CHAPTER 11

# Vertically 2-D Nearshore Circulation Model

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**Abstract**

A two layer undertow model is developed which consists of surface and inner layer. The surface layer defines breaking wave dynamics and the inner layer defines the mean flow(circulation) and turbulence fields. The interface between two layers is determined by time and depth averaging of the mean water level and wave height in the surf zone (interface model), in which Reynolds stresses are taken into consideration as well as radiation stresses. The system of equations in the inner layer is derived by time averaging the mass and momentum equations over one wave period. Time and space averaging of these equations in the surface layer defines the surface boundary conditions of the mean flow field in the inner layer. Turbulence in the inner layer is discribed by the standard $k-\varepsilon$ model.

The numerical calculation method is also discussed and model calibration is performed by comparing with the experiments by Stive and Wind (1985).

## 1. Introduction

The vertical circulation occuring in the surf zone consists of both the shoreward mass transport due to breaking wave and the offshore-directed bottom current (undertow). Combining the 2-D vertical and horizontal models, it may be possible to construct a 3-D model of the nearshore circulation system.

Svendsen (1984)[6] developed a theoretical model using the first order approximation technique in describing breaking waves. Hansen and Svendsen (1984) [2] considered the effect of the bottom boundary layer in the undertow. This model was examined by using Stive and Wind's experimental data (1985)[4]. It was shown that the undertow is suppressed by the shear stress at the trough level, the static pressure induced by set-up, and the constraint of zero net flow.

Madsen and Svendsen (1979) [3] developed a theory of vertically integrated conservation equations for breaking waves in the surf zone by introducing the concept of time and depth averaging of mass, momentum and energy between the bottom and mean water level (M.W.L.). From their treatment an idea came

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to mind to define the total depth by the surface and inner layers, thus allowing a theoretical treatment of the breaking waves in the surface layer.

In this study, a simplified two layer model is proposed, in which the surface layer is introduced to describe breaking wave dynamics and to obtain the time-averaged boundary conditions for dynamics in the inner layer. The interface between these two layers is set by M.W.L. The breaking wavesgenerate turbulence and the turbulent kinetic energy conservation is considered in the model for the inner layer. Therefore, the boundary conditions for the $k-\varepsilon$ equations in the inner layer are defined by modelling the dynamics of surface layer. Mass, momentum, and energy conservation laws are formulated by employing Madsen and Svendsen's model. The motion in the inner layer is decomposed into time-averaged mean flow and turbulence. The governing equations of mean flow motion are expressed in terms of the vorticity and stream function, which are derived from the mass and momentum conservation equations. The standard $k-\varepsilon$ model is employed as the governing equation for turbulent motion. The coordinate transformation (conformal mapping) method developed by Wanstrath, Whitaker and Reid (1976)[7] , is used to numerically calculate the 2-D vertical circulation pattern in arbitrary depth. Calibration of the numerical model is performed by a comparison with the experimental data of Stive and Wind (1985) [4].

## 2. Model Outline and Basic Equations

Using the coordinates and variables shown in **Fig.1**, the governing equations are derived. The inner layer is defined as the region extending from the bottom to the mean water level (wave set-up) $\bar{\zeta}$. While the surface layer extends from the trough of the breaking wave to its crest.

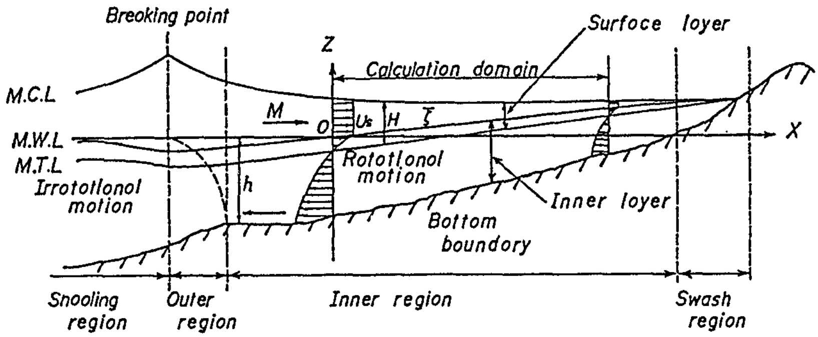



**Fig.1** Schematic explanation of the model and coordinate system

The velocities $u_i$ are decomposed into three modes, i.e. the mean flow $\bar{u}_i$, waves $u_{wi}$ and fluctuations $u_i'$. Other quantities, $\sigma_{ij}, p, s_{ij}$ are also decomposed in the same manner, as:

$$u_i = \bar{u}_i + u_{wi} + u'_i, \qquad \sigma_{ij} = \bar{\sigma}_{ij} + \sigma_{wij} + \sigma'_{ij},$$

$$p_{ij} = \bar{p}_{ij} + p_{wij} + p'_{ij}, \qquad s_{ij} = \bar{s}_{ij} + s_{wij} + s'_{ij} \tag{1}$$

The characteristic time-scales of the three components are assumed to be quite different, therefore no correlation between them is considered. Applying these operations to the mass and momentum equations, the basic equations for the mean flow are obtained. When the Boussinesq's eddy-viscosity assumption is used to describe the Reynolds stresses $\overline{u'_i u'_j}$ by means of velocity gradients, the closure problem becomes a matter of how to determine the eddy viscosity $\nu_t$, which is defined by:

$$\overline{u'_i u'_j} = -\nu_t \left( \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) + \frac{2}{3} k \delta_{ij}, \qquad \nu_t = C_\mu \frac{k^2}{\varepsilon}, \qquad k = \frac{1}{2} \overline{u'_i u'_i} \tag{2}$$

### (a) Inner layer: mean-flow equations

By differentiating $x$ and $z$ momentum equations and eliminating the pressure term $\bar{p}$, we get

$$\frac{\partial^2}{\partial x \partial z}(\bar{u}\bar{u} + \overline{u_w u_w} + \overline{u'u'}) + \left( \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial x^2} \right) (\bar{u}\bar{w} + \overline{u_w w_w} + \overline{u'w'})$$

$$- \frac{\partial^2}{\partial x \partial z}(\bar{w}\bar{w} + \overline{w_w w_w} + \overline{w'w'}) = \nu \nabla^2 \left( \frac{\partial \bar{u}}{\partial z} - \frac{\partial \bar{w}}{\partial x} \right) \tag{3}$$

When small amplitude wave theory is assumed the momentum fluxes of the total wave component, $\overline{u_w u_w}$, $\overline{u_w w_w}$ and $\overline{w_w w_w}$, become zero in Eq.(3).

The stream function $\psi$ and vorticity $\Omega$ are now defined as

$$\frac{\partial \psi}{\partial z} = \bar{u}, \qquad \frac{\partial \psi}{\partial x} = -\bar{w} \quad \text{and} \quad \Omega = \frac{\partial \bar{u}}{\partial z} - \frac{\partial \bar{w}}{\partial x} \tag{4}$$

In terms of the stream function and vorticity defined above, the basic equations are rewritten as:

$$\nabla^2 \psi = \Omega \tag{5}$$

and

$$\left( \frac{\partial \psi}{\partial z} \right) \left( \frac{\partial \Omega}{\partial x} \right) - \left( \frac{\partial \psi}{\partial x} \right) \left( \frac{\partial \Omega}{\partial z} \right) - \frac{\partial^2}{\partial x \partial z} \left( 4 \nu_t \frac{\partial^2 \psi}{\partial x \partial z} \right)$$

$$+ \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} \right) \left\{ \nu_t \left( \frac{\partial^2 \psi}{\partial z^2} - \frac{\partial^2 \psi}{\partial x^2} \right) \right\} = \nu \nabla^2 \Omega \tag{6}$$

### (b) Inner layer: standard $k - \varepsilon$ model

The $k$-equation is rewritten in the $x - z$ plane as:

$$\bar{u} \frac{\partial k}{\partial x} + \bar{w} \frac{\partial k}{\partial z} = \frac{\partial}{\partial x} \left\{ \left( \frac{\nu_t}{\sigma_k} + \nu \right) \frac{\partial k}{\partial x} \right\} + \frac{\partial}{\partial z} \left\{ \left( \frac{\nu_t}{\sigma_k} + \nu \right) \frac{\partial k}{\partial z} \right\} + \text{Prod} - \varepsilon \tag{7}$$

where

$$\mathrm{Prod} = \nu_t \left\{ 2\left(\frac{\partial \bar{u}}{\partial x}\right)^2 + 2\left(\frac{\partial \bar{w}}{\partial z}\right)^2 + \left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x}\right)^2 \right\} \tag{8}$$

The $\varepsilon$-equation is also rewritten as

$$\bar{u}\frac{\partial \varepsilon}{\partial x} + \bar{w}\frac{\partial \varepsilon}{\partial z} = \frac{\partial}{\partial x}\left\{\left(\frac{\nu_t}{\sigma_\varepsilon} + \nu\right)\frac{\partial \varepsilon}{\partial x}\right\} + \frac{\partial}{\partial z}\left\{\left(\frac{\nu_t}{\sigma_\varepsilon} + \nu\right)\frac{\partial \varepsilon}{\partial z}\right\} + \frac{\varepsilon}{k}(C_{1\varepsilon}\mathrm{Prod} - C_{2\varepsilon}\varepsilon) \tag{9}$$

### (c) Interface model

The boundary conditions for the governing equatious in the inner layer are discussed here. The surface boundary is the mean water level $\bar{\zeta}$ which is an unknown value in the two layer system. Previous investigations of surf zone wave height and set-up show fairly good predictability of both quantities. Therefore, Svendsen's (1984)[6] model is employed to determine the wave set-up $\bar{\zeta}$ and breaking wave height variation. Svendsen also presented the mass, momentum, and energy dissipation due to breaking waves in his model by employing the first order approximation of motion, the effect of the surface roller and the wave shape parameter $B_o$. Consequently, Svendsen's model is needed to determine the wave energy dissipation (production in the $k$ equation), mass flux and shear stress acting on the interface (surface) boundary.

On the other hand, the boundary condition for the stream function $\psi$, the vorticity $\Omega$, the turbulent kinetic energy $k$, and it's dissipation $\varepsilon$ must be determined. For this problem, Madsen and Svendsen's (1979)[3] formulation of the mass and momentum conservation equations are employed.

The momentum equation in the $x$-direction is integrated over depth and then averaged over the wave period $T_w$ as

$$\frac{\partial}{\partial x}\int_{-h}^{\bar{\zeta}} \rho \bar{u}\bar{u}\mathrm{d}z + \frac{\partial}{\partial x}(S' + S_w) = -\rho g(h + \bar{\zeta})\frac{\partial \bar{\zeta}}{\partial x} \tag{10}$$

where $S'$ is the depth-integrated Reynolds stress and $S_w$ the radiation stress defined by respectively

$$S' = \int_{-h}^{\bar{\zeta}} \rho \left(2\nu_t \frac{\partial \bar{u}}{\partial x} - \frac{2}{3}k\right)\mathrm{d}z \tag{11}$$

$$S_w = \int_{-h}^{\bar{\zeta}} \rho \bar{u}_w \bar{u}_w \mathrm{d}z + \int_{-h}^{\bar{\zeta}} \overline{p_o}\mathrm{d}z - \frac{1}{2}\rho g \bar{\zeta}_w \bar{\zeta}_w \tag{12}$$

where $p_o$ is the dynamic pressure given by

$$p_o = \rho g(z - \bar{\zeta}) + p \tag{13}$$

In Eq.(10), the inertia term, and the depth-integrated Reynolds stress term are added to the usual momentum balance equation. The latter term described by Eq.(11) is the interaction term between a mean flow distribution and $k$-equation in the inner layer.

The energy equation is given by

$$\frac{\partial B_*}{\partial x} = D_* \tag{14}$$

where $B_*$ is the mean wave energy flux and $D_*$ the wave energy dissipation. The non-dimensional wave energy flux $B$ and energy dissipation $D$ are introduced respectively, as:

$$B = \frac{B_*}{\rho g C H^2} \quad \text{and} \quad D = D_* \frac{4hT_w}{\rho g C H^3} \tag{15}$$

where $C$ is the propagation speed of the breaker. In the surf zone, $C$ is approximated by $C \sim \sqrt{g(h+\bar{\zeta})}$. Using Eqs.(14) and (15), the equation for the wave height variation is finally botained as

$$\frac{H}{H_r} = K_S \bigg/ \left(1 - \frac{H_r}{8C_r B_r T_w} \int_{x_r}^{x} \frac{DK_S^3}{h+\bar{\zeta}} \mathrm{d}x\right) \tag{16}$$

where the subscript $r$ refers to some chosen reference points, and $K_S$ is the shoaling coefficient.

Solving Eqs.(10) and (16) simultaneously, the wave set-up and breaker wave height variation are obtained. However, in these equations, three parameters must be estimated, $B, S_w$ and $D$. Svendsen (1984) [5] evaluated these parameters assuming that (i) the mean velocity in the roller is equal to the propagation speed $C$, (ii) the cross sectional area $A$ of the roller is equal to $0.9H^2$, and (iii) wave energy dissipation is analogous to that of hydraulic jump, these are respectively given as:

$$S_w \simeq \rho g \left(\frac{3}{2}B_o + 0.9\frac{h+\bar{\zeta}}{L}\right) H^2 \tag{17}$$

$$B = B_o + 0.45\frac{h+\bar{\zeta}}{L} \tag{18}$$

$$D = -\left[\left(1 + \frac{\zeta_c}{H}\frac{H}{h+\bar{\zeta}}\right)\left\{1 + \frac{H}{h+\bar{\zeta}}\left(\frac{\zeta_c}{H} - 1\right)\right\}\right]^{-1} \tag{19}$$

where $L$ is the wave length, $\zeta_c$ :the crest elevation and $B_o$: wave shape parameter defined by

$$B_o = \frac{1}{T_w}\int_0^{T_w} \left(\frac{\zeta}{H}\right)^2 \mathrm{d}t \tag{20}$$

## (d) Surface layer model

The boundary conditions for the stream function $\psi$ and the vorticity $\Omega$ equations are considered here. To determine the boundary conditions at the surface $\bar{\zeta}$ of the inner layer, the equations of mass and momentum are derived in the surface layer. The kinematic boundary condition at $z = \bar{\zeta}$ is derived from the conservation of mass in the surface layer as

$$\bar{u}(\bar{\zeta})\frac{\partial \bar{\zeta}}{\partial x} - \bar{w}(\bar{\zeta}) = -\frac{\partial}{\partial x}\int_{\zeta}^{\bar{\zeta}} u \mathrm{d}z \tag{21}$$

The dynamic boundary condition at $z = \bar{\zeta}$ is derived from the conservation of momentum in the surface layer. The pressure at $z = \bar{\zeta}$ is obtained by integrating the vertical momentum equation between $\zeta$ and $\bar{\zeta}$ and time-averaging over the wave period $T_w$, as

$$\frac{\bar{p}(\bar{\zeta})}{\rho} = -\overline{w'(\bar{\zeta})w'(\bar{\zeta})} + \overline{u'(\bar{\zeta})w'(\bar{\zeta})\frac{\partial \bar{\zeta}}{\partial x}} - \overline{\bar{w}(\bar{\zeta})\frac{\partial}{\partial x}\int_{\zeta}^{\bar{\zeta}} u \mathrm{d}z} \tag{22}$$

where the viscous term is neglected. The time-averaged horizontal momentum conservation in the surface layer is derived from the following definition.

$$\overline{\int_{\zeta}^{\bar{\zeta}} \text{momentum} \mathrm{d}z} = \overline{\int_{-h}^{\zeta} \text{momentum} \mathrm{d}z} - \int_{-h}^{\bar{\zeta}} \text{momentum} \mathrm{d}z \tag{23}$$

The horizontal momentum equation is given by

$$\frac{\partial}{\partial x}\overline{\int_{\zeta}^{\zeta}\left(u^2 + \frac{p}{\rho}\right)\mathrm{d}z} = -\frac{\bar{p}(\bar{\zeta})}{\rho}\frac{\partial \bar{\zeta}}{\partial x} - \overline{u'(\bar{\zeta})u'(\bar{\zeta})\frac{\partial \bar{\zeta}}{\partial x}} + \overline{u'(\bar{\zeta})w'(\bar{\zeta})} - \bar{u}(\bar{\zeta})\frac{\partial}{\partial x}\overline{\int_{\zeta}^{\zeta} u \mathrm{d}z} \tag{24}$$

Substituting the pressure Eq.(22) into Eq.(24), the right-hand side of the equation becomes:

$$\begin{aligned}\text{RHS} = {} & \left\{\overline{w'(\bar{\zeta})w'(\bar{\zeta})} - \overline{u'(\bar{\zeta})u'(\bar{\zeta})}\right\}\frac{\partial \bar{\zeta}}{\partial x} + \left\{\bar{u}(\bar{\zeta}) + \bar{w}(\bar{\zeta})\frac{\partial \bar{\zeta}}{\partial x}\right\}\frac{\partial M_S}{\partial x} \\ & - \overline{u'(\bar{\zeta})w'(\bar{\zeta})\left(\frac{\partial \bar{\zeta}}{\partial x}\right)^2} + \overline{u'(\bar{\zeta})w'(\bar{\zeta})}\end{aligned} \tag{26}$$

where

$$M_S = \overline{\int_{\zeta}^{\zeta} u \mathrm{d}z} \tag{27}$$

When the orders of $\bar{u}(\bar{\zeta})$ and $\partial\bar{\zeta}/\partial x$ are $O(1)$ and $O(\delta)$, the orders of the following terms can be estimated by the kinematic boundary condition, Eq.(21).

$$\bar{w}(\bar{\zeta}) \sim O(\delta), \qquad \frac{\partial M_S}{\partial x} \sim O(\delta) \tag{28}$$

In the surface layer, strong shear flow is generated by the mass transport shoreward due to the breaker. Then the following order estimation may be reasonable.

$$\overline{u'(\bar{\zeta})w'(\bar{\zeta})} \sim \frac{\partial \bar{u}}{\partial z} \sim O(1) \tag{29}$$

This order estimation leads to the brief expression of Eq.(26), as

$$\text{RHS} = \overline{u'(\bar{\zeta})w'(\bar{\zeta})} + O(\delta) \tag{30}$$

On the other hand, LHS of Eq.(24) can be estimated as follows. The local pressure is given by integrating the vertical momentum equation between $z$ and $\zeta$ as

$$\frac{p(z)}{\rho} = g(\zeta - z) + \frac{\partial}{\partial x}\int_z^{\zeta} uw dz - \rho w^2(z) \tag{31}$$

Assuming that $u^2$ is decomposed as

$$u^2(z) = \bar{u}^2(z) + u_w^2(z) + u'^2(z) \tag{32}$$

then, substituting these equations into the left-hand side of Eq.(24) yields

$$\begin{aligned} \text{LHS} = \frac{\partial}{\partial x}\Bigg\{ & \frac{g}{2}\overline{(\zeta - \bar{\zeta})^2} + \overline{(\zeta - \bar{\zeta})\int_z^{\zeta} uw\text{d}z} - \overline{\int_{\bar{\zeta}}^{\zeta} ww\text{d}z} \\ & + \overline{(\zeta - \bar{\zeta})\left(\bar{u}^2(\bar{\zeta}) + u_w^2(\bar{\zeta}) + u'^2(\bar{\zeta})\right)}\Bigg\} \end{aligned} \tag{33}$$

It is then assumed that in the surface layer, the large scale turbulence exists whose velocity and surface fluctuation have dominant frequency in the spectral domain. If this dominant frequency is a harmonics of the wave, and other spectral components are negligiblly small, the turbulence components can be treated in the same manner as wave components. However, this assumption may be too bold to apply at this time since turbulence is usually characterized by multi-phase motion. However, eddies generated by breaking waves seem to have the same dominant frequency which is strongly related to the wave motion. If the above assumption is allowed, the brief expression of Eq.(33) can be obtained, as

$$\text{LHS} = \frac{\partial}{\partial x}\left\{\frac{g}{2}(\overline{\zeta_w\zeta_w} + \overline{\zeta'\zeta'})\right\} \tag{34}$$

Finally the dynamic boundary condition at the surface of the inner layer is

$$\overline{u'(\bar{\zeta})w'(\bar{\zeta})} = \frac{\partial}{\partial x}\left\{\frac{g}{2}(\overline{\zeta_w\zeta_w} + \overline{\zeta'\zeta'})\right\} \tag{35}$$

In the equation, the first term on the right-hand side is approximated by Eq.(20) and, for the second term, the following assumption is made.

$$\overline{\zeta'\zeta'} \simeq \left(\frac{A}{L}\right)^2 \tag{36}$$

Furthermore, the shear stress due to turbulence is described by

$$\overline{u'(\bar{\zeta})w'(\bar{\zeta})} = -\frac{\tau_S}{\rho} \simeq -\nu_t \frac{\partial \bar{u}}{\partial z}\bigg|_{z=\bar{\zeta}} \tag{37}$$

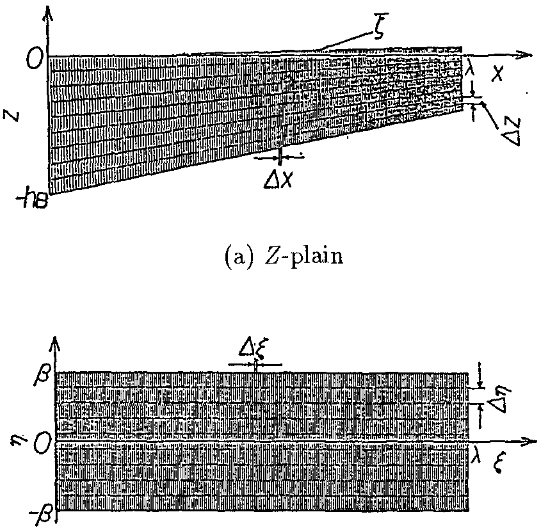
Consequently, the dynamic surface boundary condition, which is expressed by breaking wave quantities, is given as

$$\nu_t \frac{\partial \bar{u}}{\partial z}\bigg|_{z=\bar{\zeta}} = -\frac{\partial}{\partial x}\left\{\frac{g}{2}\left(H^2 B_0 + \frac{A^2}{L^2}\right)\right\} \tag{38}$$

## 3. Numerical Model

### a. Coordinate transformation

In the numerical model the conformal mapping method (Wanstrath, Whitaker and Reid (1976)[7]) is employed to make the model applicable over a wider range of surf zone topography. The governing equations derived in the $Z$-plane are transformed to the $\zeta^*$-plane( **Fig.2**). The arbitrary surface and bottom boundaries in $Z$-plane are transformed into straightline ($\eta = \pm\beta$) in the $\zeta^*$-plane.




(a) $Z$-plain

(b) $\zeta^*$-plane

**Fig.2** $Z$ and $\zeta^*$-plane in the coordinate transformation

$$x(\xi,\eta) = \xi + \sum_{n=1}^{N}(b_n \sinh nk_c\eta + c_n \cosh nk_c\eta)\sin nk_c\xi \tag{39}$$

$$z(\xi,\eta) = b_0 + \eta + \sum_{n=1}^{N}(b_n \cosh nk_c\eta + c_n \sinh nk_c\eta)\cos nk_c\xi. \tag{40}$$

The Fourier coefficients $b_n(n = 0, 1, 2, ..., N)$, and $c_n(n = 1, 2, 3, ..., N)$ are determined by matching the surface and bottom boundaries at $\eta = \pm\beta$, using the least square error. The transformation function $J^2$ is:

$$\frac{\partial(x,z)}{\partial(\xi,\eta)} = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2 \equiv J^2. \tag{41}$$

Using these definitions, the relations of partial differentiations of function $\phi$ with respect to $(x, z)$ and $(\xi, \eta)$ are

$$\frac{\partial \phi}{\partial x} = -\frac{1}{J^2}\left(\frac{\partial z}{\partial \xi}\frac{\partial \phi}{\partial \eta} - \frac{\partial x}{\partial \xi}\frac{\partial \phi}{\partial \xi}\right) = a\frac{\partial \phi}{\partial \xi} - b\frac{\partial \phi}{\partial \eta} \tag{42}$$

$$\frac{\partial \phi}{\partial z} = -\frac{1}{J^2}\left(-\frac{\partial z}{\partial \xi}\frac{\partial \phi}{\partial \xi} - \frac{\partial x}{\partial \xi}\frac{\partial \phi}{\partial \eta}\right) = a\frac{\partial \phi}{\partial \eta} + b\frac{\partial \phi}{\partial \xi} \tag{43}$$

where

$$a \equiv \frac{1}{J^2}\left(\frac{\partial x}{\partial \xi}\right), \qquad b \equiv \frac{1}{J^2}\left(\frac{\partial z}{\partial \xi}\right). \tag{44}$$

Finally, operators of the first order partial differentiation are given by

$$\frac{\partial}{\partial x} = a\frac{\partial}{\partial \xi} - b\frac{\partial}{\partial \eta} \equiv D_x, \qquad \frac{\partial}{\partial z} = a\frac{\partial}{\partial \eta} + b\frac{\partial}{\partial \xi} \equiv D_z. \tag{45}$$

The second order partial differentiations are

$$D_{xx}(\ ) = aD_{\xi\xi}(a,\ ) + bD_{\eta\eta}(b,\ ) - aD_{\xi\eta}(b,\ ) - bD_{\eta\xi}(a,\ ) \tag{46}$$

$$D_{zz}(\ ) = aD_{\eta\eta}(a,\ ) + bD_{\xi\xi}(b,\ ) + aD_{\eta\xi}(b,\ ) + bD_{\xi\eta}(a,\ ) \tag{47}$$

$$D_{xz}(\ ) = D_{zx}(\ ) = aD_{\xi\xi}(b,\ ) - bD_{\eta\eta}(a,\ ) + aD_{\xi\eta}(a,\ ) - bD_{\eta\xi}(b,\ ) \tag{48}$$

where

$$D_{\xi\xi}(a,\ ) = \frac{\partial}{\partial \xi}\left(a\frac{\partial}{\partial \xi}\right), \quad D_{\xi\eta}(a,\ ) = \frac{\partial}{\partial \xi}\left(a\frac{\partial}{\partial \eta}\right),$$

$$D_{\eta\xi}(a,\ ) = \frac{\partial}{\partial \eta}\left(a\frac{\partial}{\partial \xi}\right), \quad D_{\eta\eta}(a,\ ) = \frac{\partial}{\partial \eta}\left(a\frac{\partial}{\partial \eta}\right). \tag{49}$$

The governing equations derived in the $Z$-plane are transformed to the $\zeta^*$-plane by applying the above mentioned operators. The stream function and vorticity equations are rewritten as:

$$\nabla^2_{(\xi,\eta)}\psi = J^2\Omega \tag{50}$$

$$D_z(\psi)\left(a\frac{\partial \Omega}{\partial \xi} - b\frac{\partial \Omega}{\partial \eta}\right) - D_x(\psi)\left(a\frac{\partial \Omega}{\partial \eta} + b\frac{\partial \Omega}{\partial \xi}\right) + F^*(\Omega, \nu_t) = \frac{\nu_t}{J^2}\nabla^2_{(\xi,\eta)}\Omega \tag{51}$$

where

$$F^*(\Omega, \nu_t) = -4D_{xz}\psi D_{xz}\nu_t + \{D_{zz}(\nu_t) - D_{xx}(\nu_t)\}\{\Omega - 2D_{xx}(\psi)\} \tag{52}$$

Similarly, the $k$ and $\varepsilon$-equations are transformed to the $\zeta^*$-plane as, respectively:

$$a\frac{\partial}{\partial\xi}(\bar{u}k)-b\frac{\partial}{\partial\eta}(\bar{u}k)+a\frac{\partial}{\partial\eta}(\bar{w}k)+b\frac{\partial}{\partial\xi}(\bar{w}k)-\left\{a\frac{\partial}{\partial\xi}\left(\frac{\nu_t}{\sigma_k}\right)-b\frac{\partial}{\partial\eta}\left(\frac{\nu_t}{\sigma_k}\right)\right\}\left(a\frac{\partial k}{\partial\xi}-b\frac{\partial k}{\partial\eta}\right)$$
$$-\left\{a\frac{\partial}{\partial\eta}\left(\frac{\nu_t}{\sigma_k}\right)+b\frac{\partial}{\partial\xi}\left(\frac{\nu_t}{\sigma_k}\right)\right\}\left(a\frac{\partial k}{\partial\eta}+b\frac{\partial k}{\partial\xi}\right)-\frac{\nu_t}{\sigma_k J^2}\nabla^2{}_{(\xi,\eta)}k=\text{Prod}-\varepsilon \tag{53}$$

$$a\frac{\partial}{\partial\xi}(\bar{u}\varepsilon)-b\frac{\partial}{\partial\eta}(\bar{u}\varepsilon)+a\frac{\partial}{\partial\eta}(\bar{w}\varepsilon)+b\frac{\partial}{\partial\xi}(\bar{w}\varepsilon)-\left\{a\frac{\partial}{\partial\xi}\left(\frac{\nu_t}{\sigma_\varepsilon}\right)-b\frac{\partial}{\partial\eta}\left(\frac{\nu_t}{\sigma_\varepsilon}\right)\right\}\left(a\frac{\partial\varepsilon}{\partial\xi}-b\frac{\partial\varepsilon}{\partial\eta}\right)$$
$$-\left\{a\frac{\partial}{\partial\eta}\left(\frac{\nu_t}{\sigma_\varepsilon}\right)+b\frac{\partial}{\partial\xi}\left(\frac{\nu_t}{\sigma_\varepsilon}\right)\right\}\left(a\frac{\partial\varepsilon}{\partial\eta}-b\frac{\partial\varepsilon}{\partial\xi}\right)-\frac{\nu_t}{\sigma_\varepsilon J^2}\nabla^2{}_{(\xi,\eta)}\varepsilon=\frac{\varepsilon}{k}(C_{1\varepsilon}\text{Prod}-C_{2\varepsilon}\varepsilon) \tag{54}$$

where

$$\text{Prod}=\nu_t\left[2D_x(\bar{u})^2+2D_z(\bar{w})^2+\{D_z(\bar{u})+D_x(\bar{w})\}^2\right] \tag{55}$$

### b. Numerical scheme

Numerical solution to an advection and diffusion (A/D) equation such as the vorticity equation, suffers from the numerical (artificial) diffusion as well as wiggles. The Dennis-Chang method is known as one of methods which can reduce these numerical problems in the A/D equation with the second order accuracy. An iteration form is discretized by this scheme and calculated till the solution converges. Using the operator notations, $A(\ )$ and $D(\ )$, for advection and diffusion terms, finite difference equation are represented as:

$$A_{UD}(\Omega^{n+1})-D(\Omega)=A_{UD}(\Omega^n)-A_{CD}(\Omega^n) \tag{56}$$

where $A_{UD}(\ )$ and $A_{CD}(\ )$ are the upwind and centered difference operators, respectively, which given by:

$$A_{UD}(\Omega)=\frac{U+\mid U\mid}{2\Delta x}(\Omega_{i,j}-\Omega_{i-1,j})+\frac{U-\mid U\mid}{2\Delta x}(\Omega_{i+1,j}-\Omega_{i,j})$$
$$+\frac{V+\mid V\mid}{2\Delta z}(\Omega_{i,j}-\Omega_{i,j-1})+\frac{V-\mid V\mid}{2\Delta z}(\Omega_{i,j+1}-\Omega_{i,j}) \tag{57}$$

$$A_{CD}(\Omega)=\frac{1}{2\Delta x}\{(U\Omega)_{i+1,j}-(U\Omega)_{i-1,j}\}+\frac{1}{2\Delta z}\{(V\Omega)_{i,j+1}-(V\Omega)_{i,j-1}\} \tag{58}$$

and for the diffusion term :

$$D(\Omega)=\frac{1}{\Delta x^2}\{(\nu_t\Omega)_{i+1,j}-2(\nu_t\Omega)_{i,j}+(\nu_t\Omega)_{i-1,j}\}$$
$$+\frac{1}{\Delta z^2}\{(\nu_t\Omega)_{i,j+1}-2(\nu_t\Omega)_{i,j+1}+(\nu_t\Omega)_{i,j-1}\} \tag{59}$$

where the advection velocity $U$ and $V$ are written as:

$$U=aD_z(\psi)-bD_x(\psi),\quad V=-\{aD_x(\psi)+bD_z(\psi)\} \tag{60}$$

For stable calculation, the dumping coefficient $\alpha_*$ is used at each iteration as

$$\Omega^{n+1} = \alpha_* \Omega^n \tag{61}$$

In the cases of Dirichlet boundary problem, Woods boundary condition is applicable as

$$\Omega_0 = -\frac{1}{2}\Omega_1 - \frac{3}{\Delta x^2}(\Omega_1 - \Omega_0) + O(\Delta x^3) \tag{62}$$

where the suffix 0 indicates the value at the boundary, and 1 indicates the value just inside the boundary.

**c. Interface boundary conditions**

At the surface, both the kinematic and dynamic boundary conditions for mean flow variables, $\psi$ and $\Omega$, are determined by the approximate expressions of the momentum and mass conservation equations in the surface layer.

The kinematic boundary condition in the $Z$-plane determines the stream function in the transformed coordinates. One possible example can be given as follows.

The velocity components $\bar{u}^*, \bar{w}^*$ in the $\zeta^*$-plane can be transformed to

$$\bar{u}^* = -\frac{\partial x}{\partial \eta}(\bar{w}_s - \bar{u}_s \frac{\partial \bar{\zeta}}{\partial x}) = -\frac{\partial M_S}{\partial \eta}, \quad \bar{w}^* = \frac{\partial x}{\partial \xi}(\bar{w}_s - \bar{u}_s \frac{\partial \bar{\zeta}}{\partial x}) = \frac{\partial M_S}{\partial \xi} \tag{63}$$

In these equations $-M_S$ is the stream function in the transformed coordinates system, which gives the surface boundary condition for the $\psi$-equation.

The boundary condition for the vorticity equation is

$$\Omega_s^* = \left.\frac{\partial \bar{u}^*}{\partial \eta}\right|_s - \left.\frac{\partial \bar{w}^*}{\partial \xi}\right|_s = \left.\frac{\tau_s}{\rho \nu_t}\right|_s - \left.\frac{\partial^2 M}{\partial \xi^2}\right|_s \tag{64}$$

The boundary condition for the $k$-equation at the surface can be given by

$$k|_s = \sqrt{\frac{\gamma_s}{C_\mu}}\left|\frac{\tau_s}{\rho}\right| \tag{65}$$

where $\tau_s$ is shear stress at the surface. The boundary condition for $\varepsilon$ is

$$\varepsilon = \gamma_s \mathrm{Prod}|_s \tag{66}$$

where

$$\mathrm{Prod}|_s = \frac{gH^3D}{4h^2T_w} + \nu_t \left.\left\{2\left(\frac{\partial \bar{u}}{\partial x}\right)^2 + 2\left(\frac{\partial \bar{w}}{\partial z}\right)^2 + \left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x}\right)^2\right\}\right|_s \tag{67}$$

where $\gamma_s$ is the coefficient to be determined by turbulence properties.

For both side boudary condtions, a third order polynomial stream function may be assumed and for the bottom boundary conditions, usual boundary conditions can be employed.

## 4. Model tests and conclusions

In order to verify the applicability of the model, test calculations was performed in the inner layer, using the following assumptions; uniform distribution of $\nu_t$, third order polynomial of stream function distribution at both side-boundaries, and the slip condition on the bottom.

**Fig.3** shows the changes in the relative values of wave celerity $C/C_B$, wave heights $H/H_B$ and mean water level $\zeta/h_B$ with the relative water depth $h/h_B$ to compare with the experiments by Hansen and Svendsen (1979) [1]. Their experimental conditions are; the beach slope $s = 1/40$, wave period $T = 1.79s$, deep water steepness $H_o/L_o = 0.032$. The figure shows that the calculations of breaking waves and mean water level are in reasonable agreement with the experiments. Furthermore, the test calculation of undertow velocity profile corresponding to Hansen and Svendsen's experimental condition is shown in **Fig.4**.

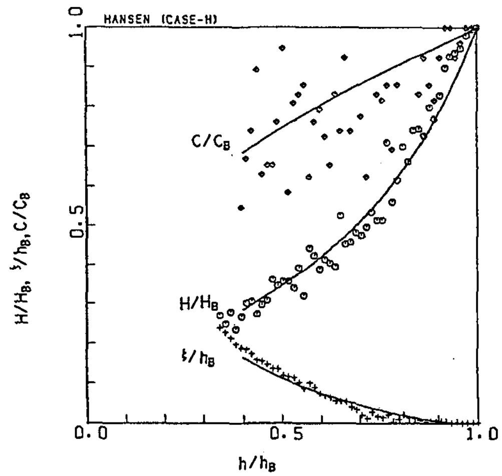



**Fig.3** Changes in the calculated wave celerity $C/C_B$, height $H/H_B$ and mean water level $\zeta/h_B$ in comparison with Hansen and Svendsen's experiments [1].

The vertical velocity distribution of undertow in the inner layer region is compared with experiments by Stive and Wind (1985) [4] in the relative water depth range of $h/h_B = 0.88.0.765, 0.647$ and $0.53$. Comparisons between the calculated velocity vectors and the experiments are shown in **Fig.5**, where solid circles indicate the experimental data of Stive and Wind. The vertical circulation pattern calculated by the 2-D vertical circulation model under their experimental condition is shown in **Fig.6**.

Because of an insufficient quantity of experimental data verification is not possible, however, the theoretical curves agree resonably well with experimens. From the comparison between Stive and Wind's laboratory experiment of undertow and our test calculations, where a uniform distribution of eddy viscosity $\nu_t$ is assumed, it could be concluded that the undertow velocity field in the inner surf zone can be calculated by the 2-DV model developed in this study. For the cases of arbitrary bathymetry, this model is more applicable than the previous 1-DV model.

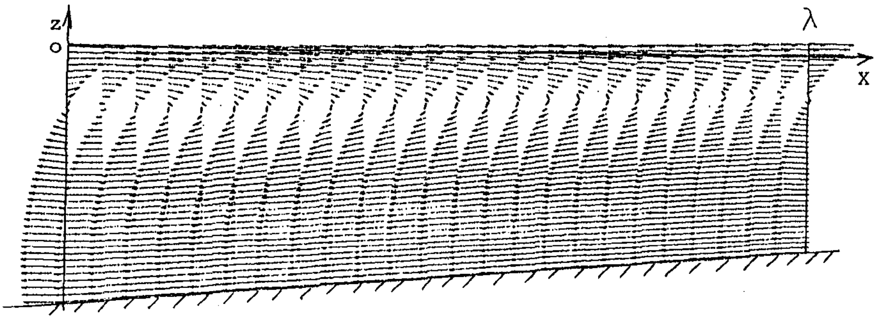



**Fig.4** Test calculation of undertow velocity profile

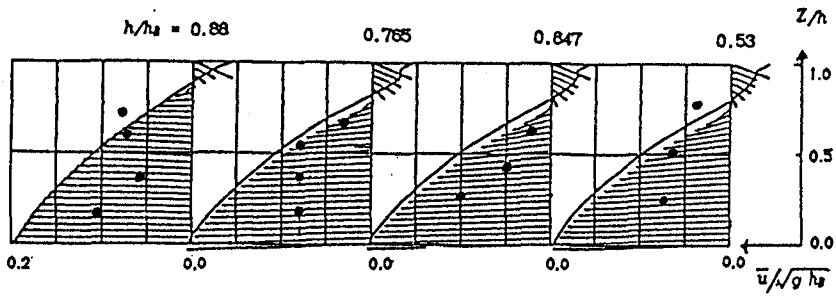



**Fig.5** The vertical distributions of undertow velocity vectors and horizontal velocity profiles (solid curves) by Stive and Wind's experiment [4], at the points of $h/h_B = 0.88, 0.765, 0.647, 0.53$.

Difficulties, however, still exist in improving the side boundary conditions which are in consistency with both mean and turbulent flow fields including bottom and surface boundary conditions. In other words, further extension of the proposed 2-DV model is required to clarify the treatment of the bottom boundary conditions which can satisfy the no-slip condition of mean flow and wall boundary condition of the $k - \varepsilon$ equations. It is emphasized that development of the 2-DV undertow model, which includs turbulent properties, contributes to the progress of surf zone investigations, such as sand transport mechanism, numerical calculation of the equiliburium beach profile, 3-D nearshore circulation model.

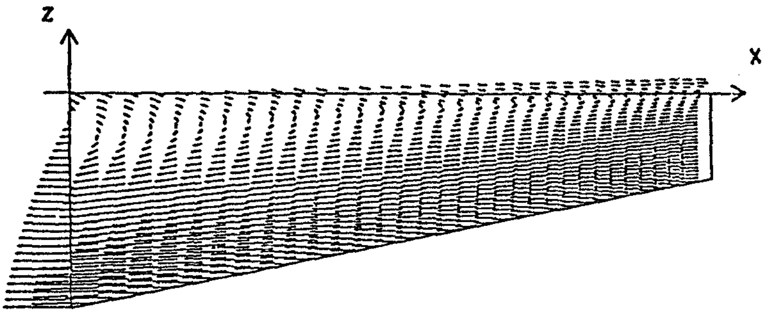



**Fig.6** The vertical circulation field calculated by the model under the experimental condition of Stive and Wind [4].

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