CHAPTER 52

SENSITIVITY ANALYSIS FOR MULTI-ELEMENT WAVEMAKERS

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1 Abstract

Lateral variations in wave height have been observed in directional wave basins when generating monochromatic plane waves. Two numerical schemes are presented to examine wave height sensitivity to perturbations in stroke length between paddles and gaps between either adjacent paddles or the sidewall of the basin. A non-uniform wavemaker front generates both the desired wave and can generate waves at an angle and at the same frequency as the main wave.

2 Introduction

Simulation of realistic sea states requires the use of directional wave basins which can produce directional spectra. These basins generally consist of numerous wavemakers along one side of the tank and either reflecting or absorbing walls for the other sides. Before realizing the full potential of these tanks one must first understand the production of uniform monochromatic waves by a multi-element wavemaker system and the errors inherent in any possible nonuniform wavemaker front.

A numerical procedure is developed to examine wave height sensitivity to gaps which may exist between paddles and variations in paddle stroke length in a wave basin with reflecting

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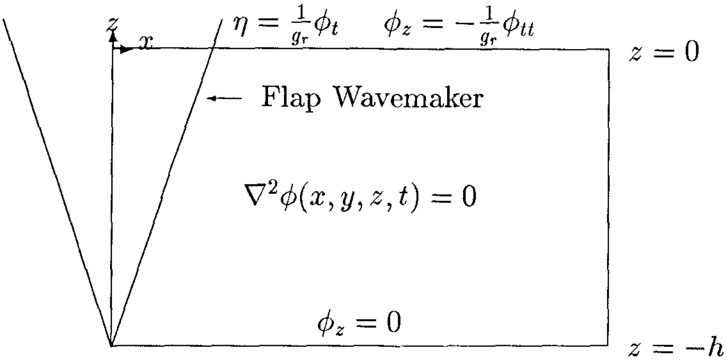

Figure 1: Boundary value problem specifications for both numerical models

sidewalls. To study variations in paddle motion for a multi-element wavemaker system a technique was developed to numerically model individual paddles. A sinusoidally varying "top hat function" consisting of the difference of two Heaviside functions is used to represent the paddle motion. This allows each paddle to have its own stroke length. For small amplitude waves the total wave height at a particular location in the tank can be found from the summation of waves produced by each individual paddle. No-flow gaps between paddles are created by positioning the "top hat functions" a small distance apart.

A second technique was used to specify the flow through gaps for uniform motion of a multi-element wavemaker system. A dual series relation is developed by matching the water particle velocity across the gaps and wavemakers and matching the velocity potential in the gaps. The perturbations in the wave field produced by the gaps is examined for two cases. The first involves an infinitely long tank with infinitely wide wavemakers placed in the middle of the tank. The second cases involves wavemakers placed a finite distance away from a fully reflecting backwall.

3 Formulation of the Boundary Value Problem

For an incompressible fluid with irrotational motion Laplace's equation

$$\nabla^2\phi(x,y,z,t) = 0$$

holds throughout the domain of the problem. Laplace's equation is elliptic and can only be solved uniquely if boundary conditions are given around the domain of the problem. For small amplitude waves the boundary conditions can be seen in Figure

1. Along the free surface the dynamic free surface and the combined free surface boundary conditions are applied. A no flow condition is specified for the bottom at $z = -h$. Applying the bottom boundary condition and the linear combined free surface boundary condition the velocity potential can be written as

$$\Phi = \phi(x, y) \cosh k(h + z) e^{-i\sigma t} \tag{1}$$

where a temporally periodic solution has been assumed. The angular frequency is given by the usual dispersion relationship

$$\sigma^2 = g_r k \tanh k(h + z)$$

where g_r is gravity, k is the wave number defined as $\frac{2\pi}{L}$ where L is the wavelength. Since we are interested in the cause of the lateral variation along the whole length of the tank, the vertically evanescent modes have been neglected because of the their rapid decay away from the paddles.

We will also specify a no flow condition through the side walls given as

$$\phi_y = 0 \text{ at } y = \pm b$$

Two families of solutions exists that satisfy the side wall condition. The first family of solutions is symmetric about the $x-$ axis and is referred to as the even modes and is given by the y portion of the velocity potential below

$$\phi_e(y) = cos\lambda_j y$$

where

$$\lambda_j \equiv \frac{j\pi}{b} \quad \text{for} \quad j = 0, 1, 2, \ldots$$

One should note that the usual plane progressive mode is the $j = 0$ mode which has no variation in wave height across the tank. The second family of solutions is anti-symmetric about the $x-$ axis and is referred to as the odd modes and the y component of the velocity potential is

$$\phi_o(y) = sin\gamma_l y$$

where

$$\gamma_l \equiv \frac{l + \frac{1}{2}\pi}{b} \quad \text{for} \quad l = 0, 1, 2, \ldots$$

In general the y component of the velocity potential is just the sum of the even and odd modes.

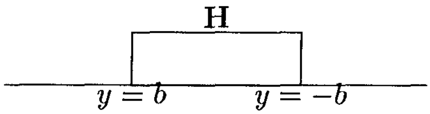

Figure 2: Top hat function generated by the difference of two Heaviside function

The x component of the velocity potential which satisfies Laplace's can be written as

$$\phi(x) = e^{i\sqrt{k^2-\alpha^2}x}$$

where α represents either the even or odd mode wave number.

The velocity potential can be separated into its even and odd contributions and be written as the sum of

$$\phi_e = \sum_{j=0}^{\infty} A_j e^{i(\sqrt{k^2-\lambda_j^2}x-\sigma t)} \cosh k(h+z) \cos \lambda_j y$$

$$\phi_o = \sum_{l=0}^{\infty} B_l e^{i(\sqrt{k^2-\gamma_l^2}x-\sigma t)} \cosh k(h+z) \sin \gamma_l y$$

This is the same solution Dalrymple (1989) found for a constant depth wave basin. To solve for the unknowns A_j and B_l we will use the wavemaker boundary conditions.

3.1 Wavemaker Boundary Conditions for Perturbations in Stroke Length

For a wave tank whose paddle spans the width of the tank the usual linearized kinematic wavemaker boundary condition can be shown to yield

$$u(0,z,t) = -iU_0 e^{-i\sigma t} \text{ for } \mid y \mid < b \tag{2}$$

where U_0 is the magnitude of the paddle velocity at $x = 0$ along the z^{th} plane and i is the $\sqrt{-1}$.

A more attractive technique exists for specifying (2) which can be expanded to account for a multi-element wavemaker. We will denote the Heaviside function as H and define $H(y-b)$ as

$$H(y-b) = \begin{cases} 0, & \text{for } y < b \\ 1, & \text{for } y \geq b \end{cases}$$

We can also combine two Heaviside functions to generate a "top hat function". Figure 2 was generated using the difference of two

Heaviside function as given below

$$\mathbf{H} = H(y+b) - H(y-b). \tag{3}$$

Multiplying the "top hat function" by $U_0(z)$ and a complex exponential of time yields the wavemaker boundary condition in terms of Heaviside functions and can be written as

$$u(0,y,z,t) = -iU_0\mathbf{H}e^{-i\sigma t}$$

The "top hat function" approach can be expanded to account for a multi-element wavemaker. The width of the Heaviside functions and placement controls the paddle width and the no flow gap width between paddles. In general the wavemaker boundary condition for the n^{th} paddle can be written as

$$u_n(x=0,y,z,t) = -iU_{0,n}\mathbf{H_n}\mathbf{e}^{-\mathrm{i}\sigma\mathrm{t}}$$

and the full wavemaker boundary condition is the sum of the contribution from each wavemaker

$$u(0,y,z,t) = \sum_{n=1}^{N} u_n \tag{4}$$

where N is the number of wavemakers and the constant $U_{0,n}$ allows for non-uniform paddle motion. For stroke length perturbations $U_{0,1} \neq U_{0,2} \neq \ldots U_{0,N}$. This is different from waves that are generated at an angle to the wavemaker in which there is a sinusoidal variation in the phase between paddles but the stroke length for each paddle is the same.

For an N wavemaker system a summation over the N paddles must also be added and the even and odd velocity potentials reduce to

$$\phi_e = \sum_{n=1}^{N}\sum_{j=0}^{\infty} A_{n,j} e^{i(\sqrt{k^2-\lambda_j^2}x-\sigma t)} \cosh k(h+z)\cos\lambda_j y \tag{5}$$

and

$$\phi_o = \sum_{n=1}^{N}\sum_{l=0}^{\infty} B_{n,l} e^{i(\sqrt{k^2-\gamma_l x}-\sigma t)} \cosh k(h+z)\sin\gamma_l y. \tag{6}$$

The constants $A_{n,j}$ and $B_{n,l}$ can be solved uniquely by applying the wavemaker boundary condition given in (4) for the n^{th} paddle and the orthogonal relationship between trigonometric series and the hyperbolic cosine function. For the even coefficients we first multiply by the $\cos\lambda_j y$ term and integrate over the width of the tank and then multiply by $\cosh k(h+z)$ term and integrate over the depth. Similarly for the odd coefficients except we multiply by the $\sin \gamma_l y$ series instead of the cosine series. The coefficients are given as

$$A_{n,j} = -\frac{\int_{-b}^{b} U_{0n}(y)\mathbf{H}_n \cos \lambda_j y dy \int_{-h}^{0} U(z) \cosh k(h+z) dz}{\sqrt{k^2 - \lambda_j^2} \int_{-b}^{b} \cos^2 \lambda_j y dy \int_{-h}^{0} \cosh^2 k(h+z) dz} \quad (7)$$

$$B_{n,l} = -\frac{\int_{-b}^{b} U_{0n}(y)\mathbf{H}_n \sin \gamma_l y dy \int_{-h}^{0} U(z) \cosh k(h+z) dz}{\sqrt{k^2 - \gamma_l^2} \int_{-b}^{b} \sin^2 \gamma_l y dy \int_{-h}^{0} \cosh^2 k(h+z) dz} \quad (8)$$

The free surface elevation can be found from the linear dynamic free surface boundary condition.

3.2 Wavemaker Boundary Condition for Flow between Paddles

In the previous formulation Heaviside functions were used to model wavemakers where a perturbation between paddle stroke length could be analyzed. In this section a numerical procedure is introduced to study the wave field created by gaps between paddles for uniform paddle motion. These gaps allow flow between the paddles unlike the previous section where no flow gaps were assumed. Since there is a transmission of fluid through the gaps, an eigenfunction expansion method describes the wave field on each side of the wavemakers.

Two cases will be analyzed for the gap problem. The first case involves an infinitely long tank with an infinitely wide wavemaker located in the center of the tank. In the second case, the gap problem is solved in a semi-infinite long tank with infinitely wide wavemakers placed next to a reflecting backwall.

3.2.1 Infinitely Long Tank

The solution technique Dalrymple and Martin (1990) applied to the problem of waves impinging on an infinite row of offshore breakwaters can be applied to the problem of wavemakers with gaps. Following the coordinate convention Dalrymple and Martin introduced, the wavemakers are modelled as infinitely thin barriers, centered at $y = \pm nb$ where $n = 1, 2, 3, \ldots, \infty$ and are separated by gaps of width $2g$. The lines $y = \pm nb$ are lines of symmetry assuming the wavemaker motion is uniform. Instead of solving the problem for an infinitely long row of wavemakers an analogous problem in which a sidewall or barrier is placed

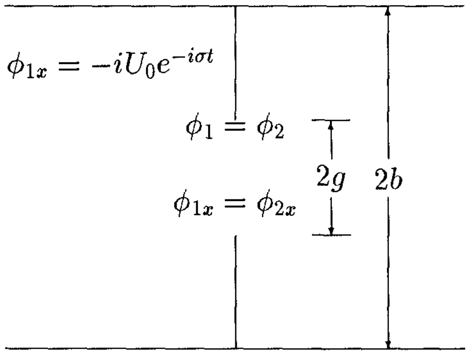

Figure 3: Boundary conditions for an infinitely long tank

along the lines of symmetry will be solved. Waves that initially travelled across these lines are now reflected back and the wave pattern seen is unchanged.

For uniform paddle motion the wave field is symmetric about the center of the wavemakers and thus only the even family of solutions should be retained in the y component of the velocity potential. The velocity potential for $x < 0$ can be written as

$$\phi_1(x,y) = \sum_{j=0}^{\infty} A_j \cos \lambda_j y e^{-i\sqrt{k^2-\lambda_j^2}x}$$

where the negative complex exponential allows waves traveling to the left and for $x > 0$

$$\phi_2(x,y) = -\sum_{j=0}^{\infty} B_j \cos \lambda_j y e^{i\sqrt{k^2-\lambda_j^2}x}$$

where the minus sign has been kept for convenience. The full velocity potential can be found using equation (1).

The wavemaker boundary conditions shown in Figure 3 will be used to solve for the constants A_j and B_j. Three conditions are specified. The first is the kinematic wavemaker boundary condition matching the wavemaker velocity with the water velocity. In the gaps we match the velocities and the velocity potentials to ensure a matching of the free surface. Setting the velocities and the velocity potentials in the gaps equal we find that

$$\sum_{j=0}^{\infty} A_j \cos \lambda_j y = 0 \text{ for } \quad | y | \leq g \tag{9}$$

Matching the velocity at the wavemaker

$$\sum_{j=0}^{\infty}(A_j\sqrt{k^2-\lambda_j^2}\cos\lambda_j y - U_0) = 0 \text{ for } g <| y |\leq b \tag{10}$$

Sneddon (1966) termed the two conditions (9) and (10) a dual series relation which can be combined to make one mixed boundary condition given as $G(y) = 0$ where

$$G(y) = \begin{cases} \sum_{j=0}^{\infty} A_j \cos\lambda_j y, & \text{for } | y |\leq g \\ \sum_{j=0}^{\infty} A_j\sqrt{k^2-\lambda_j^2}\cos\lambda_j y - U_0, & \text{for } g <| y |\leq b \end{cases} \tag{11}$$

The above expression can not be solved analytically and we will resort to using a least squares techniques to solve for the unknown coefficients A_j. This requires that the following equation be a minimum

$$\int_{-b}^{b} | G(y) |^2 \, dy. \tag{12}$$

by definition $| G(y) |^2 = GG^*$ where the * denotes the complex conjugate. If we minimize the above integral with respect to A_j we find

$$\frac{\partial}{\partial A_m}\int_{b}^{b} | G | \, dy = 0$$

or

$$\int_{b}^{b} G^* \frac{\partial G}{\partial A_m} dy = 0 \text{ for } m = 0, 1, \cdots$$

where

$$\frac{\partial G}{\partial A_m} = \begin{cases} \cos\lambda_m y & \text{for } | y |\leq g \\ \sqrt{k^2-\lambda_m^2}\cos\lambda_m y & \text{for } g <| y |\leq b \end{cases}$$

3.2.2 Semi-Infinite Tank

The above problem of an infinitely long tank only needs to be slightly modified to account for a reflecting backwall located behind the wavemakers. This problem can be seen in Figure 4 where the backwall is located a distance ℓ behind the wavemakers. To account for the backwall an additional term must be added to ϕ_1 and is shown below

$$\phi_1(x,y) = \sum_{j=0}^{\infty}(B_j e^{-i\sqrt{k^2-\lambda_j^2}x}\cos\lambda_j y + C_j e^{i\sqrt{k^2-\lambda_j^2}(x+l)}\cos\lambda_j y)$$

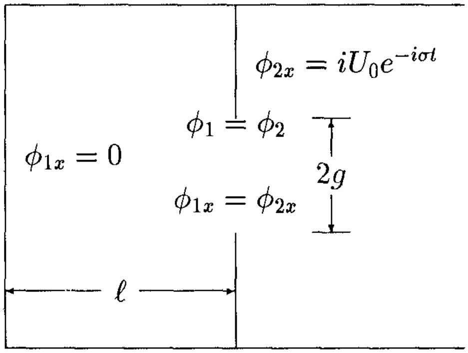

Figure 4: Boundary conditions for a semi-infinite tank

Applying the no-flow condition at $x = -\ell$ we can first find an expression for B_j in terms of C_j and re-write ϕ_1 as

$$\phi_1(x, y) = \sum_{j=0}^{\infty} B_j \cos \lambda_j y [e^{-i\sqrt{k^2-\lambda_j^2}x} + e^{i\sqrt{k^2-\lambda_j^2}(x+2l)}].$$

By matching the velocities in the gap we find

$$B_j = -\frac{A_j}{1 - e^{2i\sqrt{k^2-\lambda_j^2}l}}$$

thus ϕ_1 can be written as

$$\phi_1 = \sum_{j=0}^{\infty} -A_j \frac{\cos \lambda_j y}{1 - e^{2i\sqrt{k^2-\lambda_j^2}l}} (e^{-i\sqrt{k^2-\lambda_j^2}x} + e^{i\sqrt{k^2-\lambda_j^2}(x+2l)})$$

We will use the remaining two boundary conditions to find a new dual series relationship given as

$$G(y) = \begin{cases} \sum_{j=0}^{\infty} A_j \cos \lambda_j y(1 + \beta) = 0, & \text{for } | y | \leq g \\ \sum_{j=0}^{\infty} A_j \sqrt{k^2 - \lambda_j^2} \cos \lambda_j y - U_0, & \text{for } g < | y | \leq b \end{cases}$$

where β is

$$\beta = \frac{1 + e^{2i\sqrt{k^2-\lambda_j^2}l}}{1 - e^{2i\sqrt{k^2-\lambda_j^2}l}}$$

4 Numerical Results

4.1 Perturbed Paddle Motion

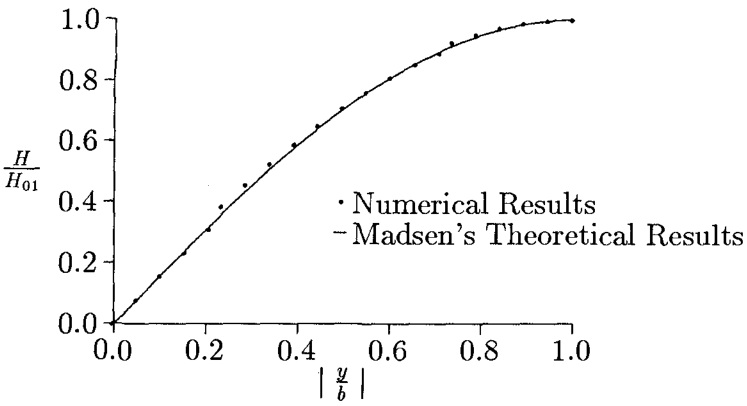

Figure 5: Relative wave height variation where H is the calculated wave height divided by the theoretical wave height H_{01} versus the dimensionless position across the tank $\frac{y}{b}$.

Madsen (1974) analyzed the results of a two wave paddle system in which the paddles were 180^o out of phase located between reflecting sidewalls and a flat impermeable bottom. Madsen's numerical results which correspond very closely with his experimental results are plotted versus the authors numerical results in Figure 5. The horizontal axis shows the nondimensional position across the tank, and the vertical axis is the nondimensional wave height given by the calculated wave height H divided by the theoretical wave height along the sidewalls. The figure shows very good agreement between the two models.

We will again use the two wavemaker system utilized by Madsen but instead of 180^o phase difference between paddles we allow a 10% variation in the stroke length between paddles. The importance of the lateral variation can be calculated by the ratio of a_{max}/a_0 where a_{max} is the combined amplitude of the non-plane modes and a_0 is the amplitude of the plane progressive mode (the $j = 0$ mode). Figure 6 shows the importance of the non-plane modes for varying kb values. This problem is purely anti-symmetric and cross tank resonance occurs at $kb = \pi/2$ and $3\pi/2$.

The usefulness of the first model is not limited to a two wavemaker system. Figure 7 was produced by a 10 paddle system with a 10% variation in the paddle stroke where the no flow gaps between paddles are 10% of the paddle width and $kb = 5.6$. The figure is a computer generated instantaneous picture of only the perturbed wave field where the wave crests are the darker regions and the wave troughs are the lighter regions. The paddles are located at the bottom of the figure and the reflecting sidewalls are

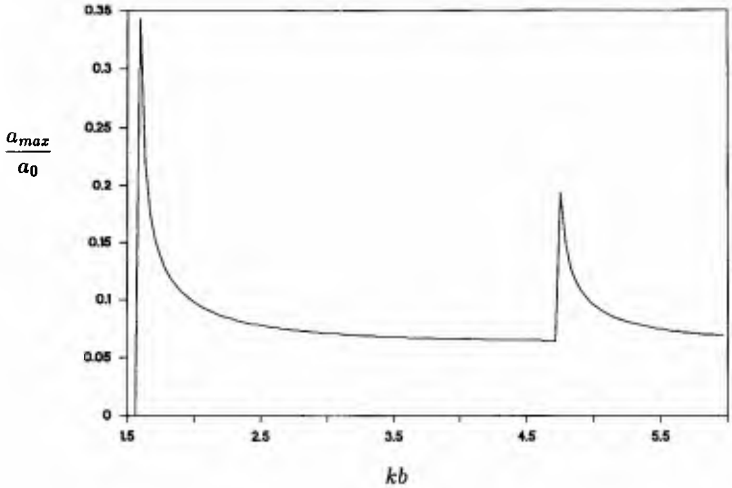

Figure 6: The lateral variation produced by a 10% stroke length difference for a two paddle system.

Figure 7: Instantaneous picture of the perturbed wave field. The paddles are located at the bottom of the figure and the sidewalls are located on the left and right margins.

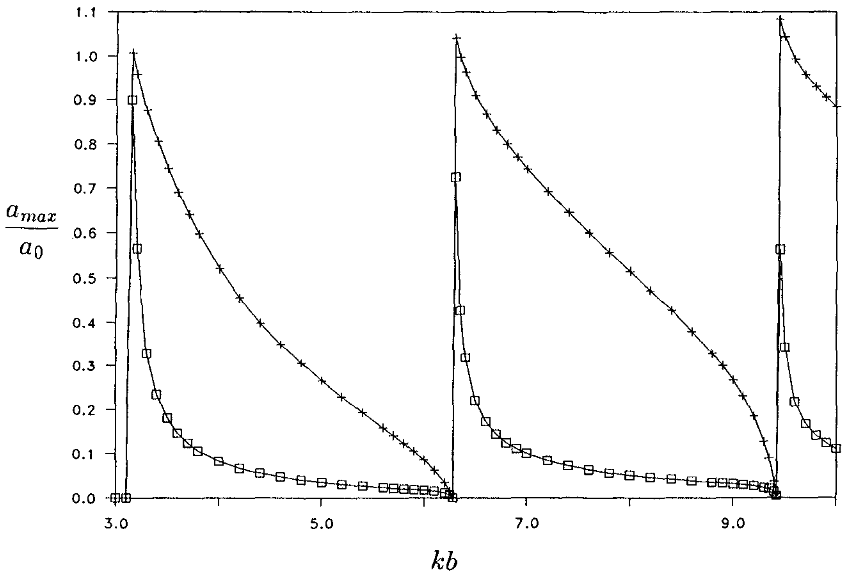

Figure 8: Resonance curves for an infinitely long tank. A gap width of 5% of the paddle width is marked with □ and and a 10% gap width is marked with +.

located on the left and right margins. A 15% variation in the waveheight was found for this case.

4.2 The Gap Problem

A lateral variation in the wave field can also be produced by flow through the gaps between paddles for an infinitely long tank. For the gap problem we assumed symmetric motion about the center of the tank and resonance should occur at $kb = n\pi$. Graphing kb versus a_{max}/a_0 defined previously, Figure 8 clearly shows the first three resonance peaks. Decreasing the gap width by 5% greatly reduces the lateral variation generated.

The wave field produced by placing a reflecting backwall just over a half wavelength behind the wavemaker can be seen in Figure 9. In this case the location of the backwall is near resonance and with $kb = 3.6$ the tank is near resonance across the tank. Graphs showing lateral variation as a function of kb for different gap sizes and backwall distances can be found in Harkins (1990).

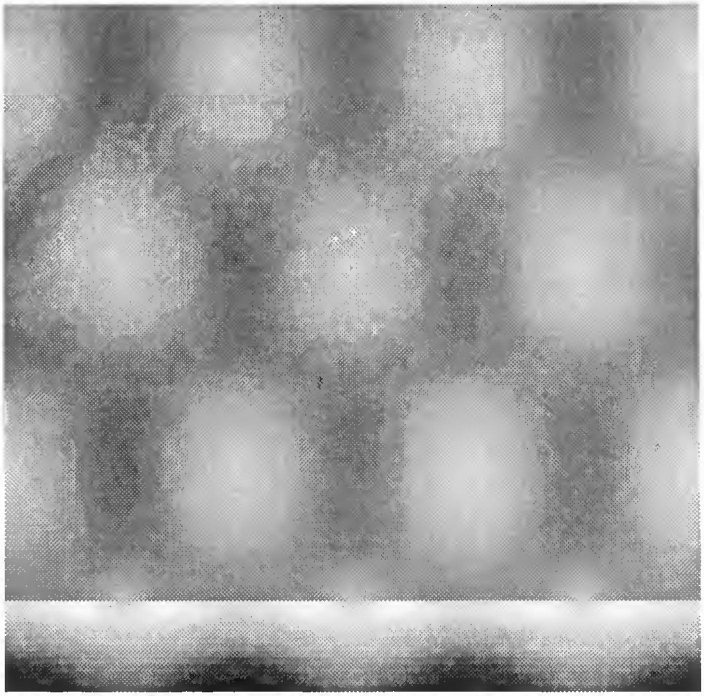

Figure 9: Instantaneous picture of the wave field produced by gaps between paddles. The backwall is located at the bottom of the figure.

5 Conclusions

Both the generated wave length and the basin geometry are important in creating a clean monochromatic plane wave. Even small perturbations in the paddle stroke length as well as flow between paddles can generate large lateral variations in the wave field especially near resonance conditions.

Heaviside functions can accurately represent discrete paddles of a multi-element wavemaker. This technique is useful in analyzing perturbations in the stroke length between paddles since the stroke length of each paddle is easily defined. Lateral variations will be present whenever the tank width is wider then the half wavelength of the generated wave and can be significant when the kb is a multiple of $\frac{\pi}{2}$. For small amplitude waves, the wave field is produced by the contribution of each paddle and thus the complexity of the wave field increase with the number of paddles.

If water is allowed to flow through the gaps between paddles, then an eigenfunction expansion can be used to describe the wave field produced. Since the wavemaker motion was assumed to be uniform, lateral variation will be present when the paddle plus gap width is greater than the wavelength of the generated wave. The lateral variation can be significant when the kb ratio is a multiple of π and when the flow through the gaps is stronger.

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