form is expressed in terms of the surface velocity, U_s :

$$S_t + h\, U_{s_x} + \frac{1}{3} h^3\, U_{s_{xxx}} = 0$$

$$U_{s_t} + gS_x = 0 \qquad (2)$$

The third form is expressed in terms of the depth-averaged velocity, $\bar{U}$:

$$S_t + h\,\bar{U}_x = 0$$
$$\bar{U}_t + gS_x - \frac{1}{3}h^2\,\bar{U}_{xxt} = 0 \qquad (3)$$

In principle, this is the form used by Abbott et al. (1984) and suggested by Whitham (1973).

The dispersion relation corresponding to a specific form of the equations can be obtained by considering solutions of constant form. It turns out that the resulting phase celerities can be expressed by

$$\frac{c^2}{gh} = \frac{1 + Bk^2h^2}{1 + (B + \frac{1}{3})k^2h^2} \qquad (4)$$

where

$$B = \begin{cases} 1/6 & \text{for Eq. (1)} \\ -1/3 & \text{for Eq. (2)} \\ 0 & \text{for Eq. (3)} \end{cases}$$

Notice that each form of the equations leads to a different celerity expression, but for small wave numbers all the derived expressions converge towards the Stokes first-order theory for waves on arbitrary depth, which for this purpose will be considered as the exact solution. As the wave number increases, the various celerity expressions become more and more inaccurate relative to the Stokes theory.

The percentage errors of the phase celerities are shown in FIG 1 as a function of h/L_0 : Firstly, Eq. (1) appears to have the poorest phase properties, and for $h/L_0 > 0.12$ solutions to the dispersion relation cannot be found. As remarked by McCowan (1987), this corresponds to the depth limitation for cnoidal wave theory. Secondly, Eq. (3) has the best phase properties of the standard Boussinesq equations. This is the form recommended by Whitham (1973) and applied by most of the existing numerical models today. The absolute water depth limit beyond which solutions to the dispersion relation cannot be found is $h/L_0 = 0.48$. However, in order to restrict phase errors to, say, 5% the practical upper limit for h/L_0 reduces to 0.22, which corresponds to the water depth limit determined numerically by McCowan (1981).

For many applications, a less restrictive water depth limitation is desirable. This requirement of an improved linear dispersion property in deeper water was addressed by Witting (1984), who presented a new set of equations valid for a single horizontal dimension. As shown by Madsen et al. (1990) the linear reduction of Witting's equations leads to a phase celerity on the form of Eq. (4) with B = 1/15. This leads to a significant improvement of the depth-limitation, and for h/L_0 as large as 0.50 celerity errors are still restricted to approximately 5% (FIG 1).

The group velocities corresponding to the various forms of the Boussinesq equations can easily be derived from Eq. (4), which yields

$$c_g = c\left[1 + \frac{Bk^2h^2}{1 + Bk^2h^2} - \frac{(B + \frac{1}{3})k^2h^2}{1 + (B + \frac{1}{3})k^2h^2}\right] \qquad (5)$$

The percentage errors compared to Stokes first-order theory are shown in FIG 2. The errors are surprisingly large even for relatively shallow water. By restricting the percentage errors to, say, 6% the practical water depth limitations become

$$h/L_o = \begin{cases} 0.055 & \text{for } B = -1/3 \\ 0.12 & \text{for } B = \ \ 1/6 \\ 0.13 & \text{for } B = \ \ 0 \\ 0.32 & \text{for } B = \ \ 1/15 \end{cases}$$

Again Witting's method is superior to the standard forms of the Boussinesq equations. Unfortunately, it turns out to be very difficult to generalize Witting's approach to two horizontal dimensions. Instead we have formulated a new set of Boussinesq equations which meet the following requirements:

a) The equations should be expressed in two-horizontal dimensions in terms of the surface elevation and the depth-integrated velocity components.

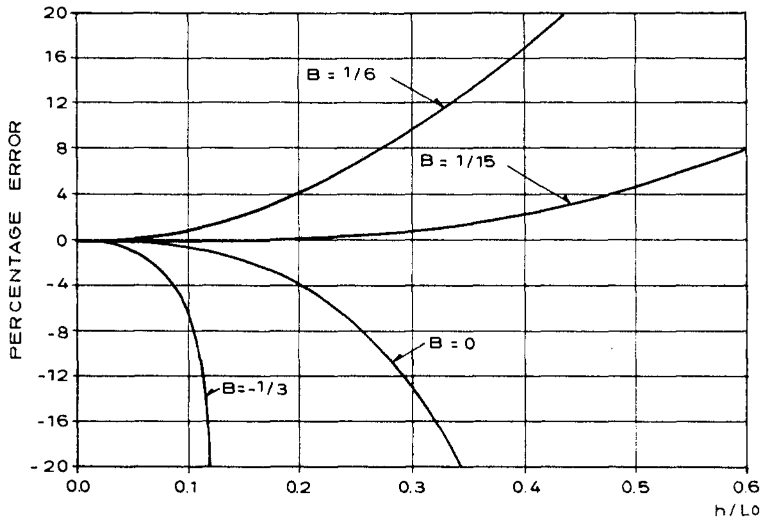


Fig. 1 Percentage error of the phase celerity.

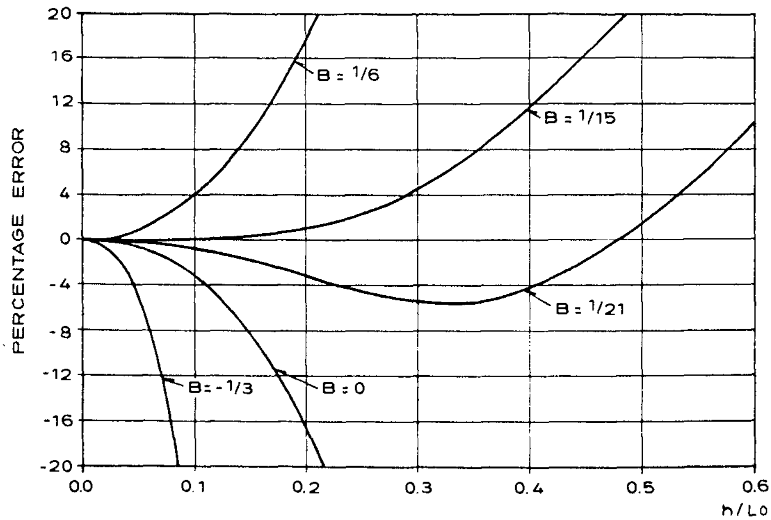


Fig. 2 Percentage error of the group velocity.

b) The resulting linear dispersion characteristics should follow Eq. (4) where the coefficient B can be chosen explicitly to improve accuracy in deeper water.

The following equations meet these requirements:

$$S_t + P_x + Q_y = 0 \tag{6a}$$

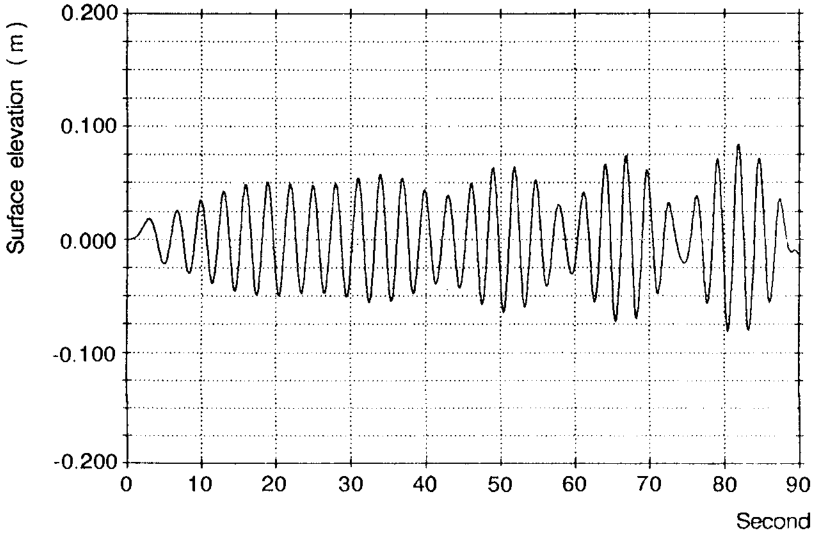
$$P_t + \left(\frac{P^2}{d}\right)_x + \left(\frac{PQ}{d}\right)_y + gdS_x - (B+\tfrac{1}{3})\ h^2\left(P_{xxt} + Q_{xyt}\right)$$
$$- Bgh^3\left(S_{xxx} + S_{xyy}\right) = 0 \tag{6b}$$

$$Q_t + \left(\frac{Q^2}{d}\right)_y + \left(\frac{PQ}{d}\right)_x + gdS_y - (B+\tfrac{1}{3})\ h^2\left(Q_{yyt} + P_{xyt}\right)$$
$$- Bgh^3\left(S_{yyy} + S_{yxx}\right) = 0 \tag{6c}$$

where d is the total water depth, h is the still water depth, and P and Q are the depth-integrated velocity components (m^2/s). Notice that in extremely shallow water, where the long wave equations are valid as a first approximation, the new equations will converge towards the standard Boussinesq equations.

FIG 3 shows the effect of applying the new equations. The bichromatic wave considered is comprised by a 2.5 s wave (h/L_o = 0.43) and a 3.0 s wave (h/L_o = 0.30) each having an amplitude of 0.05 m. In the standard model (i.e. with B = 0) the group celerity errors are -90% for the 2.5 s wave and -44% for the 3.0 s wave. Hence, especially the 2.5 s wave is slowed down so much that the resulting time series taken at a position only 12 m down the channel almost looks like a monochromatic wave, at least until the 2.5 s wave eventually reaches the point and re-establishes the bichromatic wave pattern. In the new model with the coefficient B = 1/21, the wave train travels down the channel almost undisturbed.

More details, especially concerning the numerical solution procedure, are given by Madsen et al. (1990).



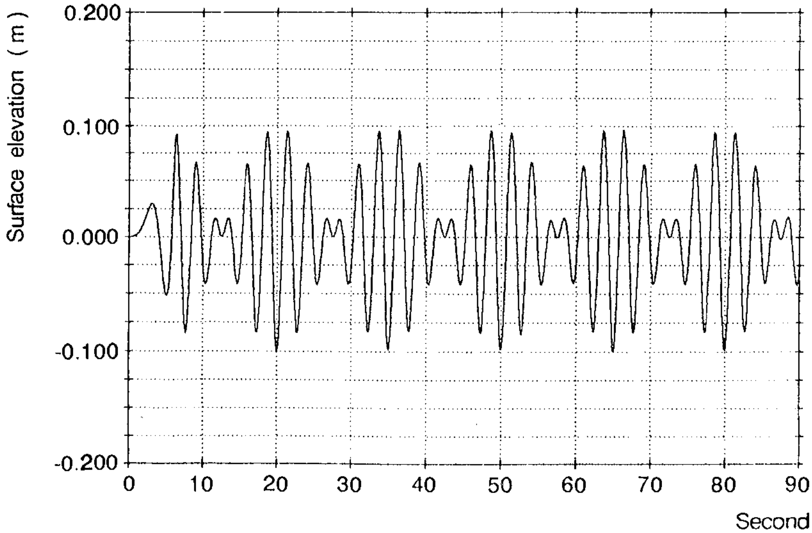


Fig. 3 Bichromatic wave test
a) Standard Boussinesq model (i.e. $B = 0$)
b) New model ($B = 1/21$)

3. SIMULATION OF WAVE-SHIP INTERACTION IN SHALLOW WATER

The second topic in this paper represents a natural step forward in the development of the Boussinesq type models by including the interaction between waves and moored or sailing ships in shallow water. Details concerning the theoretical formulations and the numerical solution procedure will not be given here but can be found in Madsen (1990). Instead a general outline of the general principles will be given below.

Depth-integrated flow equations in the time-domain are formulated to describe the free surface flow beside the ship as well as the pressurized flow under the ship. A slot-technique is used to combine the two flow regimes into a single set of equations expressed in terms of the depth-integrated velocity components and the local pressure height. This technique is similar to the one used by Preissmann and Cunge (1961) for modelling pressurized flow in conduits. The excess pressure, depth-averaged over the underkeel clearance, is used to determine the hydrodynamic forces and moments on the ship. This approximation is valid in shallow water and especially if the underkeel clearance is small. The local wave generation due to the moving ship is taken into account by distributing sources on the ship hull and including the effect in the continuity equation. The ship motion is represented by six degrees of freedom using one coordinate system, which is fixed relative to the moving ship and one, which is fixed relative to the surrounding bathymetry. Both systems are rectangular but the grid spacing can be different. First of all, the ship is described in the ship coordinate system by defining two-dimensional maps of ship draft and ship slot factors. A slot factor of 0.01 will indicate solid ship body, while a factor of 1.0 indicates an open water point. Normal vectors defining the ship elements will be determined in each ship grid point. When the ship moves around in the harbour coordinate system, the ship slot factors in this system will change from time step to time step indicating the new position of the ship. A simple interpolation routine is used to determine the successive ship configurations in the harbour coordinate system during the motion. This requires the roll and pitch angles to be small, while there is only physical limitations on the other four displacements of the ship.

The depth-integrated formulation makes the model very efficient and even though it is solved directly in the time-domain the computer economy is quite reasonable: The speed of the model is approximately 1200-1500 points/s on an IBM 4381 depending on the ratio of ship points to clear water points. A typical grid size will be from 1 to 5 m, depending on the beam of the ship, and a typical time step will be 0.1 to 0.2 s.

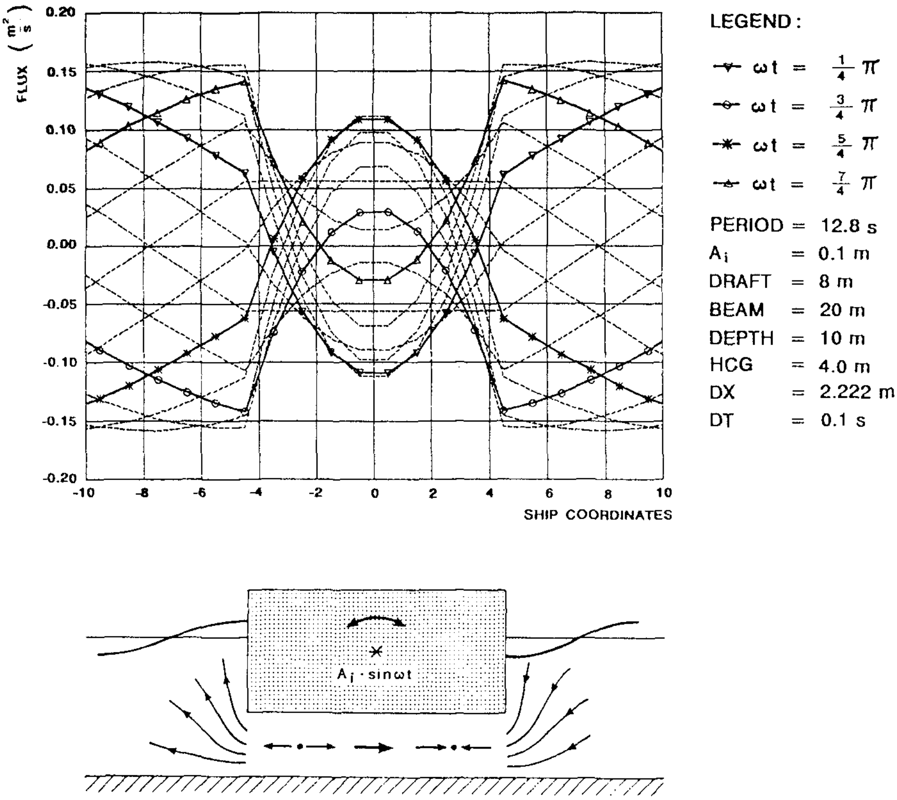


Fig. 4 Forced roll motion: Flux envelope.

One of the key problems in the model development has been the internal wave generation due to the motion of the ship. The resulting pressure and velocity distribution on the ship hull has to be accurate in order to take added mass and wave damping effects proper into account. In order to test this the forced harmonic motion of a box shaped ship has been considered. Simple analytical solutions for this problem were derived by Svendsen & Madsen (1981) and Madsen et al. (1980) and they concluded that

a) The forced sway motion leads to a linear pressure distribution and a constant flux distribution in space.

b) The forced heave motion leads to a parabolic pressure distribution and a linear flux distribution in space.

c) The forced roll motion leads to a cubic pressure distribution and a parabolic flux distribution in space.

The internal wave generation in the Boussinesq model has been verified against these analytical distributions. As an example the simulated flux envelope for forced roll motion is shown in FIG 4 comprising of sixteen instantaneous line plots covering one period in the forced motion.

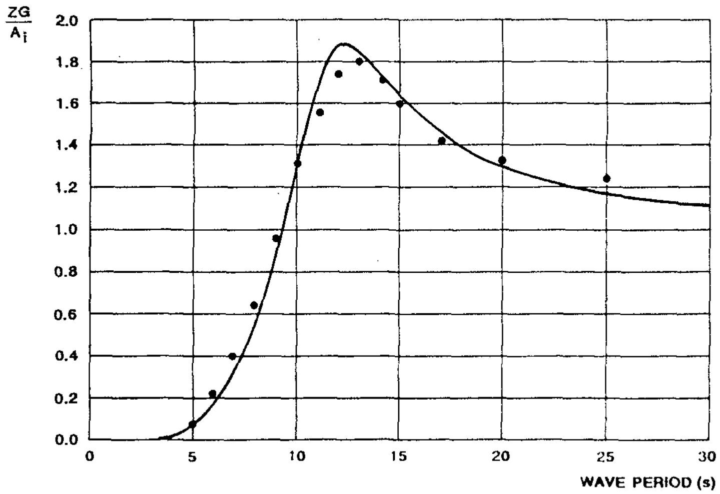


Fig. 5 Response curve for wave induced heave.
Draft = 8.0 m, Beam = 20 m, Depth = 10 m.
- Theory by Svendsen & Madsen (1981)
• Boussinesq model.

An example of wave induced motion of a moored ship is shown in Fig. 5 where the response curve for heave motion has been compared with the analytical solution by Svendsen and Madsen (1981).

Finally, Figs, 6, 7 and 8 show examples of perspective plots of the wave field surrounding a moored ship (Fig. 6), the wave field generated by yaw extinction (Fig. 7) and the wave field generated by a sailing ship which is accelerating from rest to a maximum speed of 20 knots in 14 m of water depth.

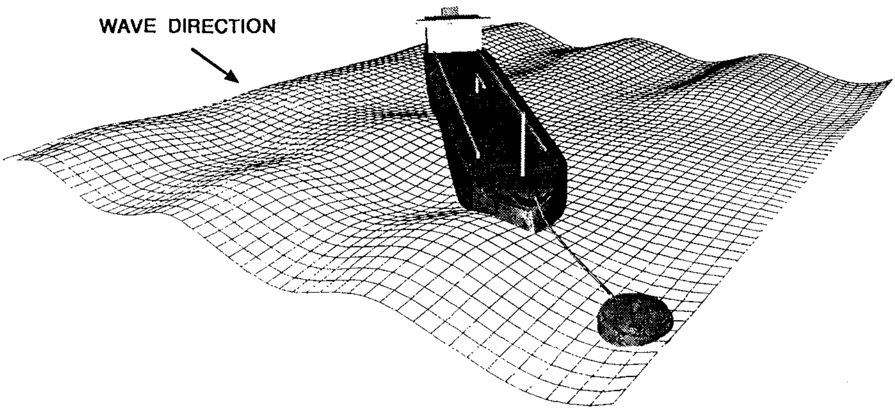


Fig. 6 Moored ship in cross waves.

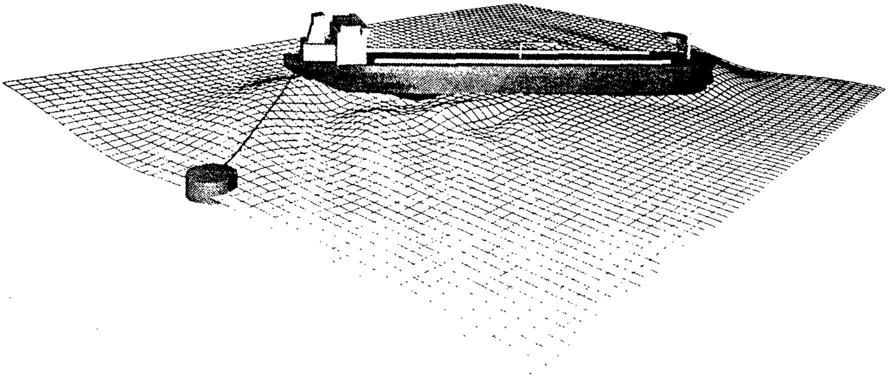


Fig. 7 Yaw Extinction test.

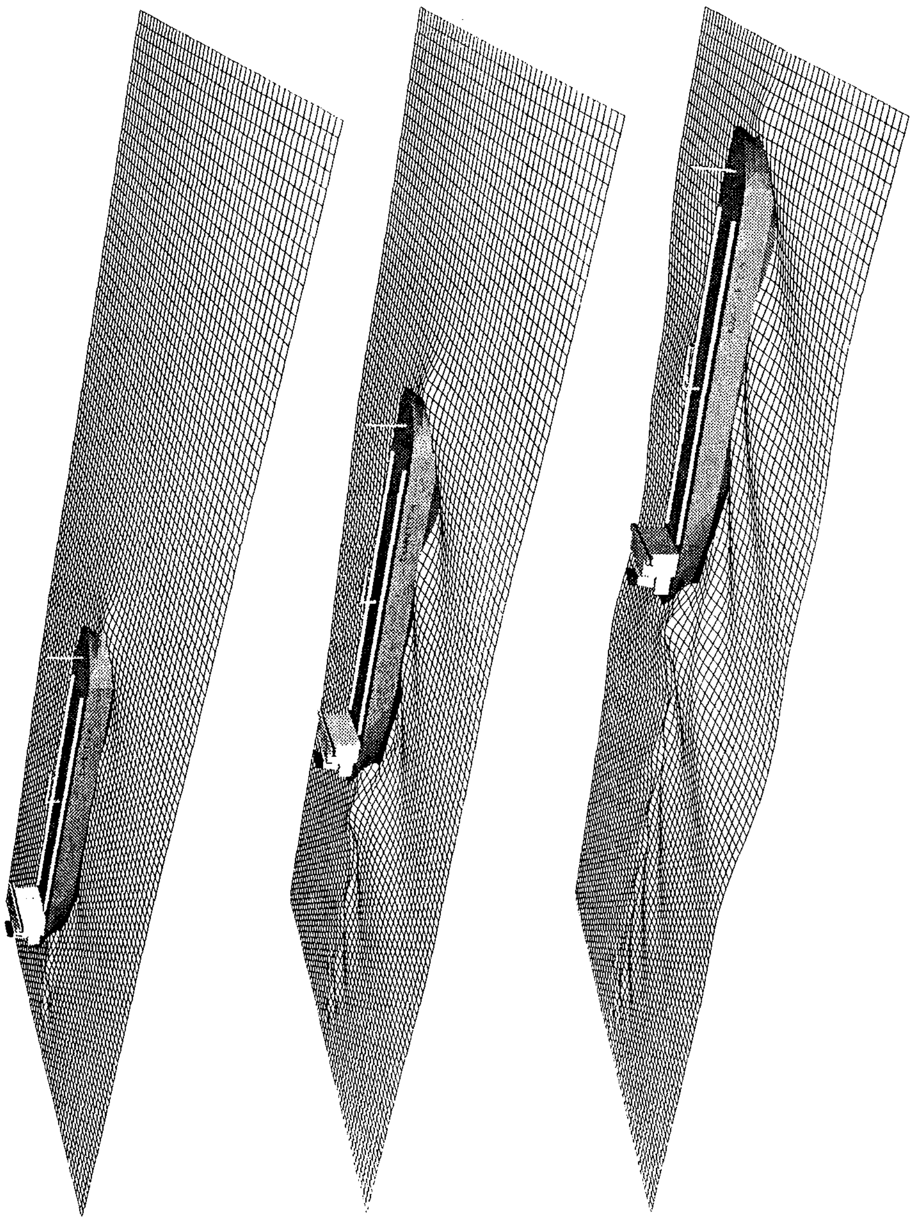


Fig. 8 Sailing ship in calm water.

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