SEALEVEL RISE : A PROBABILISTIC DESIGN PROBLEM

prof. ir. drs. J.K. Vrijling [1]
ir. I.J.C.A. Van Beurden

Abstract

An economic model is developed to calculate the optimal height of sea defences in case of sealevel rise. The optimal amount and the optimal period of heightening are found, for the single period as well as for the multi-period case. Finally the optimal strategy of heightening of sea defences in case of an uncertain sealevel rise is formulated.

1.0 Introduction

The relative sealevel rise is a well-known fact. In the period from 1682 to 1930, a rise has been measured at the official benchmarks in Amsterdam. The velocity of the rise however appeared to be a function of time. Initially the velocity was 0.04 m per century but after 1850, the sealevel rise increased up to 0.17 m per century. The Delta Committee therefore chose a value of 0.20 m per century for this phenomenon.
During the last 20 years, measurements have confirmed this estimate.

However recently experts have come to doubt this figure. In studies several causes are mentioned that may increase the velocity of the sealevel rise :

- melting of the polar ice pack.
- thermal expansion of the oceans.
- tectonic movement of the crust of the earth.
- settlement of alluvial soils due to dewatering.

Estimates of the future sealevel rise based on these ranging from 0.20 to 1.20 m per century are mentioned.

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1) Professor hydraulic engineering,
T.U Delft, Stevinweg 1, 2628 CN, Delft

facts differ considerably. For the year 2050 values However these extreme values have not yet been confirmed by observations.

If the sealevel starts to rise, it will pose a significant threat to the low lying countries. The question is when and to what extent should these countries strengthen their sea defences, considering the uncertainty in the relative sealevel rise.

2.0 An economic model

Although the question of the strengthening of sea defences allowing for sealevel rise is multi-facetted, it is readily schematised to an economic decision problem. Solutions for planning period and height are found by minimising the total costs TC consisting of the investment in heightening of the sea defences I and the present value of the expected loss in case of inundation R :

$$TC = I + R \tag{1}$$

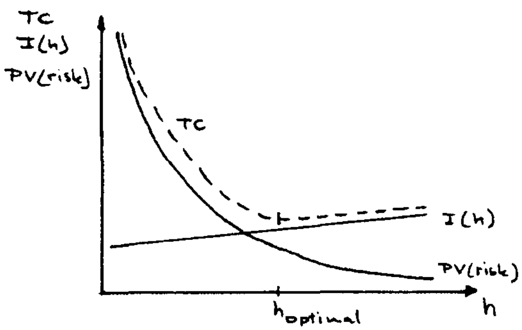

figure 1
Optimalisation of the height of a sea defence.

In this paper the problem as stated by Van Dantzig[1960] is used as a starting point for the analysis.

2.1 Base case

To allow for sealevel rise, it will be necessary to heighten the sea defence after a certain period. If only one planning period is considered the investment I is schematised by :

$$I = I_0 + I_1(H - H_0) \tag{2}$$

where I_0 = initial cost of heightening [DFl]
I_1 = cost per meter heightening [DFl]
H = height after heightening [m]
H_0 = initial height [m]

The present value of the expected loss in case of inundation R is given by :

$$R = \int_0^{T_p} W P_f(H)\, e^{-rT} dt = \frac{W P_f(H)}{\gamma}\left(1 - e^{-\gamma T_p}\right) \tag{3}$$

where W = damage by inundation [DFl]
T_p = planning period [year]
$P_f(H)$ = probability of failure, schematised by :

$$P_f(H) = e^{-\frac{H-\eta t-A)}{B}} \tag{4}$$

η = sealevel rise [m/year]
A, B = parameters of the distribution [m]
γ = $r - \eta/B$
r = real rate of interest [/year]

It is assumed that the parameters A en B of the probability of failure, are not affected by the sealevel rise.
The optimal probability of failure can be found by solving :

$$\frac{\partial TC(H, T_p)}{\partial H} = 0 \tag{5}$$

which yields :

$$P_{f_{opt}}(T_p) = e^{-\frac{H_{opt}-A}{B}} = \frac{I_1 B}{W} \frac{\gamma}{1 - e^{-\gamma T_p}} \tag{6}$$

In the following calculations values for the various parameters according to Van Dantzig have been used :

I_0	= $40 \cdot 10^6$	[DFl]	I_1	= $1 \cdot 10^6$	[DFl]
H_0	= 3.00	[m]	W	= $24 \cdot 10^9$	[DFl]
A	= 1.96	[m]	B	= 0.33	[m]
r	= 0.015	[/year]			

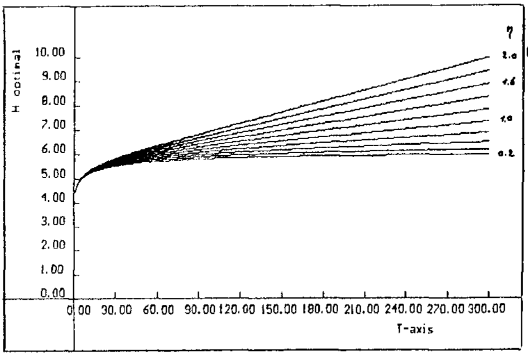

figure 2
Optimal height of
for single period.

In figure 2 the optimal height derived from $P_{f.opt}(T)$ is shown.
The additional heightening $(H - H_{(\eta=0)})$ of the sea defence to allow for sealevel rise can also be expressed as a fraction of the occurring rise during the planning period ($\cdot T_p$). This is shown in figure 3. From this figure it can be seen that to allow for sealevel rise not the total occurring rise during the planning period has to be added to the height of the sea defence as should be expected, but only a part of it.

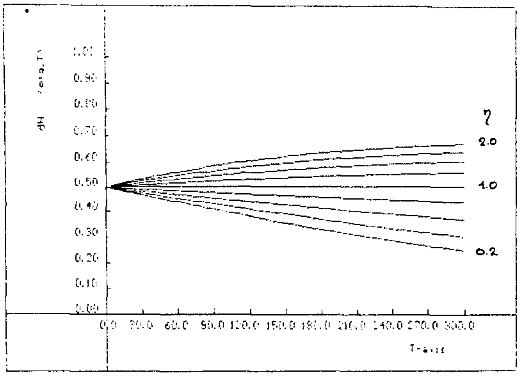

figure 3
Additional heightening to allow for sealevel rise as fraction of occurring sealevel rise

Additionally the condition for an economical solution is $TC_0 >= TC$:

$$P_f(H_0) \ge \frac{\gamma}{W}\frac{(I_0+I_1(H-H_0))}{1-e^{-\gamma T_p}} + P_f(H) \tag{7}$$

Substitution of the optimal value of the height given by equation (6) yields :

$$P_f(H_0) \geq \frac{I_0+I_1(H-H_0+B)}{1-e^{-\gamma T_p}} \frac{\gamma}{W} \tag{8}$$

2.2 Regular heightening

For a consistent solution of the optimal strategy however, not only the first heightening of the sea defence has to be taken into account, but also the subsequent future adaptations. In this case terms have to be added to I and R for future investments and risks. If all following planning periods are considered, this yields :

$$I = I_0+I_1(H-H_0)+(I_0+I_1\eta T_p)\frac{1}{e^{rT_p}-1} \tag{9}$$

$$R = \frac{W}{\gamma} P_f(H) \frac{1-e^{-\gamma T_p}}{1-e^{-rT_p}} \tag{10}$$

The condition for an economical solution can be found by substitution of $T_p = \infty$ in equation (8) :

$$P_f(H_0) \geq (I_0+I_1(H-H_0+B))\frac{\gamma}{W} \tag{11}$$

Optimum height

By solving equation (5), the optimal value for the probability of failure as a function of the planning period is found :

$$P_{f_{opt}} = \gamma \frac{BI_1}{W} \frac{1-e^{-rT_p}}{1-e^{-\gamma T_p}} \tag{12}$$

In figure 4 the optimal height computed from $P_{f.opt}$ and the optimal minimum height given by $H_{min} = H_{opt} - \eta T$ are shown for several values of η. From this figure it can be seen that for higher sealevel rises and longer planning periods, the optimal height increases and the optimal minimum height decreases. In figure 5 the relative extra-heightening to allow for sealevel rise is shown.
The allowance factor is in the order of 0.5. Only for extreme values of the sealevel rise and long planning periods this factor is significantly exceeded.

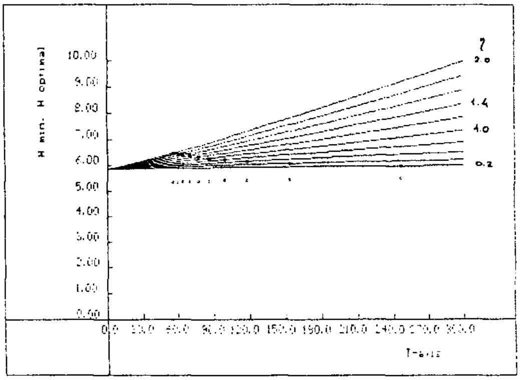

figure 4
Optimal height and
optimal H_{min}

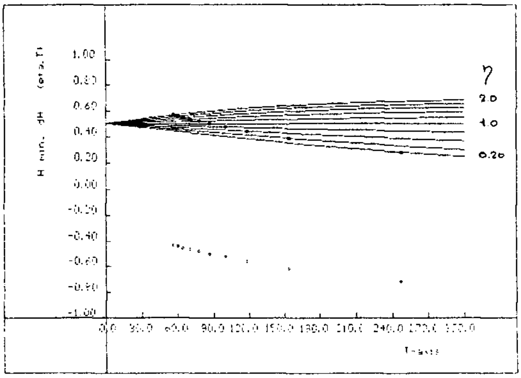

figure 5
Additional heightening
as fraction of occurring sealevel rise
(regular heightening)

Optimum planning period

Additionally total cost can be optimized by differentiating TC with respect to the planning period. Algebraically a solution cannot be found, however for the examples above, values for the optimal planning period $T_{p.opt}$ have been found numerically as shown by dots in figure 4 and figure 5.

In figure 6 total cost TC is shown as a function of the planning period. Early non-optimal adaptations appear to be more expensive than later non-optimal adaptations, especially for less severe sealevel rises.

As a result of the optimal height and the optimal planning period, the crest of the sea defence should be kept in a band between the optimal design level and a certain minimum. In figure 7 is shown an optimal scenario.

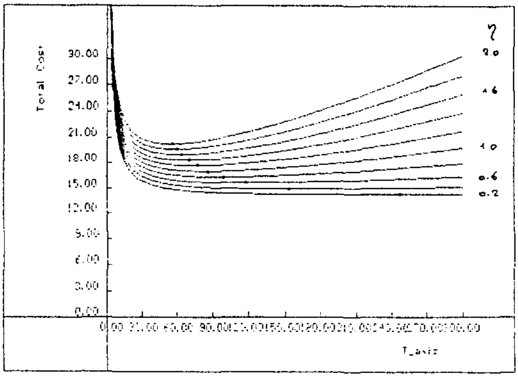

figure 6
Total cost as function of the planning period.

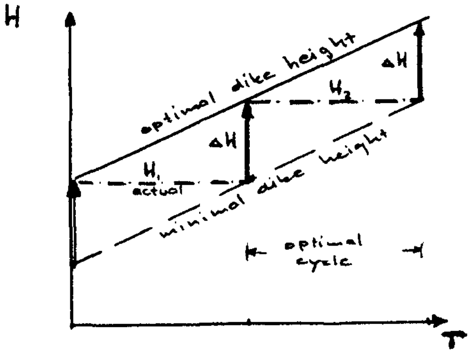

figure 7
Optimal scenario for heightening.

2.3 Optimum waiting period

If the sea defence already has an non-optimal height or if some of the parameters change, the question arises, when and to what extent the sea defences should be heightened. To solve this problem, equations (9) and (10) have to be changed into:

$$I = [I_0+I_1(H+\eta T_w-H_0)+(I_0+I_1\eta T_{opt})\frac{1}{e^{rT_{opt}}-1}]\,e^{-rT_w} \tag{13}$$

$$R = \frac{W}{\gamma}P_f(H_0)\,(1-e^{-rT_w})+\frac{W}{\gamma}P_f(H) \tag{14}$$

where T_w = waiting period

Again the optimal height can be solved by differentiating TC with respect to H. This leads to the same result as given before.
The optimal waiting period has not been solved

algebraically, but numerical solutions give the impression that a simple solution of the following form exists :

$$T_w = T_{p,opt} - \frac{H_{opt} - H_0}{\eta} \qquad (15)$$

Optimal scenarios based on this equation are shown in figure 8. As for severe sealevel rises, optimal heights are higher and minimal heights are lower, in case of sudden increase of the sealevel rise immediate action will not be necessary. Only when the lower bound of the new regime is reached the sea defence should be heightened.

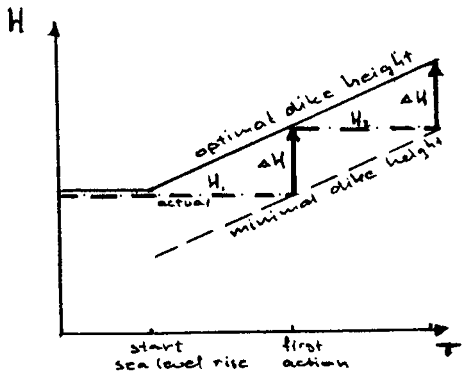

figure 8
Optimal action

2.4 Optimal design sealevel rise

So far the optimal scenario has been calculated for a certain known value of the sealevel rise. However as mentioned above, this value is uncertain. The occurring sealevel rise may vary from 0.20 to 1.20 meter per century according to various estimates. It will be clear that this will affect the total cost of the regular heightening.
In the following calculations the actual value of the occurring sealevel rise is supposed to be known at the moment the sea defences need to be heightened due to this sealevel rise.

Total cost in the case of a different assumed sealevel rise and occurring sealevel rise is given by :

$$TC(\eta_a, \eta_o) = I(\eta_a, \eta_o) + R(\eta_a, \eta_o) \qquad (16)$$

where :

$$I(\eta_a,\eta_o) = I_0 + I_1(H_{opt_{\eta_a}} - H_0) + (I_0 + I_1\eta_o T_{opt_{\eta_o}}) \frac{1}{1-e^{-rT_{opt_{\eta_o}}}} e^{-rT_w} \quad (17)$$

$$R(\eta_a,\eta_o) = \frac{W}{\gamma}(P_f(H_{opt_{\eta_a}})(1-e^{-\gamma T_w}) + P_f(H_{opt_{\eta_o}}) \frac{1-e^{-\gamma T_{opt_{\eta_o}}}}{1-e^{-rT_{opt_{\eta_o}}}} e^{-rT_w}) \quad (18)$$

T_w is can be calculated using equation (15) :

$$T_w = T_{opt_{\eta_o}} - \frac{H_{opt_{\eta_o}} - H_{opt_{\eta_a}}}{\eta_o} \quad (19)$$

In figure 9 the extra total cost due to different values for the design and occurring sealevel rise is shown for a design sealevel rise of 20 cm per century. As can be expected, the extra cost is higher for lower occurring sealevel rises.

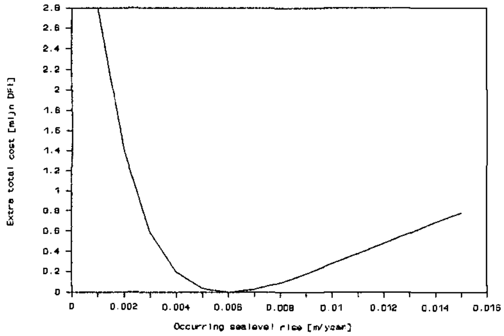

figure 9
Extra total cost for
η_d = 0.006 m/year

Assuming a certain probability density function for the occurring sealevel rise $f(\eta_0)$ the expected total cost for an assumed value of the sealevel rise can be calculated :

$$TC_{unc}(\eta_a) = \int_0^\infty TC(\eta_a,\eta) f(\eta) d\eta \quad (20)$$

In case of a discrete probability density function this equation can be written as:

$$TC_{unc}(\eta_a) = \sum_{i=1}^{N} TC(\eta_a,\eta_i) p(\eta_i) \quad (21)$$

The optimal design sealevel rise η_d can be found by minimising $TC_{unc}(\eta_d)$.
Calculations have been made for three probability density functions (figure 10) :

1) : Uniform distribution
2) : Normal distribution :
3) : Gumbel distribution :

All distributions having $\mu = 0.006$ and $\sigma = 0.002$ m/year.

The value of the expected total cost as a function of the sealevel rises is shown in figure 11.

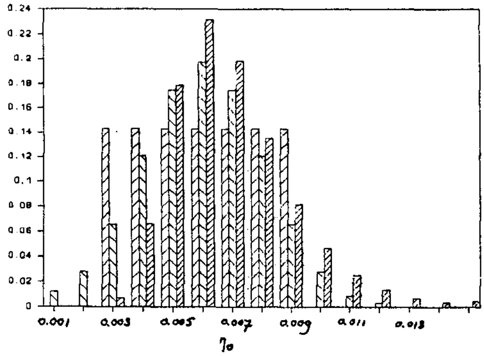

figure 10
The probability density functions of sealevel rise

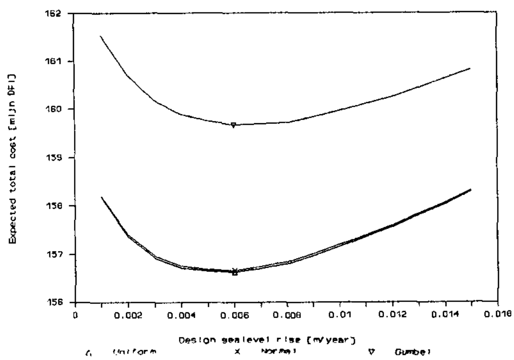

figure 11
Expected value of the total cost.

The minimal value of the expected cost is approximately found at the average value of the sealevel rise.
Further it is observed that the cost is less sensitive for an over-estimate of the occurring sealevel rise than for an under-estimate. It seems therefor advisable to choose a safe value for the design sealevel rise, if the sea defences have to be heightened.

3.0 Conclusions

The question when and to what extent the sea defences of low-lying countries should be strengthened in case of an increasing sealevel rise can be answered by an economic probabilistic model.

To allow for sealevel rise not the occurring sealevel rise has to be added to the height of the sea defence but only a part of it. This part is depending on the several parameters in the order of 0.5. Only in case of very high values of the sealevel rise this factor is significantly exceeded and approaches 0.7.

If the sea defences are well maintained, immediate action is not advised in case of an increase of sealevel rise. Action should be taken when the lower bound of the new regime as defined in this paper is reached.

In case of action under uncertainty of the future occurring sealevel rise, do not under-estimate this value, as the total cost of a sea defence that is constructed to low, exceeds the cost of a sea defence constructed sufficiently high.

Appendix 1 : Reference

Van Dantzig (1960) The economic decision problem concerning the security of the Netherlands against storm surges.

Appendix 2 : Mathematical tools

The following formulas have been used in this paper :

Present value of a future amount A_T

$$PV(A_T) = A_T\, e^{-rT} \tag{22}$$

Present value of a flow F

$$PV(F) = \int_0^T F e^{-rt} dt = \frac{F}{r}\,(1- e^{-rT}) \tag{23}$$

Present value of a future flow F_f

$$PV(F_T) = \int_T^{T+\Delta T} F_T e^{-rt} dt = \frac{F_T}{r} e^{-rT}(1- e^{-r\Delta T}) \tag{24}$$

Present value of a returning future cost

$$PV(A,T) = \sum_{n=0}^{\infty} A\, e^{-rnT} = A\,\frac{1}{1- e^{-rT}} \tag{25}$$