CHAPTER 190

BOUNDARY CONDITIONS IN FINITE ELEMENT MODELING OF STRATIFIED COASTAL CIRCULATION

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ABSTRACT

The effect of boundary conditions on numerical computations of stratified flow in coastal waters is examined. Clamped, free radiation and sponge layer conditions are implemented in a two-layer finite element model and the results of simple tests in a two-layer stratified basin are presented.

INTRODUCTION

Numerical models of coastal circulation require the specification of boundary conditions at the sea boundaries, where the computational domain is artificially terminated. In 2-D depth-averaged models conditions are normally imposed on the vertical displacement of the free surface at the grid nodes located on the boundary. Most commonly, the surface displacements are prescribed in a desired manner, according to either a known tidal input or to an elevation time series available from field data. This type of b.c. is referred to as clamped, since the position of the surface at the boundary is absolutely defined a priori for the entire computation. Other types of b.c. often employed include the Sommerfeld free radiation conditions in various forms, which allow disturbances created within the domain to travel freely outwards, or sponging functions, which serve as filters absorbing outgoing waves. The importance of boundary conditions has been recognized in recent years and the development of improved formulations as well as their implementation in discretized form has been the subject of several investigations (Orlanski, 1976; Enquist and Majda, 1977; Miller and Thorpe, 1981; Roed and Cooper, 1986). Evidence from recent research (Blumberg and Kantha, 1985; Roed and Cooper, 1986) indicates

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that the type of boundary conditions used may have an appreciable influence on the computed circulation patterns, especially for transient problems.

During the summer, many coastal water bodies become stratified, primarily due to substantial temperature differences over the water column. Stratification may also develop in estuaries receiving large freshwater inflows, due to salinity differences. The simplest way to study such water bodies is by a two-layer idealization. However, in a two-layer system the problem of boundary conditions becomes more complicated, since conditions have to be specified not only for the free surface, but also for the interface, where data are usually lacking. The effect of the treatment of b.c. on the resulting circulation of layered systems has received little attention (Lorrenzzetti et al, 1985), although it may be at least as important as in the 2-D case.

In an effort to assess the importance of boundary conditions in hydrodynamic modeling of stratified coastal waters, the present paper examines the influence of some common types of b.c. on the wind-driven circulation of a two-layer stratified basin. In the following, the implementation of the various boundary conditions and the results obtained for some simple test problems are presented and discussed.

GOVERNING EQUATIONS AND NUMERICAL MODELING

For the purposes of this study an available two-layer finite element circulation model, code named CAFE-2 (Wang and Connor, 1975; Pagenkopf et al, 1976) was employed as a tool, after some suitable modifications for handling the various types of b.c. considered. It is felt, however, that the results obtained would be equally applicable, at least qualitatively, to other numerical models as well. The model used solves the following set of governing equations for the two-layer system, where the subscript 1 denotes the bottom layer and the subscript 2 the top layer (see Fig. 1):

$$\frac{\partial H_1}{\partial t} + \frac{\partial q_{1x}}{\partial x} + \frac{\partial q_{1y}}{\partial y} = q_{01} \tag{1}$$

$$\frac{\partial q_{1x}}{\partial t} + \frac{\partial}{\partial x}(\bar{u}_1 q_{1x}) + \frac{\partial}{\partial x}(\bar{u}_1 q_{1y}) - f q_{1y} = -\frac{\partial F_{1p}}{\partial x} + \frac{\partial F_{1xx}}{\partial x} + \frac{\partial F_{1xy}}{\partial y} + \frac{1}{\rho_1}(\tau_x{}^i - \tau_x{}^b) + \frac{1}{\rho_1}\left(p^i \frac{\partial \zeta_1}{\partial x} - p^b \frac{\partial \zeta_o}{\partial x}\right) \tag{2}$$

$$\frac{\partial q_{1y}}{\partial t} + \frac{\partial}{\partial x}(\bar{u}_1 q_{1y}) + \frac{\partial}{\partial y}(\bar{v}_1 q_{1y}) + f q_{1x} = -\frac{\partial F_{1p}}{\partial y} + \frac{\partial F_{1xy}}{\partial x} + \frac{\partial F_{1yy}}{\partial y} + \frac{1}{\rho_1}(\tau_y{}^i - \tau_y{}^b) + \frac{1}{\rho_1}\left(p^i \frac{\partial \zeta_1}{\partial y} - p^b \frac{\partial \zeta_o}{\partial y}\right) \tag{3}$$

$$\frac{\partial H_2}{\partial t} + \frac{\partial q_{2x}}{\partial x} + \frac{\partial q_{2y}}{\partial y} = q_{02} \tag{4}$$

$$\frac{\partial q_{2x}}{\partial t} + \frac{\partial}{\partial x}(\bar{u}_2 q_{2x}) + \frac{\partial}{\partial y}(\bar{v}_2 q_{2x}) - f q_{2y} = - \frac{\partial F_{2p}}{\partial x} + \frac{\partial F_{2xx}}{\partial x} + \frac{\partial F_{2xy}}{\partial y} + \\ + \frac{1}{\rho_2}(\tau_x{}^s - \tau_x{}^i) + \frac{1}{\rho_2}(p^s \frac{\partial \zeta_2}{\partial x} - p^i \frac{\partial \zeta_1}{\partial x}) \tag{5}$$

$$\frac{\partial q_{2y}}{\partial t} + \frac{\partial}{\partial y}(\bar{u}_2 q_{2y}) + \frac{\partial}{\partial y}(\bar{v}_2 q_{2y}) + f q_{2x} = - \frac{\partial F_{2p}}{\partial y} + \frac{\partial F_{2xy}}{\partial x} + \frac{\partial F_{2yy}}{\partial y} + \\ + \frac{1}{\rho_2}(\tau_y{}^s - \tau_y{}^i) + \frac{1}{\rho_2}(p^s \frac{\partial \zeta_2}{\partial y} - p^i \frac{\partial \zeta_1}{\partial y}) \tag{6}$$

where ζ_o, ζ_1, ζ_2 are the elevations of the bottom, interface and free surface from an arbitrary datum, respectively.

H_1 = $\zeta_1 - \zeta_o$ = bottom layer thickness

H_2 = $\zeta_2 - \zeta_1$ = top layer thickness

ρ_1, ρ_2 are the layer densities.

$\bar{u}_i$, $\bar{v}_i$ are the x,y components of the mean layer velocity (i=1,2)

q_{xi}, q_{yi} are the x,y fluxes ($q_x = \bar{u}H$, $q_y = \bar{v}H$)

q_{oi} is any external input, eg. rainfall

f = $2\omega \sin\varphi$, is the Coriolis parameter

p^s, p^i, p^b denote the pressure at the free surface, interface and bottom, respectively.

τ^s, τ^i, τ^b denote the shear stress at the free surface, interface and bottom, respectively, which are modeled by quadratic laws.

F_{pi} is the total pressure force over the layer thickness

F_{xxi}, F_{xyi}, F_{yyi} are the integrated internal stresses due to molecular viscosity, turbulent velocity fluctuations and spatial velocity nonuniformity over the vertical, which are modeled by the eddy viscosity approximation.

By employing linear triangular elements for the spatial discretization, the above partial differential equations are reduced to a system of ordinary differential equations, which is further discretized in time as follows (i=1,2):

$$\underset{\sim}{M} \frac{\underset{\sim}{H}_i^{n+\frac{1}{2}} - \underset{\sim}{H}_i^{n-\frac{1}{2}}}{\Delta t} = \underset{\sim}{P}_i \; (H^{n-\frac{1}{2}}, \; Q^n) \tag{7}$$

$$\underset{\sim}{M}' \frac{\underset{\sim}{Q}_i^{n+1} - \underset{\sim}{Q}_i^{n}}{\Delta t} = \underset{\sim}{P}_i'(H^{n+\frac{1}{2}}, Q^n) \tag{8}$$

where $\underset{\sim}{H}_i$ are the vectors of layer thickness at the nodes

$\underset{\sim}{Q}_i(\underset{\sim}{q}_{xi}, \underset{\sim}{q}_{yi})$ are the vectors of fluxes at the nodes

$\underset{\sim}{M}, \underset{\sim}{M}'$ are matrices related to the geometry of the grid

and $\underset{\sim}{P}, \underset{\sim}{P}'$ include all other terms, i.e. pressure gradient, friction, eddy viscosity, Coriolis, convective terms etc.

The time-stepping scheme employed solves for the dependent variables at alternate time levels, i.e. for the layer thicknesses $\underset{\sim}{H}_1$, $\underset{\sim}{H}_2$ at times n+1/2, n+3/2 and for the fluxes $\underset{\sim}{Q}_1$, $\underset{\sim}{Q}_2$ at times n, n+1, n+2 etc.

IMPLEMENTATION OF BOUNDARY CONDITIONS

For the numerical tests an idealized basin was considered, shown in Fig. 2. It has a rectangular shape, with dimensions about 40x60 Km, and an island near the center. The water depth was assumed constant, at 100 m, and the initial position of the interface was set at a depth of 35 m. The density difference between the layers was assumed to be 2‰. As shown in the figure, a regular grid with 86 nodes and 129 triangular elements was established. Three types of boundary conditions were considered for the wind-driven circulation, as follows:

(a) Clamped boundaries: Zero displacement was specified for both the free surface and the interface at all nodes on the sea boundary, so that the layer thicknesses were fixed:

$$H_1 = \bar{H}_1 = 65\text{m} \tag{9a}$$

$$H_2 = \bar{H}_2 = 35 \text{ m} \tag{9b}$$

(b) Free radiation: Extending the well-known Sommerfeld radiation condition (Roed and Cooper, 1986) and considering as celerity that of long gravity waves, the following equations are applicable for the displacements of the free surface and the interface at the boundary nodes:

$$\frac{\partial \zeta_2}{\partial t} + \sqrt{g(H_1+H_2)} \ \frac{\partial \zeta_2}{\partial \eta} = 0 \tag{10a}$$

$$\frac{\partial \zeta_1}{\partial t} + \sqrt{g' \frac{H_1 H_2}{(H_1+H_2)}} \ \frac{\partial \zeta_1}{\partial \eta} = 0 \tag{10b}$$

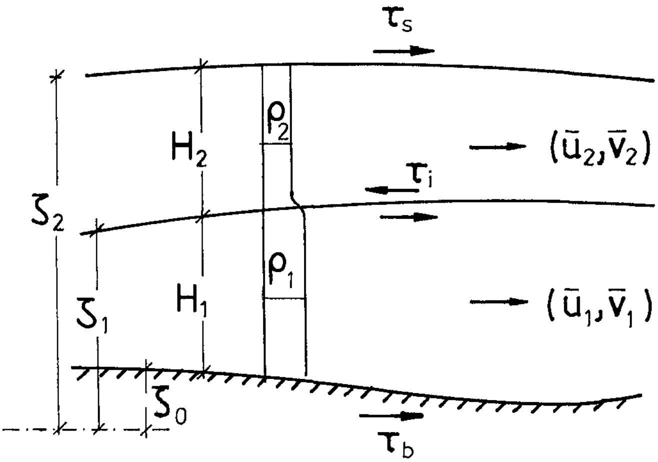

Figure 1. Notation sketch

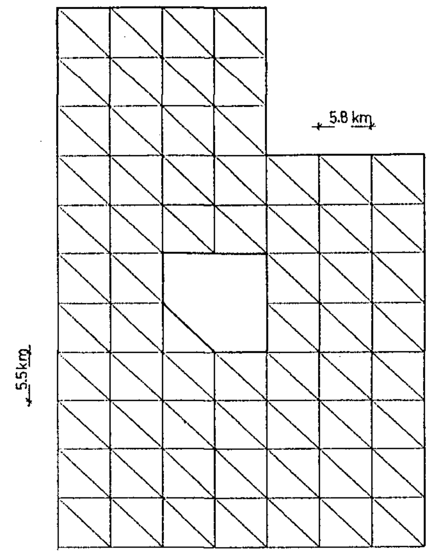

Figure 2. Idealized basin for the numerical tests

where $g' = g(\rho_1 - \rho_2)/\rho_1$

and η is the direction of the outward normal to the boundary.

To obtain the boundary values at each time step, the above equations were discretized by employing forward differences in time and backward in space, as follows.

$$\frac{\zeta_{2,b}^{n+1} - \zeta_{2,b}^{n}}{\Delta t} + \sqrt{g(H_{1,b}^{n} + H_{2,b}^{n})}\ \frac{\zeta_{2,b}^{n} - \zeta_{2,a}^{n}}{\Delta\eta} = 0 \qquad (11a)$$

$$\frac{\zeta_{1,b}^{n+1} - \zeta_{1,b}^{n}}{\Delta t} + \sqrt{g'\frac{H_{1,b}^{n}\ H_{2,b}^{n}}{(H_{1,b}^{n} + H_{2,b}^{n})}}\ \frac{\zeta_{1,b}^{n} - \zeta_{1,a}^{n}}{\Delta\eta} = 0 \qquad (11b)$$

where the subscript b denotes the boundary node, the subscript a refers to the adjacent node to the interior of the domain, and $\Delta\eta$ is the distance between these two nodes.

(c) Sponge layer: The displacements of the boundary nodes were set to zero, as in case (a), but the elevation fields were sponged within a layer adjacent to the boundary, following the scheme proposed by (Lorrenzetti et al, 1985). To accommodate the sponge layer of a chosen thickness $L=2\Delta s$ the grid was locally refined in this case, so that four element were formed in this zone. The integration scheme (eq.7) was modified as follows:

$$\underset{\sim}{M}\ \frac{\underset{\sim i}{H}^{n+\frac{1}{2}} - \underset{\sim i}{H}^{n-\frac{1}{2}}}{\Delta t} + \underset{\sim}{K}\ \frac{\underset{\sim i}{H}^{n-\frac{1}{2}} - \underset{\sim i}{\bar{H}}^{n-\frac{1}{2}}}{\Delta t} = \underset{\sim i}{P} \qquad (12)$$

where $\underset{\sim}{\bar{H}}$ includes the specified depths at the boundary

and $\underset{\sim}{K}$ is a damping function depending on the ordinate η normal to the boundary:

$$K(\eta) = A\left(\frac{\eta-\eta_o}{L-\eta_o}\right)^x \qquad \text{for} \qquad \eta_o < \eta \leqslant L \qquad (13a)$$

$$K(\eta) = 0 \qquad \text{for} \qquad \eta \leqslant \eta_o \qquad (13b)$$

The behaviour of the sponge layer depends to some extent on the values of the parameters A, x, in eq. (13a). The results shown below correspond to values of A=1.0 and x=0.5.

RESULTS AND DISCUSSION

The b.c. described above were used for studying the circulation in the rectangular basin of Fig. 2 under uniform wind forcing. For these tests, the non-linear convective terms and the Coriolis terms were omitted in the computations. Preliminary experiments indicated that considerable oscillations of the interface persist for several days after a sudden application of the wind field. Therefore to reduce the impact of sudden forcing, the wind was introduced gradually, using a linear increase from zero to 20 knots over a period of 90,000 sec; the wind speed was kept constant thereafter.

Figures 3, 4, 5 show the circulation patterns calculated at 360,000 sec for a 20-knot wind normal to the open boundary, directed inwards,for each of the three types of b.c. described previously. It is seen that in all cases an inflow is established in the top layer and outflow in the bottom layer, as expected for this particular wind direction. However, there are significant differences between the three cases. For the clamped boundaries, an anticlockwise gyre around the island forms in the top layer,with stronger velocities on the right side of the basin, and a reverse gyre in the bottom layer. Somewhat similar results are observed for the sponge layer conditions, but with comparable intensity of flow on both sides of the basin. However, the free radiation conditions generate a markedly different flow field, with nearly uniform inflow over the entire top layer and outflow at the bottom layer. Substantial differences are also noted in the displacement of the interface, which under the free radiation conditions descends much lower throughout the domain compared to the other two tests.

Fig. 6 presents the evolution of the differential volume of each layer relative to the initial position of the free surface and the interface. It is evident from the figure that under clamped or sponge boundary conditions, the resulting flow fields attain macroscopically nearly the same steady state after about 2 days of wind forcing, while no steady state is approached after 4 days for the free radiation conditions. In fact, there seems to be a gradual filling of the top layer and a corresponding emptying of the bottom layer. This result is qualitatively similar to that of Blumberg and Kantha (1985), who also observed in their 2-D model a continuous emptying of the domain when using free radiation conditions.

Figures 7, 8, 9 show the circulation pattern for a 20-knot wind parallel to the open boundary. Under clamped b.c., more intense currents are observed in this case. The flow generally tends to follow the wind direction on both sides of the island and in both layers, and outflow occurs on the opposite end of the sea boundary. Rather similar results are obtained under the sponge conditions. But again, the free radiation conditions yield a quite different flow pattern, where there is inflow along the entire opening in the top layer and outflow in the bottom layer. The behaviour of the differential volumes in this case is shown in Fig. 10.

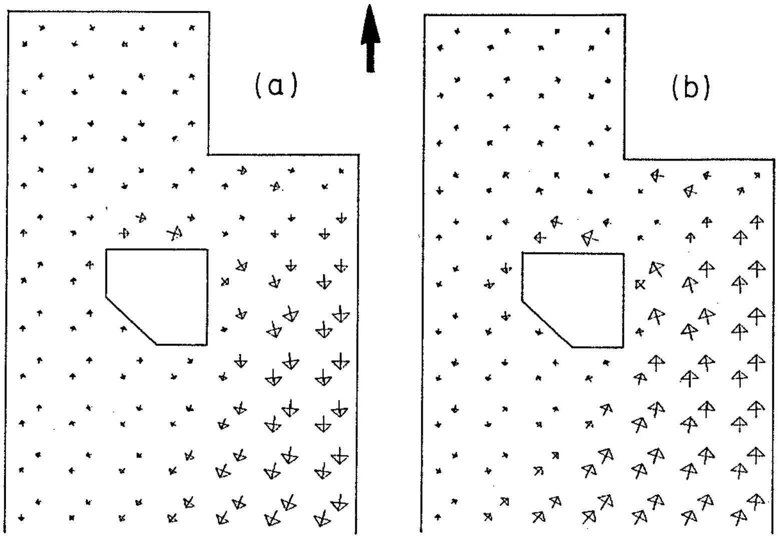

Figure 3. Flow field for wind normal to the open boundary, with clamped conditions: (a) Bottom, (b) Top layer

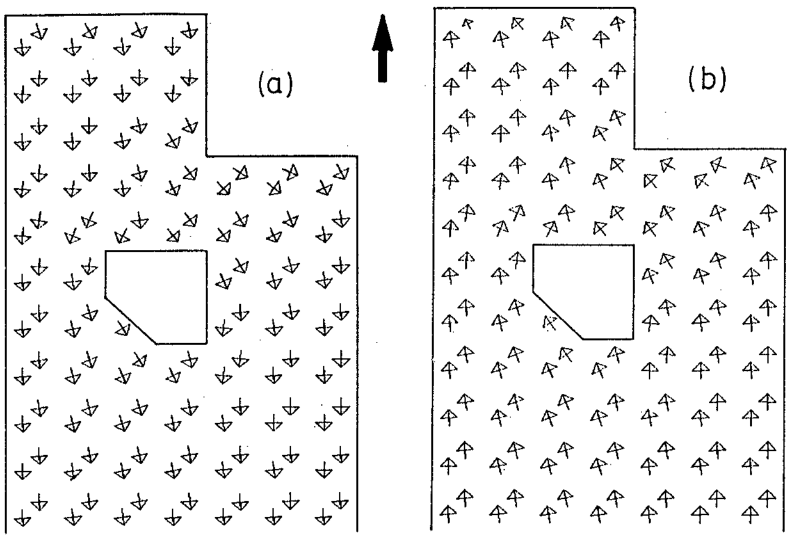

Figure 4. Flow field for wind normal to the open boundary, with free radiation conditions: (a) Bottom, (b) Top layer

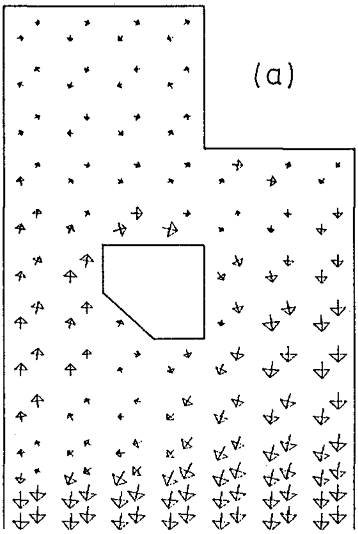

Figure 5. Flow field for wind normal to the open boundary, with sponge layer conditions: (a) Bottom, (b) Top layer

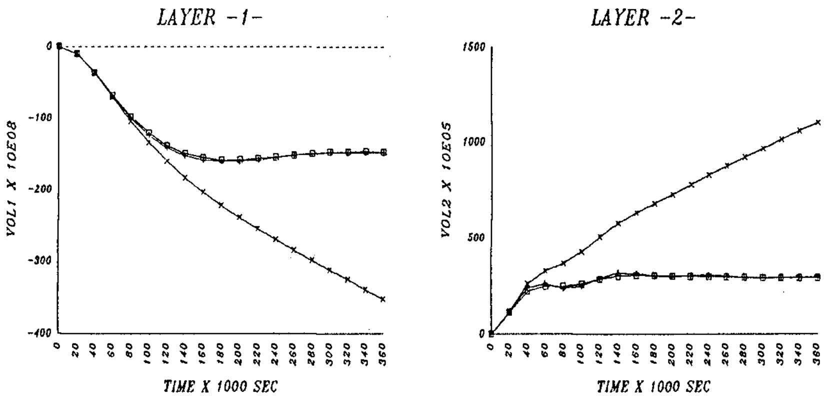

Figure 6. Time change of layer volumes for wind normal to the open boundary; + clamped conditions, x free radiation, □ sponge layer.

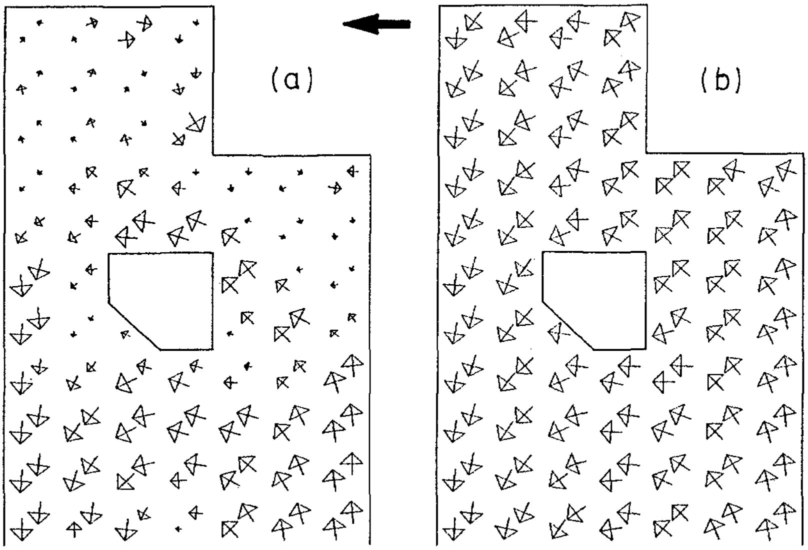

Figure 7. Flow field for wind parallel to the open boundary, with clamped conditions:(a) Bottom, (b)Top layer

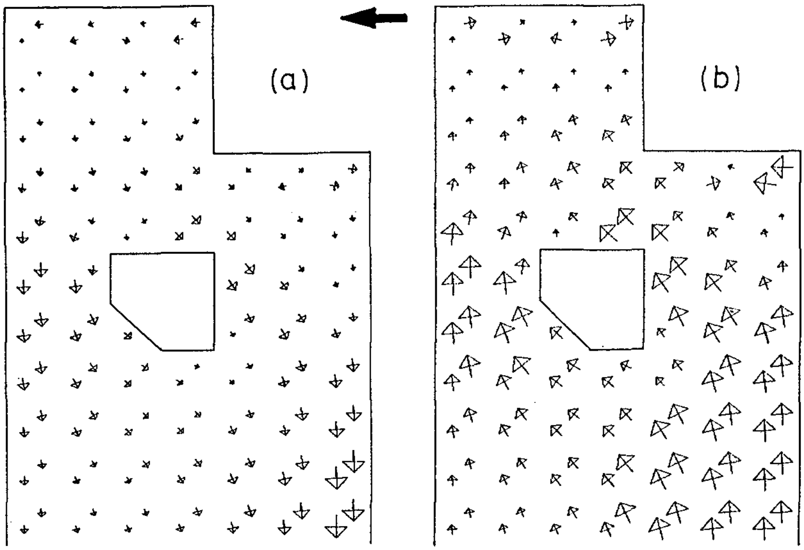

Figure 8. Flow field for wind parallel to the open boundary, with free radiation conditions: (a) Bottom, (b) Top layer

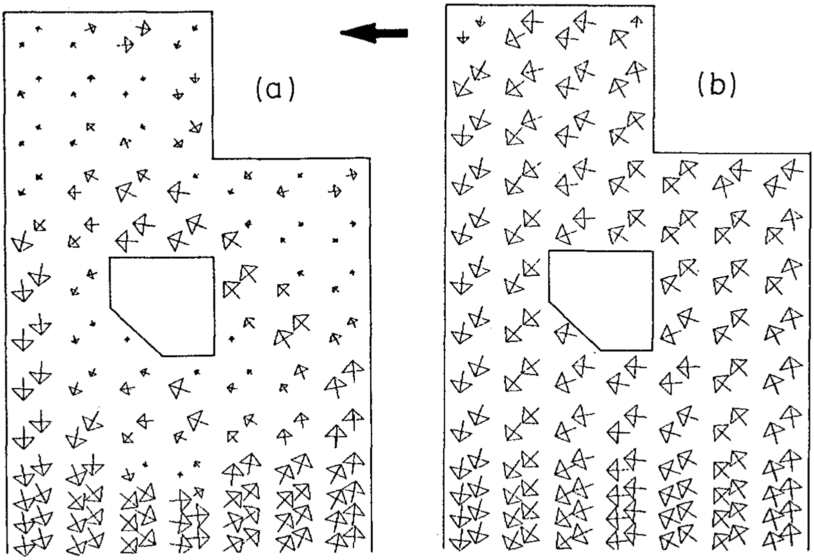

Figure 9. Flow field for wind parallel to the open boundary, with sponge layer conditions. (a) Bottom, (b) Top layer

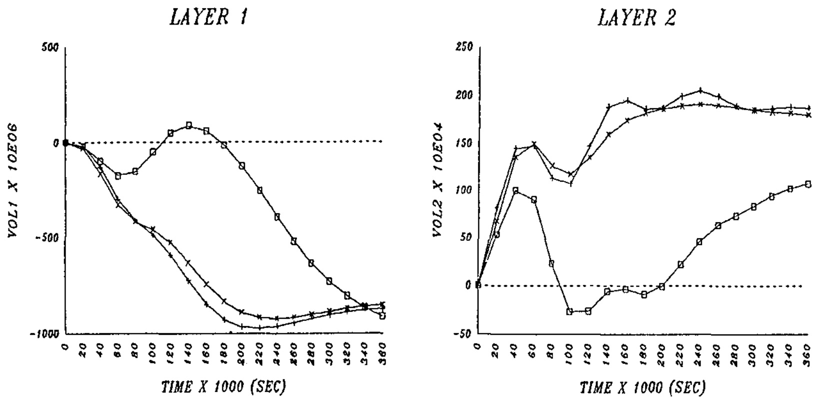

Figure 10. Time change of layer volumes for wind parallel to the open boundary: + clamped conditions, □ free radiation, x sponge layer

Substantial oscillations are observed for both layers, which under clamped conditions persist for almost 4 days, while stabilization occurs much earlier for the sponge conditions. Still, the results of these two types of b.c. are close to each other, both during transient and near steady state. On the contrary, the free radiation conditions yield much different global features and, after about 2 days of oscillatory transients, seem to establish again a progressive emptying of the lower layer and filling of the top layer.

It is evident from the above that the way of forcing has also some effect on the response of the two-layer system. Most boundary conditions are ideally designed to handle disturbances travelling normal to the opening. Thus, the enhanced oscillations observed in the second test are conceivably due to the wind direction being parallel to the boundary, indicating the difficulty of the system to accommodate the oblique incidence of waves.

The degree to which the above observations depend on the morphological features of the basin is currently being studied by additional tests made on a natural water body, namely the Saronic Gulf in Greece, which has dimensions of the same order of magnitude as those of the idealized basin but a variable bathymetry and complex geometry. Preliminary results obtained with the Coriolis terms included show also major differences for the wind-driven circulation under the three types of b.c., indicating qualitatively the same behaviour as above.

Therefore, it may be concluded that the clamped and sponge conditions give approximately the same results concerning the wind-driven circulation of a two-layer system. The former is simplest to implement, but the latter has the advantage of faster damping of the initial transient oscillations, so that the computation time to steady state is reduced. The free radiation conditions could be more appropriate for transient problems since they allow interior disturbances to exit, however they should not be used when a steady state solution is sought. Still, the computation of steady state might often be of academic rather than of practical interest. Taking into account the slow time response of a large stratified basin, it is practically impossible for a meteorological forcing to prevail for the duration necessary to create a steady flow field. In simulating events of finite duration, a combination of free radiation and clamped conditions of the form proposed in 2-D problems by Blumberg and Kantha (1985) may prove successful for stratified problems as well. In any case, the observed substantial influence of the b.c. on the computed flow patterns points out the need for further research into the subject.

CONCLUSIONS

Three different types of boundary conditions that may be specified along the sea boundaries of numerical models for stratified coastal circulation were examined, namely clamped, free radiation and sponge layer conditions. These boundary conditions were implemented in an idealized two-layer basin and numerical experiments for wind forcing were carried out. It was found that the b.c. generally have a very pronounced effect on the flow field, not only close to the boundary, but essentially over the entire domain. Results obtained for clamped and sponge boundaries are to some extent similar, but free radiation conditions give very different patterns. Besides, in the latter case steady state is not reached for a long time, even after gradual introduction of the wind forcing.

ACKNOWLEDGMENT

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