CHAPTER 192

Bore front modeling in terms of Burgers equation and its numerical calculation method

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Abstract

Bore front hydraulics are investigated in terms of Burgers equation to clarify the dynamics of a moving discontinuity in water flow. Burgers equation has been derived from the one dimensional open channel equation with the horizontal turbulent diffusion term. The derived equation system consists of Burgers with respect to dynamic characteristic and hyperbolic equation with respect to water surface elevation, which satisfys Jeffery-Vedernikov condition $F_r = 2$ through discontinuity. It has been verified from the experiments that the condition $F_r = 2$ is a good approximation of bore front dynamics. The numercal calulation method of Burgers equation employing Cole-Hopf transformation and QUICKEST algorithm was also proposed and confirmed its efficiency.

1. INTRODUCTION

Bores, which can often be observed in rivers, estuaries and coastal zones, i.e. hydraulic jumps, flood waves, breaking waves, tidal bores etc., are well known for their complicated flow structures consisting of typical time scales of order of mean flow, waves and turbulence. From the coastal engineering view point, developing our knowledge of bore dynamics may lead to a better understanding of breaker dynamics and flood wave hydraulics. From the view point of numerical calculation for bore front propagation on beaches, dry lands or rivers, more theoretical studies for its propagation speed and hydraulics are needed. Example numerical calculations include flooding simulation of tsunamis, storm surges and run-up of coastal waves. Sometimes an artificial (physically meaningless?) moving boundary condition is forced to apply to numerical models of this type. The so-called threshold depth is defined in the numerical cells in order to establish an artificial moving boundary.

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The basic concept in treating this problem is made by employing the hydraulic jump analogy, where a weak solution of the physical conservation laws is allowed. There are, in general, three conservation laws, mass, momentum and energy, from which we get three sets of jump conditions. For the derivation of shallow water theory, the incompressible fluid approximation is assumed and only two conservation laws are used through the discontinuity.

On the other hand, the so-called Burgers equation is well known as an approximation for dissipative fluid systems, and has been used in analyzing shock-like fluid motions. In this equation, nonlinearity of propagation and diffusive effects are balanced. The main aim of this paper is to determine a method to apply this simple equation to the problem of modeling bore front propagting on the river and dryland.

It is also well known that the advantage in using Burgers equation is that it can be easily solved by transforming it into the linear heat equation by applying the Cole-Hopf transformation. In this paper, we attempt to describe bore front motion by application of Burgers equation, and we propose a solution method which is wiggle-free with no numerical viscosity. Consequcntly, the simple formulation of bore motion by Burgers equation, and its transformed linear version, advection/diffusion equation, can be solved accurately by the QUICKEST algorithm.

We carried out experiments of bore propagation as a dam-break problem. Our purpose was to examine the bore front profile while focusing on the criterion of undular bore creation in the relationship to bore amplitude and water depth, including extremely small depth (dry land condition). Furthermore, the derivation of Burgers equation, with respect to the propagation speed, is accomplished by using the equations of mass and momentum consevation, in which the Gardner-Morikawa (G-M) transformation is employed Finally, we proposed an effective numerical calculation method for the Burgers equation by employing both the Cole-Hopf transformation and the QUICKEST numerical algorithm.

2. BORE FRONT PROPAGATION

2.1 Flood Wave Theories

Assuming a wide and rectangular channel, where the hydrostatic pressure law is valid, the 1-D open channel flow equations are,

$$h_t+(hu)_x = 0 \tag{1}$$

$$u_t+uu_x+g'h_x = g'S-C_f\frac{u^2}{h} \tag{2}$$

where subscripts indicate partial derivatives, $u(x,t)$ and $h(x,t)$ are depth-averaged current, and water depth, respectivly, and $g'=g\cos\theta$, $S=\tan\theta$, θ is bottom slope angle(see Figure 1).

First of all, some consideration on flood waves with

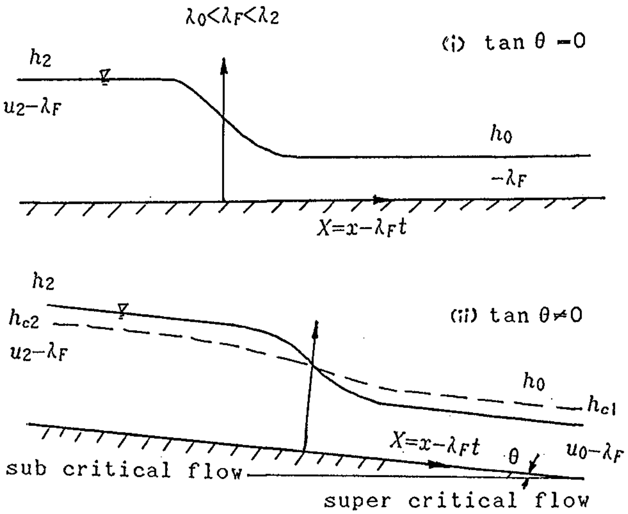

Figure 1. Coordinates and definition of variables

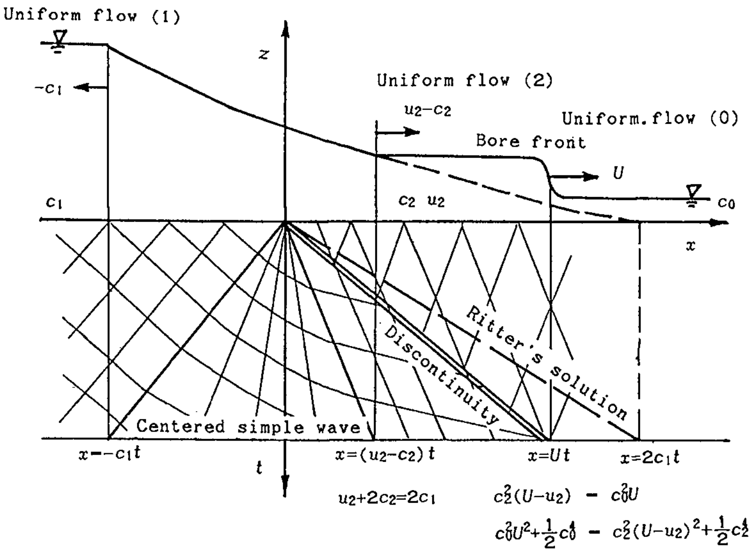

Figure 2. Characteristics of dam break problem.

smooth surface profile are made. If flow motion varies gradually, the kinematic wave approximation is derived, which is called Kleitz-Seddon formula.

$$h_{0t}+\frac{3}{2}u_0h_{0x} = 0, \qquad u_0 = \sqrt{\frac{g\cdot Sh}{C_f}} \tag{3}$$

Whitham(1973) showed the monoclinal flood wave by assuming boundary conditions of kinematic approximation at both up and down stream ends, in the reference frame moving at the speed of λ_F as,

$$h = h(X), \qquad u = v(X), \qquad X = x-\lambda_F t \tag{4}$$

Eqs.(1) and (2) can be written by

$$\frac{dh}{dX} = -\frac{(Q-\lambda_F h)^2 C_f - g' h^3 S}{g' h^3 - Q^2} \tag{5}$$

where $Q = h_1(\lambda_F-u_1) = h_2(\lambda_F-u_2)$ is total discharge, subscripts 1 and 2 indicate the uniform state quantities in the up and down stream directions, respectively. The behavior of the solution of Eq.(5) depends on the sign of the denominator $g' h^3-Q^2$. The numerator must have roots of $h = h_1, h_2$ to be zero at both boundaries, which are the two roots of the cubic. The third root is

$$h_3 = \frac{C_f Q^2}{Sg' h_1 h_2} = \frac{h_1 h_2}{(h_1^{1/2}+h_2^{1/2})^2} \tag{6}$$

We are seeking the solution in the range of $h_1 < h < h_2$. Moreover, since $h_3 < h_1$ and $\lambda_F > u$, the sign of dh/dX depends on the denominator. At the stage of $h_2 \rightarrow h_1$, we get the front propagation speed as,

$$\lambda_F = \frac{3}{2}u_1 \tag{7}$$

This means the smallest amplitude flood wave is the kinematic wave, and a smooth surface solution (monoclinal flood wave) exists when the denominator is positive, which is satisfied in the condition of

$$\frac{3u_1}{2} < \lambda_F < u_1+\sqrt{g' h_1} \tag{8}$$

The smooth disturbance propagates faster than the kinematic wave but slower than the dynamic wave in the flow ahead.

When the disturbance's speed exceeds the downstream dynamic wave's

$$\lambda_1 = u_1+\sqrt{g' h_1} < \lambda_F < \lambda_2 = u_2+\sqrt{g' h_2} \tag{9}$$

no smooth surface solution exists. Flow changes from supercritical to subcritical in the reference frame (see Figure

1), which means the initiation of a breaking front (bore front).
Now we consider the higher order contibution of the advection term by introducing a small paramcter ε perturbed from the kimematic solution of u_0, h_0, and assuming constant S and C_f,

$$u = u_0 + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \ldots..$$

$$h = h_0 + \varepsilon h^{(1)} + \varepsilon^2 h^{(2)} + \ldots.. \qquad (10)$$

In the order of $O(\varepsilon)$, we get

$$h_t^{(1)} + u_0 h_x^{(1)} + h_0 u_x^{(1)} = 0 \qquad (11)$$

$$u_t^{(1)} + u_0 u_x^{(1)} + g' h_x^{(1)} + g' S\left(\frac{2u^{(1)}}{u_0} - \frac{h^{(1)}}{h_0}\right) = 0 \qquad (12)$$

From Eqs.(11) and (12) eliminating $u^{(1)}$, we get the single equation for $h^{(1)}$ in the form

$$\left(\frac{\partial}{\partial t} + c_+ \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c_- \frac{\partial}{\partial x}\right) h^{(1)} + \frac{2g' S}{u_0}\left(\frac{\partial}{\partial t} + \frac{3}{2} u_0 \frac{\partial}{\partial x}\right) h^{(1)} = 0 \qquad (13)$$

where $c_\pm = u_0 \pm \sqrt{g' h_0}$. In Eq.(13) the first term indicates dynamic characteristics and the second term the kinematic one.

If the approximation of $\partial/\partial t \sim 3u_0/2\,\partial/\partial x$ is applied in Eq.(13), we get the linear Burgers equation as,

$$h_t^{(1)} + \frac{3}{2} u_0 h_x^{(1)} = \frac{u_0}{2g' S}\left(g' h_0 - \frac{u_0^2}{4}\right) h_{xx}^{(1)} \qquad (14)$$

The right hand side of Eq.(14) is the diffusion term. If its sign is a positive, solution of this equation is a diffusive type, and the negative sign leads to an explosive solution which means there is no solution balancing advection and diffusion. This criterion $F_r \gtreqless 2$ is called Jeffery-Vedernikov condition, where F_r is Froude number defined as,

$$F_r = \frac{u_0}{\sqrt{g' h_0}} = 2 \qquad (15)$$

In other words, this condition depicts the criterion of the balancing of nonlinearity and dissipation as well as the initiation of the bore front.

After the discontinuity appears, the weak solution is assumed to the conservation laws, such as the hydraulic jump analysis. However, we sometimes need information of flow quantities inside the discontinuity. For this we need a more complicated constitution equation system including turbulence dynamics.

On the other hand, the simplest equation which describes flow field inside the discontinuity is Burgers equation. As long as the depth-averaged quantities are allowed, it might be easy-to-use the field equation. From the consideration of higher order effects in the flood wave, it is supposed that Burgers equation should describe the local balance of nonlinearity and dissipstion effects through the discontinuity.

2.2 Dam break problem

To make sure the bore front propagation speed we considered here the dam break problem shown in Figure 2. Assuming the invisid fluid flow with hydrostatic pressure distribution on the horizontal bottom, the open channel flow equations (1) and (2) can be rewritten by,

$$\left\{\frac{\partial}{\partial t}+(u\pm c)\frac{\partial}{\partial x}\right\}(u\pm 2c) = 0 \tag{16}$$

where $c = \sqrt{gh}$, u is the depth-averaged flow velocity. If $c_0 = 0$ (dry land), the Ritter's solution is obtained, which consists of two regions, zone of silence and centered simple wave. In the case of $c_0 \neq 0$, the only possible configuration consists of three uniform flow regions and a centered simple wave, as shown in Figure 2. The discontinuity must occure in the region between uniform flow regions (0) and (2) shown in the figure. For determination of the unknown propagation speed of the discontinuity $\lambda_F = U$, c_2 and u_2 in the uniform flow region (2), we need additional conservation law. Other than mass conservation we could use the momentum or energy conservation law but both. Our choice for this problem is that of momentum.

One more relation can be obtained on the characteristic of $x = (u_2-c_2)t$ where the Riemann invariant $u+2c = 2c_1$ in the region of centered simple wave is given by the known quantity c_1. Consequently we get three equations, as

$$u_2+2c_2 = 2c_1 \tag{17}$$

$$c_2^2(U-u_2) = c_0^2 U \tag{18}$$

$$c_0^2U^2+\frac{1}{2}c_0^4 = c_2^2(U-u_2)^2+\frac{1}{2}c_2^4 \tag{19}$$

Combining Eqs.(18) and (19), third order polynomial equation with respect to U is obtained.

$$U^3-2u_2U^2+\{(u_2)^2-c_0^2\}U+\frac{1}{2}u_2c_0^2 = 0 \tag{20}$$

From Eqs.(17) and (18), the third order polynomial equation with respect to u_2 is derived.

$$u_2^3-(4c_1+U)u_2^2+4c_1(c_1+U)u_2-4U(c_1^2-c_0^2) = 0 \tag{21}$$

To solve the above set of two equations, we consider Eq.(20), the second order polynomial equation with respect to u_2 and two its solutions $u_{2(\pm)}$ are

$$u_{2(\pm)} = \frac{1}{4U}\left\{4U^2 - c_0^2 \pm c_0\sqrt{c_0^2+8U^2}\right\} \tag{22}$$

At the limit of $c_1 \to c_0$, u_2 should be zero. This condition is satisfied by $u_{2(-)}$. Substituting $u_{2(-)}$ into Eq.(22) and sloving the resultant equation numerically in the range of $c_0 < U < 2c_1$, we get the configulation of solutions of U, u_2, c_2 as shown in Figure 3, where three regions of breaking front with undular component, strong bore front and Burgers front are zoned in terms of $\beta = c_0/c_1$. In addition to these other information such as $F_r = U/c_2$, $F_{r2} = u_2/c_2$ and U/c_0 are shown.

2.3 Experiment of bore front propagation

Experiments were conducted as a dam break problem by using a flume and gate system with dimensions of 0.7m in width and height (Figure 4). Surface profiles were measured by 5 wave gauges of capacitance type. Flow visualization techniques werc employed for measuring the mean velocity fields near the bore front. The initial water depth in front of the gate was varied in order to examine the effects of downstream boundary condition to the bore propagation. The experimental conditions are presented in Table 1, where h_0, h_1 and h_2 are initial water depths of up/down-stream sides of the gate and water depth in the uniform region formed just after bore front, *i.e.* uniform flow (2) in Figure 2.

Table 1 Experimental conditions

Run No.	h_1 (cm)	h_0 (cm)	c_0/c_1	h_2 (cm) crest-mean	c_2/c_1 crest-mean
1	19.8	1.0	0.225	5.14	0.557
2	20.9	3.8	0.426	7.71-8.0	0.608-0.62
3	22.6	9.8	0.659	6.00-9.0	0.515-0.63
4	25.6	12.8	0.707	6.94-10.5	0.521-0.64
5	19.3	2.2	0.338	6.51	0.581
6	24.8	0.8	0.180	6.64	0.517
7	14.7	0.8	0.233	6.43	0.661
8	8.0	0.9	0.335	3.94	0.702

Figure 5 shows two typical examples of measured bore profiles above the initial still water level (Run No.3 and 6). It can be seen that undular bores are observed in the wave profiles of the case whose initial still water is deep, however there is no undular bore in the case of an extremery shallow initial depth. This fact can be explained by using the results shown in Figure 3. From the consideration of characteristics in the uniform region (2), the undular bore criterion is $u_2 = c_2$, which corresponds to the intersection of the curves c_2/c_1 and u_2/c_1. Circles in Figure 3 indicate the experimental results of U/c_1 and c_2/c_1. Numbers indicate Run No. in Table 1. Solid circles represent the cases of undular bores and white ones are strong bore fronts without undular components. Figure 6 shows Froude numbers F_{r2} calculated by measured currents and bore heights. It can be confirmed that Froude numbers exist in the range, $1 < F_{r2} < 2$.

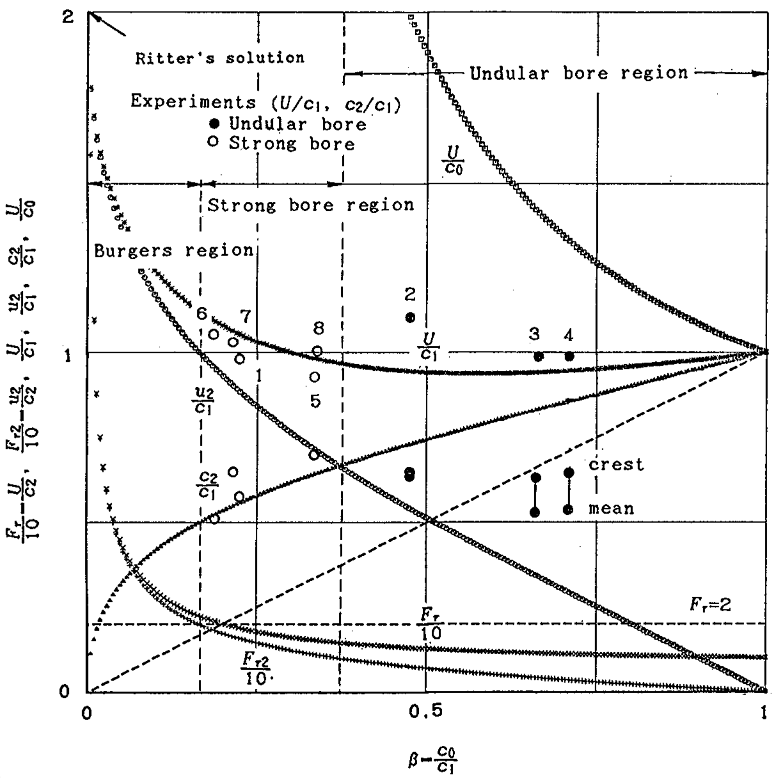

Figure 3. Solutions of dam break problem.

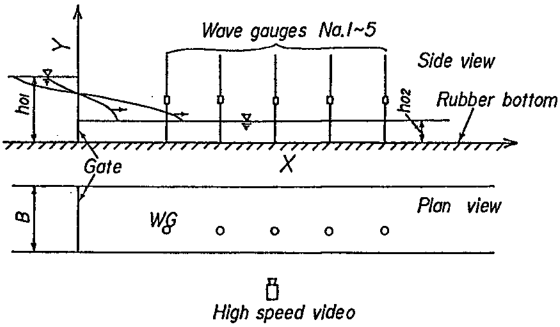

Figure 4. Set-up of experiment.

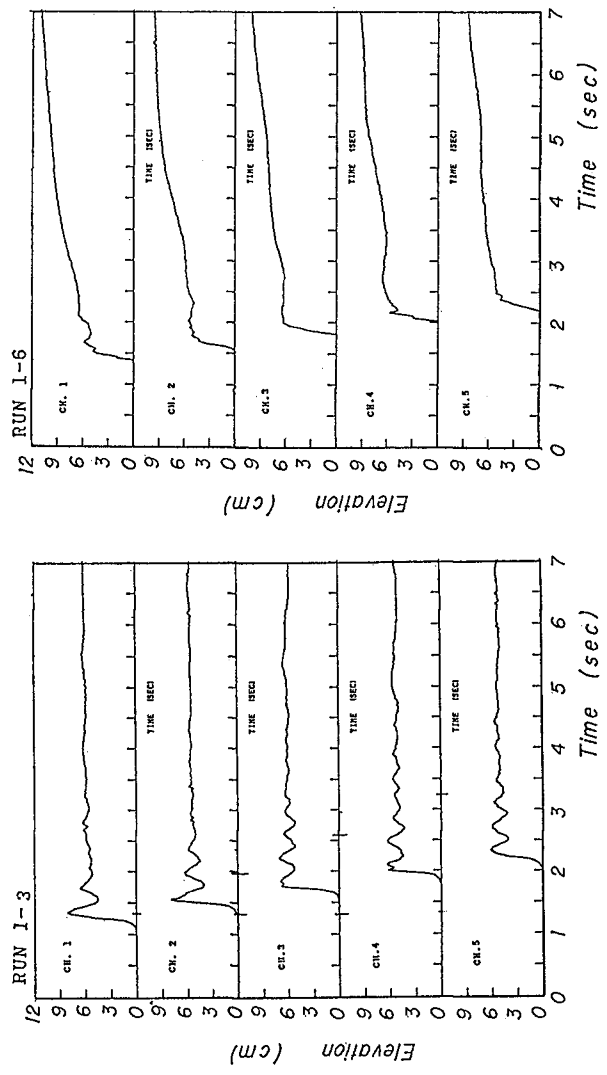

Figure 5. Examples of bore front surface profiles.
(above the still water level)

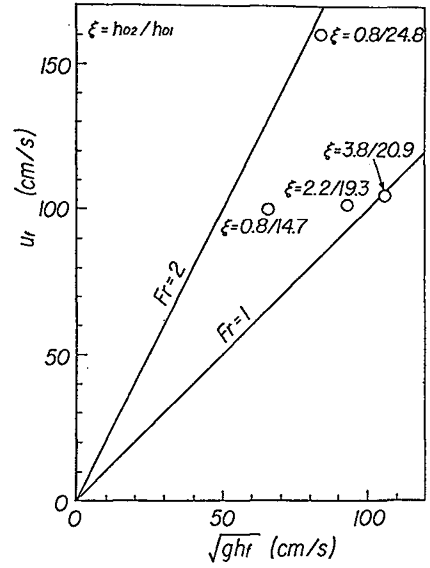

Figure 6. Experimental results of bore front Froude number.

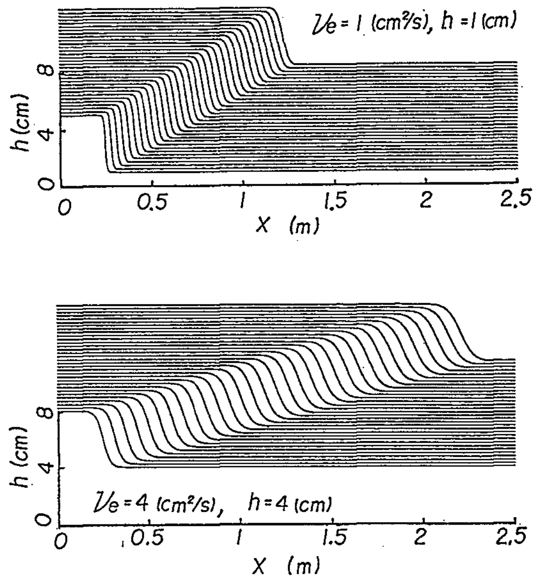

Figure 7. Test runs of numerical calculation of Burgers eq.

In this figure it should be noted that bore propagating speed U/c_1 keeps almost constant up to the point of $F_{r2} = 2$, which is corresponding to Jeffery-Vedernikov condition. Futhermore experimental results of U/c_1 have good agreement with the curve calculated by nonlinear long wave theory, especially in the region of strong bore. Unfortunately there is no data in the region of $F_{r2} > 2$, however, it might be that the nonlinearity of the propagation and turbulent mixing effects balance each other to keep the relation of $F_{r2} = 2$ in this region. This is the basic conception of Burgers approximation which will be elaborated on in next section. It is also mentioned in the figure that the dispersion effect can not be neglected in the region of $F_{r2} < 1$, in which experimental reults in c_2/c_1 are smaller than the theoretical one. This result indicates that undular bore components sustain their energy in the form of wave motion while some of them are transported in the backward direction.

3. BORE FRONT MODELING BY BURGERS EQUATION

3.1 Derivation of Burgers Equation

To introduce the balance effect of nonlinearity and dissipation, the horizontal momentum diffusive effect, $\nu_e u_{xx}$, is considered in the momentum equation (2).

$$u_t+uu_x+g'h_x = g'S-C_f\frac{u^2}{h}+\nu_e u_{xx}, \tag{23}$$

where ν_e is the horizontal turbulent eddy viscosity.

First of all, for the 0-th order(basic) flow condition, uniform flow, still water or dry land must be defined. For the open channel flow balancing between gravitational force and bottom friction the kinematic wave condition is the lowest order of the flood wave formulation called Kleitz-Seddon law. In addition to this it is possible to apply Burgers equation to the analysis of a discontinuity propagating on the 0-th order condition of still water or horizontal dry land. To achieve this aim the higher order effects of nonlinearity of propagation and turbulent diffusivity should be considered in the momemtum equation, which should be described by Burgers equation. Employing the formulation of nonlinear long wave theory in terms of Riemann invariant, two variables of the propagation speed of disturbance defined by $c = \sqrt{gh}$ and the characteristic speed defined by $\lambda = u+c$ are choosen. The basic equations are rewritten using these variables as,

$$c_t+(\lambda-\frac{3}{2}c)c_x+\frac{1}{2}c\lambda_x = 0 \tag{24}$$

$$\lambda_t+(\lambda-\frac{1}{2}c)_x+\frac{3}{2}cc_x=g\tan\theta\frac{C_f u_0^2}{h_0}+\nu_e\lambda_{xx}+\nu_e c_{xx} \tag{25}$$

Based on the 0-th order solutions (kinematic waves or still water) 1-st order (Burgers discontinuity) and higher order components are introduced by

$$\lambda = \lambda_0+\varepsilon\lambda^{(1)}+\varepsilon^2\lambda^{(2)}+\cdots$$

$$c = c_0+\varepsilon c^{(1)}+\varepsilon^2 c^{(2)}+\cdots \tag{26}$$

where ε is defined by the parameter of magnitude of discontinuity, *i.e.* amplitude of discontinuity

$$\varepsilon = \frac{(\lambda_{-\infty}-\lambda_{\infty})}{\lambda^{(1)}_{-\infty}}. \tag{27}$$

From the similitude of the Gardner-Morikawa transformation balancing equation, the nonlinearity of propagation and diffusivity are achieved by the following transformation (Taniuchi & Nishihara (1977)).

$$\xi = \varepsilon(x-\lambda_0 t), \qquad \tau = \varepsilon^2 t \tag{28}$$

Applying the G-M transformation to the fundamental equations (24),(25) and substituting Eq(26), we get Burgers equation

$$\lambda_\tau^{(1)}+\lambda^{(1)}\lambda_\xi^{(1)} = \frac{2\nu_e}{3}\lambda_{\xi\xi}^{(1)} \tag{29}$$

and hyperbolic equation of $s^{(1)}$

$$c_\tau^{(1)}+\lambda^{(1)}c_\xi^{(1)} = 0, \qquad \lambda^{(1)} = \frac{3}{2}u^{(1)} \tag{30}$$

These equations (29) and (30) are the Burgers equation system derived here for the problem of bore front propagation. Notice that discontinuity h or c propagates by the speed of $3u/2$ and the dynamic characteristic speed λ is described by Burgers equation (29). This also means that λ satisfys the Jeffery-Vedernikov condition which describes the intersection of the dynamic and kinematic characteristics through the discontinuity. As the discontinuity becomes greater, horizontal turbulent diffusivity increases in order to decrease the propagation speed, which is the balancing of effects of nonlinearity of propagation and horizontal diffusivity of momentum. The bottom friction does not contribute to the propagation of Burgers components.

For the practical purpose, we can calculate the front propagation speed $\lambda = \lambda_0+\varepsilon\lambda^{(1)}$ and surface elevation h and current u by using the resultant set of equations when the up-stream boundary condition $\lambda_{-\infty}$ and 0-th order flow (basic flow) condition are given. If a bore propagates on the horizontal dry land and uniform flow exists in the up-stream region, $\xi = -\infty$, applying the basic flow condition, $\lambda_0 = 0$, the bore front speed can be given by $\lambda_F = \lambda_{-\infty}/2$.

The bore front formulation by using Burgers equation mentioned here might have following advantages.

(1) Continuous flow structures through the discontinuity can be calculated by the depth-averaged quantities.

(2) Bore front propagation speed and profile (front length) are also evaluated.

(3) Burgers equation can be transformed into the linear

advection/diffusion equation by the nonlinear transfomation, which makes thc numerical calculation easy; this will be refered in the following scction.

4. NUMERICAL CALCULATION METHOD OF BURGERS EQUATION

Burgers cquation has an interesting property, in that it can be transformed into a linear equation (heat equation) by the Cole-Hopf transformation

$$h = -2\nu\varphi_x/\varphi, \quad \nu = \frac{2\nu_e}{3} \tag{31}$$

Applying this transformation to the derived Burgers equation (35) in the fixed coordinates, the advection/diffusion linear PDE is obtained

$$\varphi_t + \lambda_0\varphi_x = \mu\varphi_{xx}, \qquad \mu = \frac{2\nu_e}{3} \tag{32}$$

Leonard(1979) has developed a third order accurate numerical scheme for this equation called the QUICKEST algorithm. Basco(1984) has shown that using the quardratic upstream interpolation operator for the advection term of Eq.(5) along with a forward time operator and centered second derivative operator, is identically equal to a forward-time and centered advection operator(FTCS) plus removal of truncation error by employing a special technique. The resulting difference equation is

$$\varphi_j^{n+1} = \varphi_j^n + \{\rho(1-C_r) - \frac{C_r}{6}(C_r^2-3C_r+2)\}\varphi_{j+1}^n - \{\rho(2-3C_r) - \frac{C_r}{2}(C_r^2 -2C_r-1)\}\varphi_j^n + \{\rho(1-3C_r) - \frac{C_r}{2}(C_r^2-C_r-2)\}\varphi_{j-1}^n + \{\rho C_r + \frac{C_r}{6}(C_r^2 -1)\}\varphi_{j-2}^n \tag{33}$$

where $C_r = c_0\Delta t/\Delta x$, $\rho = \mu\Delta t/\Delta x^2$ and $c_0 = \sqrt{gh_0}$.
The abovementioned method of solving Burgers equation is wiggle-free, accurate and low cost.

Example testing of this method was performed by using propagation test of a bore whose profile was given by the kink solution of

$$h = h_0 + a\tanh\{-\frac{a}{2\nu}(x-x_0)\} \tag{34}$$

where a is half amplitude x_0 is initial position of bore, and the corresponding transformed profile is given by

$$\varphi = C\exp\{-\frac{1}{2\nu}\int h dx\} \tag{35}$$

Using this initial condition and the boundary conditions of constant water depth in the far field, example solutions of Burges equation were calculated as shown in

Figure 7.

5. CONCLUSIONS

Bore front hydraulics were invcstigated in terms of Burgers equation and experiments were carried out to make clear the dynamics of moving discontinuity of water flow. The maim results obtained are:
(1) Burgers equation has been derived from the one dimensional open channel flow equation with horizontal momentum mixing term. The derived equation system consists of Burgers with respect to dynamic characteristic and hyperbolic equation with respect to water surface, which satisfys the Jeffery-Vedernikov relation $F_{r2} = 2$ through discontinuity.
(2) It is made clear from experiments that in the region of $1 < F_{r2} < 2$ nonlinear long wave theory has a good applicability for bore front propagation, however, in the region of $F_{r2} > 2$ turbulent diffusivity of momentum has to be considered in terms of Burgers equation. Moreover, in the region of $F_{r2} < 1$ the dispersion effect is not negligible in the observed undular bores.
(3) An accurate numerical calculation method to solve Burgers equation has also been proposed by employing the Cole-Hopf transformation and QUICKEST algorithm.

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