CHAPTER 46

A comparison of the performance of three mathematical models of wave disturbance in harbour approaches

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Abstract

In many design studies for coastal harbours mathematical models are used to define wave conditions in the vicinity of their entrance. There is a wide variety of models available for this purpose, each with their own characteristics. This paper compares the performance of three models which are in use for this type of study. The advantages and drawbacks of each of the models is considered by comparison with results from a physical model of a typical harbour approach bathymetry.

Introduction

Mathematical models of wave disturbance are in frequent use for predicting wave conditions both within and in the approaches to a harbour. Information on wave activity in the approaches to a harbour will be required to make a full assessment of ship manoeuvrability, and the movement of sediments. In the case of a harbour entered by a dredged channel, estimating wave disturbance will often be of importance in studies directed towards optimising channel depth and alignment.

There are presently available a wide variety of models which can be used to estimate wave disturbance. In this work attention is directed towards situations in which wave conditions are to be calculated at many locations, rather than at a few isolated points, in the harbour approaches. The performance of the models within a harbour is not considered here, either because their behaviour in such circumstances has already been examined, or they are unsuited to that type of application.

In developing wave disturbance models comparisons are often made between their results, and those from analytical solutions to

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idealised problems. In this case the technique which is used is to compare the results from the three mathematical models, with those from a physical model for a typical harbour approach bathymetry. Four sets of results from the physical model were available covering different incident wave conditions. In addition to the accuracy of the representation achieved by the mathematical models, consideration was also given to the speed of the calculation procedure.

The three mathematical models used in the comparison were a wave ray model and two finite difference models. The three models, and the physical assumptions inherent in their mathematical derivation, are described in detail in the following section. In subsequent sections the physical model test case is described and the performance of the models examined. In the final section the overall conclusions resulting from this work are given.

Description of the mathematical models

The three models used in this comparison were PORTRAY (see Smallman, 1987) and two finite difference wave models based on alternative formulations of the mild slope equation. FORTRAY, which is based on a ray tracking technique, was developed at Hydraulics Research, and is in frequent use in site specific studies. Both of the finite difference models were developed at UK Universities and were transferred to Hydraulics Research during 1987. The finite difference models are based on hyperbolic, see Copeland (1985), and parabolic, see Dodd (1988), approximations to the mild slope equation. The mild slope equation is given by

$$\underline{\nabla} \cdot (\mathbf{c} \ \mathbf{c}_{\varphi} \ \underline{\nabla} \phi) + \mathbf{w}^2 \ \phi \ \mathbf{c}_{\varphi} / \mathbf{c} = 0$$

where $\phi(\mathbf{x}, \mathbf{y})$ is the velocity potential, c is the phase velocity, c_g the group velocity and ω the radian frequency. The equation, first derived by Berkoff (1972), describes the propagation of periodic, small amplitude surface gravity waves over a seabed of mild slope and will represent the combined effects of refraction, shoaling and diffraction. A more detailed description of each of the models follows.

PORTRAY is based on a ray tracing technique developed from the theory of light. Under the assumptions that the waves are linear, and that a wave in water of local depth, d, will behave similarly to a wave in water of constant depth d, wave refraction and shoaling can be shown to be governed by Snells law. Rays are tracked in the direction of wave propagation, and wave heights are calculated using the principle of conservation of energy between neighbouring rays. This approach has a limitation in that diffraction, ie a lateral transfer of wave energy, which can be caused by rapid gradient changes in the bed, is not included explicitly in the governing assumptions of the model. This can lead to some difficulties in applying the model, particularly in areas where relatively long period waves are incident along the line of a dredged channel. The model has, however, been extensively validated against physical models and in most situations, particularly in harbour wave disturbance studies, found to provide accurate results, see Smallman (1987).

A set of linear hyperbolic equations to model refraction and diffraction processes in coastal zones was put forward by Ito and Tanimoto (1972) at about the same time as the elliptic mild slope equation was derived. Later Copeland (1985) derived similar equations from the transient form of the mild slope equation. This hyperbolic form can be written as

 $\frac{\partial Q}{\partial t} + \frac{c^2}{n} \nabla(n \eta) = 0$ $\frac{\partial \eta}{\partial t} + \nabla Q = 0$ Here the water surface elevation η is
(2)

 $n = A(x,y)e^{-i(\chi-\omega t)}$

where A is the amplitude of the water surface fluctuation and χ is the phase angle.

Also, $n = \frac{c_g}{c}$ and Q is a dummy variable representing the flow rate, defined as a vertically integrated function of particle velocity. The equations can represent diffraction, refraction and reflections under the assumptions made in their derivation.

By creating a hyperbolic form from the original equation the mild slope problem has been embedded in a larger space (x,y,t). This appears to be an unnecessary complication as the time dependence, $e^{-1\omega t}$, is known in advance. Therefore time stepping will produce only a phase change. If it does not then there is a basic inconsistency in the derivation.

This point has been explored by Madsen and Larsen (1987). They make the observation that the time stepping is actually only an iteration towards the steady state, and that only the steady state solution is a solution to the mild slope equation. This accounts in part in the difficulties which are known to occur in getting the hyperbolic form to converge to the steady state. A difficulty which needs to be resolved more satisfactorily before the method can be used reliably in practice.

The time taken to solve the elliptic mild slope equation computationally, and the mathematical uncertainties of the hyperbolic form, means that attention has been given to the parabolic approximation. This will be computationally efficient to solve and mathematically more rigorous in its derivation. However, this is achieved at the expense of accuracy in the representation of physical problems. That is, whilst refraction and diffraction are represented in the parabolic approximation, reflections are not. A consequence of this is that it is therefore not suitable for representing wave disturbance in harbours where reflected waves will be important.

The first comprehensive account of a parabolic approximation to the mild slope equation was given by Radder (1979). The equation modelled here is based on an improvement to this work given by Booij (1981). The derivation assumes that the reflected wave field is negligibly small so that only forward travelling waves are considered. This leads to the equation

 $\frac{\partial \phi}{\partial x} = \frac{i}{2k} \frac{\partial^2 \phi}{\partial y^2} + \left\{ ik - \frac{1}{2k} \frac{\partial k}{\partial x} \right\} \phi$ (3)

where x is the main direction of wave propagation, y is the transverse direction. Deviations from the x direction are considered in the equation as oblique amplitude modifications. The parabolic approximation will allow refraction, shoaling and seabed diffraction to be modelled. The approximation works best where the important effects occur in the direction of wave propagation, as transverse effects are only included in a weak sense. A detailed account of the deviation of the governing equation for the parabolic model used in this work, and its application to a number of test problems, is given in Dodd (1988).

The equation (3) can be solved numerically using an evolutionary finite difference technique; this type of method only requires storage of one or two adjacent rows of solution points and, as a consequence, is considerably less expensive in terms of cost and storage than the equivalent numerical solution to an elliptic equation. Thus, the main advantage of the parabolic equation is that it permits a more rapid and straightforward method of solution than would be possible for the elliptic equation.

Finally, all three of the models were modified to include the effects of wave breaking. This was done using an empirical formulation put forward by Weggel (1972).

Physical model

The physical model layout is shown in Figure 1. In the tests carried out random waves tests for two spectra from each of two directions were carried out. The characteristics of the incident conditions are given in Table 1.

Test	Significant wave	Peak wave period	Direction	
No	height H _a (m)	T _n (s)	(°)	
1	4.3	^P 8.6	0	Storm
2	1.9	6.0	0	Typical
3	6.0	10.0	25	Storm
4	3.2	7.5	25	Typical

Table 1. Incident wave conditions for physical model tests.



Figure 1 Physical model layout

All of the physical model tests were run at a fixed still water level of +1.9m CD. Measurements of wave height (H_s) were made in the physical model at the ten locations shown on Figure 1. These positions were selected to be representative of conditions within and at the sides of the channel. This particular bathymetry is typical of a dredged harbour approach channel and, as such, will provide a good test of the capabilities of the models for a range of incident wave conditions.

Mathematical model set up

All of the mathematical models require the bathymetry to be described as a set of depth values over a regular grid. Firstly, the bed contours were digitised and then output on a 7m (prototype) mesh. This mesh size gave at least ten points per wave length to the incident wave conditions given in Table 1. This choice of mesh size satisfied the accuracy of representation constraints of the two finite difference models. PORTRAY is not reliant on such grid constraints, so a second set of grid data, each node 35m (prototype) apart, was:generated without significant loss of resolution in the description of the bathymetry. PORTRAY was run both for the 35m and 7m mesh layouts.

The mathematical model grid systems were deliberately aligned to one of the wave propagation directions. This was done because, in the derivation of the parabolic model, the coordinates are assumed to be in the main wave propagation direction. It is possible for the model to include wave effects in directions up to 45° different from the main direction, but with its accuracy reducing as the angle increases. This point will need to be borne in mind when considering the parabolic model results for the 25° direction cases.

All of the mathematical models are monofrequency, unidirectional models. It is possible to run each of them repeatedly for different frequency components, and then use linear superposition to achieve a spectral description of wave propagation. This would give a true comparison with results from a random wave physical model. However, for this study monofrequency runs were made using the peak period of the incident wave spectrum to represent the random wave train. This is not ideal, but has been found in many applications to give a reasonable approximation to an incident wave spectrum. It is intended that this research will be extended subsequently to include mathematical models with random wave incident conditions.

For the hyperbolic finite difference model it remained to select a time step. To satisfy the stability criterion of this model, time steps were selected such that 25 elapse in each wave period. To achieve convergence for this model it was typically necessary to run it for 10 wave periods. Neither the parabolic model nor PORTRAY are time dependent models, so no time step was required for either.

Comparison of results

For the purposes of discussion of the results each of the conditions in Table 1 will be referred to as 'typical' or 'storm' waves with an associated direction. For example, the condition storm (25°) will refer to results from test 3.

For each of the four test cases wave height coefficients at analysis points corresponding to the probe positions of the physical model were calculated. These are presented in Tables 2 to 5 for the physical model, each of the mathematical models on the 7m grid and PORTRAY over the 35m grid (referred to as PORTRAY CG). Each of the tabulated values is averaged over nine grid points centred at the analysis point, thus providing values which can be confidently accepted as representative of wave conditions at the given locations.

Discussion of results

Before examining the numerical results in more detail it is worth observing some of the more quantitative aspects of the results which are shown in Figure 2.

Figure 2 represents the calculated significant wave heights for each of the model tests for the storm (0°) condition. Figures 2a and 2b represent the results of PORTRAY for the fine and coarse grid solutions respectively. Figures 2c and 2d represent the calculated wave field for the hyperbolic and parabolic models.

Considering Figures 2a and 2b, it can be seen that PORTRAY models the wave effects on a fine and course mesh in a similar manner. The ray approximation means that diffraction by the channel is not represented, and the rays are reflected at the channel sides. This leads to the formation of caustics, and consequently excessively large wave heights at the channel sides. Physically, there will be an area of higher wave activity in this location, but the lack of representation of seabed diffraction in ray models will exaggerate this effect. This behaviour is is more clearly seen in the finer grid case, where distinctive regions of large wave heights can be seen. Figure 2b shows how PORTRAY over a coarser grid has led to a more even distribution of the wavefield. This is because the averaging procedure (Southgate, 1984), used to calculate wave heights in the ray models, has effectively introduced a type of numerical diffraction. Also of note are the regions of very low wave heights. Since the PORTRAY model is based on tracing out wave orthogonals there will be some regions where no rays have been able to penetrate. Wave heights in these regions will be physically low, but PORTRAY predicts zero or negligibly small wave heights. It should be observed that, as far as the ray models are concerned, the storm (0°) was the most stringent of the four cases tested. This is because it represents long period waves incident along the line of a dredged channel, ie the case where physical diffraction will be an important mechanism. It will be seen from the numerical results that its performance improves for shorter period waves (typical (0°)), and more markedly for waves from 25°.

Considering the two finite difference models, Figures 2c and 2d, both hyperbolic and the parabolic models produce a similar



DISTANCE IN METRES Figure 2b. Waveheight contours. PORTRAY_CG(Coarse Grid),Storm (0°).



Figure 2d. Waveheight contours. Parabolic. Storm(0°).

wave pattern to that of PORTRAY, and compare more favourably with the coarse grid solution. Diffraction of waves in the vicinity of the channel results in a more even distribution of wave heights. At the channel sides waveheights are generally larger than in the channel region itself. It can be seen that the patterns of wave activity produced by these two models differ away from the channel. The hyperbolic model appears to concentrate energy close to the channel whereas the parabolic model spreads this diffractive energy out more evenly over a larger region.

Examination of the tabulated results allows a quantitative comparison with the measured data. The following discussion will consider positions along the centre, to the left and to the right of the channel, for each test case, in turn. The results shown in Table 2 correspond to measured and predicted waveheight coefficients at the prediction points, see Figure 1, for the illustrated storm (0°) condition, see Figure 2. At positions 3 and 6, along the centre of the channel, all of the models underpredict the waveheights compared with the measured values. The finite difference models are more reliable, within 30% of the measured data, than the PORTRAY solutions. This will be due to an insufficient number of rays penetrating into this region. Further along the channel, shorewards, at position 10, each model predicts significantly higher waveheights than was measured, although the predictions lie within 20% of their combined average. Considering positions to the left of the channel, at points 1, 2, 5, 8 and 9, the finite difference models' results are more consistent with the measured data, notably at position 1 and 5 where the results lie within 10% of the recorded values. Elsewhere, for all the models there are some discrepancies for which there is no apparent trend. To the right of the channel at point 4 each model overpredicts slightly compares with the measured value. However, at position 7 the numerical models underpredict conditions compared with those measured.

The tabulated results in Table 3 corresponding to the typical (0°) condition indicate that each model's performance will change for different incident wave periods. Only some of the trends discussed above are repeated. At position 3 improvement in the predictions is limited to PORTRAY-CG, although the finite difference models' prediction is only slightly worsened. The underprediction shown at position 6 for the storm (0°) condition is reversed for the typical (0°) , and at position 10 the finite difference models' prediction is much improved at just over 20% difference. Only at position 1 to the left of the channel do one of the PORTRAY solutions differ from the measured values by more than the finite difference models do. However, the largest variations from the measured value oscillates between the models. The trend for the storm (0°) condition at position 4 and 7 is repeated for the typical (0°); the trends worsened at 4 but improved at 7.

It is clear from Table 3 that the performance of the finite

Analysis	Physical	PORTRAY	Parabolic	Hyperbolic	PORTRAY_CG	
1	0.9	0.6	1.0	1.0	0.7	
2	0.9	1.3	1.2	1.7	1.2	
3	0.7	0.1	0.5	0.5	0.0	
4	0.6	0.6	0.7	0.7	0.8	
5	0.7	0.5	0.7	0.7	1.5	
6	0.7	0.3	0.6	0.5	0.5	
7	0.6	0.5	0.3	0.2	0.0	
8	0.8	0.8	0.9	1.1	0.7	
9	0.7	0.6	1.1	1.1	1.1	
10	0.6	1.3	1.4	1.5	1.7	
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Table 2. Waveheight coefficients , $Storm(0^{\circ})$						
Anolucio	Dhugi an 1	DODEDAY	Devebalia	Numerhalia	DODERAY CC	
point	model	PORTRAI	Parabolic	nyperbolic	PORINAL_CG	
1	1.1	0.7	1.0	1.0	0.8	
2	1.4	1.4	1.4	1.7	1.5	
3	0.3	0.0	0.4	0.4	0.2	
4	0.6	0.9	0.9	1.1	0.9	
5	1.0	1.0	0.9	1.0	1.2	
6	0.8	1.9	1.0	1.3	1.5	
7	0.4	0.4	0.4	0.4	0.2	
8	0.7	1.1	0.9	1.0	0,9	
9	0.9	1.1	1.4	1.2	1.0	
10	0.9	1.5	0.7	0.7	2.2	
Table 3.	Waveheig	ht coeffi	cients , Typ	ical(0 ⁰)		
	5					
Analvsis	Physical	PORTRAY	Parabolic	Hyperbolic	PORTRAY CG	
point	mode1				_ 1	
1	0.6	0.6	0.9	1.1	0.8	
2	0.5	0.5	0.9	1.1	0.6	
3	0.4	0.9	1.2	1.5	0.8	
4	0.6	0.2	0.7	0.8	0.4	
5	0.6	0.5	0.9	1.0	0.5	
6	0.5	0.4	1.0	1.2	0.5	
7	0.4	0.2	0.8	0.9	0.4	
8	0.6	0.5	0.8	1.0	0.5	
9	0.6	0.4	0.9	1.0	0.5	
10	0.4	0.9	1.1	1.2	0.6	
Table 4.	. Waveheight coefficients ,Storm(25 ⁰)					
Analysis	Physical	PORTRAY	Parabolic	Hyperbolic	PORTRAY_CG	
point	mode1					
1	0.9	1.0	1.0	1.0	1.0	
2	0.9	0.9	0.9	1.0	0.9	
3	1.1	1.5	1.6	1.4	1.0	
4	0.9	0.8	0.7	0.9	0.6	
5	0.8	0.9	1.1	1.0	0.9	
6	0.8	0.8	1.2	1.2	0.9	
7	0.8	0.2	0.8	0.7	0.7	
8	0.9	0.8	0.9	1.0	0.9	
9	0.9	0.8	1.1	1.1	0.8	
10	0.6	1.3	1.3	1.2	1.0	
	Waveheight coefficients ,Typical(25 ⁰)					

difference models for the storm (25°) conditions is considerably worse than that of the PORTRAY solutions with the parabolic model performing slightly better, but results which vary considerably from the measured values are commonplace. The results also indicate that the PORTRAY-CG solution offers the best comparison with the physical model result. Along the centre of the channel the finite difference models show the largest discrepancies with the measured values. To the left of the channel PORTRAY-CG and PORTRAY compare well with the measured data with PORTRAY-CG marginally better at 5, 8 and 9. Predictions by the finite difference models offer slightly better solutions to the left of the channel than along the centre, but still differ markedly from the measured values. Along the positions to the right of the channel, these discrepancies are reduced at position 4, but are again large at 7. Again the performance of PORTRAY-CG is best overall.

The tendency for the finite difference models to overpredict the waveheights for the storm (25°) is not apparent for the typical (25°) condition. Along the centre of the channel the PORTRAY-CG offers the best solution. To the left of the channel all the models perform well with results rarely more than 20% different, and in general closer to 10% different. To the right of the channel at positions 4 and 7 the largest deviation from the measured data are predicted by PORTAY-CG and PORTRAY respectively, with the results within 20% from the measured data at position 4, and 10% at position 7. The results from the finite difference models in this region are in good agreement with the physical model values.

Summary of results

The numerical models in general tend to exaggerate the physical features. In particular they display large spatial variations in wave height, which are not seen in the physical model. However, the mathematical models all give a reasonable representation of the overall physical behaviour.

For waves incident along the line of the channel, the finite difference models perform better in general than PORTRAY, although for incident waves in the off normal direction, notably the storm (25°) condition, the accuracy of the difference models is reduced. This confirms the expectation implied in the assumption made in the derivation of the parabolic approximation, but is less easy to explain in the case of the hyperbolic model.

Running the PORTRAY model over a carefully selected coarser grid has the effect of introducing numerical diffraction into the solution. This leads to a better representation, compared with the physical model, than the fine grid case.

In general the hyperbolic model took considerably longer, of the order of 8 to 10 times, to run than the parabolic and PORTRAY

models. The formulation of the hyperbolic model, which introduces the time variable, also gives rise to difficulties in determining the converged solution. For the test cases described here the parabolic and PORTRAY models took approximately 30 minutes elapsed time on a SUN 3/50 workstation.

Conclusions

All of the mathematical models tested give a reasonable description of the overall physical effects of waves propagating in the vicinity of a dredged channel. This was certainly good enough to justify their use in comparison of harbour approach schemes. Further calibration of the models, against a physical model or site specific measurements, is recommended if they are to be used for calculation of absolute values.

The PORTRAY model performed better for shorter period waves, and incident directions not along the line of the channel: For longer period waves, and where the waves were directly incident along the channel, the parabolic model gave a good representation, but does require that the grid is aligned in the main wave propagation direction. The combination of these two models appears to encompass most of the important physical features. The hyperbolic model did not appear to offer any significant advantages in terms of accuracy over the parabolic model and PORTRAY in these situations, and its run-times were significantly longer.

These conclusions are based on the comparisons made with the physical model tests described here, but are consistent with the anticipated behaviour inherent with the governing assumptions made for each of the mathematical models. Further work is in progress to examine improvements in the mathematical models' performance when random incident conditions are represented.

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