CHAPTER 104

LONGSHORE CURRENT AND TRANSPORT ACROSS
NON-SINGULAR EQUILIBRIUM BEACH PROFILES

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ABSTRACT

The longshore current and longshore sediment transport
distributions are described across an equilibrium beach
profile comprised of an intersecting planar foreshore
and a concave-up profile.   Such a profile shape avoids
the singularity associated with the infinite-slope at
the shoreline described by traditional equilibrium
profile forms and allows prediction of beach processes
at and above the shoreline.   The mathematical expres-
sions which describe the distributions are simplified
and can be more readily applied relative to expressions
previously presented in the literature.   The findings
are in general agreement with similar previous analytic
studies and indicate that the current and transport
maxima are generally located at about the intersection
of the planar and concave-up portions of the profile.

INTRODUCTION

Recent analytical solutions and data indicate that the
longshore current and transport distributions across
non-planar beaches can be markedly different than that
predicted for planar beaches (see Sawaragi and Deguchi,
1978; Berek and Dean, 1982; Sternberg et al., 1984,
Bodge and Dean, 1987; Fulford, 1987; Kraus, Rosati and
Gingerich, 1988; Bodge, in press; among others).   Since
few natural beaches are planar in profile, the solution
of the longshore current and transport distributions
across non-planar beaches is relevant to improved under-
standing and modelling of littoral processes for physi-
cally realistic beach profiles.

BACKGROUND

Bruun (1954) and Dean (1977) have suggested that equi-
librium beach profiles are described by the form

$$h = a \, x^{2/3} \tag{1}$$

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where h=depth, x=distance offshore, and a is a
dimensional coefficient. A disadvantage of the
concave-up profile described by this equation is the
infinite beach slope at the shoreline ($x=0$) which leads
to (i) a singularity in the description of the
longshore current and transport at the shoreline, and
(ii) no description of the swash zone. In contrast,
most equilibrium profiles in nature are characterized
by a relatively planar beachface which extends from the
berm to just below the shoreline.

McDougal and Hudspeth (1984) analytically considered
the wave-induced set-up and longshore current and trans-
port across beach profiles described by a planar fore-
shore and an $ax^{2/3}$-type profile seaward of the fore-
shore. These authors considered both "stress" and
"energetics" longshore transport models after Komar
(1977) and Bagnold (1963), respectively.

In the present study, the set-up, longshore current and
longshore transport across such a combined planar-fore-
shore/concave-up profile is analytically described
using (1) the longshore current solutions described by
Longuet-Higgins (1970) and in part by McDougal and
Hudspeth (1984), respectively, across the planar and
concave-up portions of the profile; (2) a simplistic
assumption that the local wave height is proportional
to the local set-up water depth, and (3) a distributed
longshore transport model described by Bodge and Dean
(1987).

THE NON-SINGULAR EQUILIBRIUM BEACH PROFILE

Consider a beach profile such as shown in Figure 1 with
x-axis directed offshore and origin $x=0$ at the still
water shoreline. The profile is composed of two mathe-
matically described segments: (1) a planar foreshore of
slope m_f, and (2) a concave-up profile of the form
$h=ax^{2/3}$, where h is the still-water depth and a is a
shape parameter. The profiles intersect at the "match-
point" $x=x_m$. The usual origin of the concave-up pro-
file (i.e., at depth $h=0$) is displaced seaward of the
x-axis origin by the distance δ. Specifically,

$$h = \begin{cases} m_f\, x & x \leq x_m \\ a(x-\delta)^{2/3} & x \geq x_m \end{cases} \tag{1}$$

Requiring that the bottom depths and slopes are identi-
cal at the match-point x_m yields

$$x_m = \frac{4}{9}\frac{a^3}{m_f^3} \tag{2}$$

and

$$\delta = \frac{x_m}{3} \tag{3}$$

That is, the location of the match-point can be simply
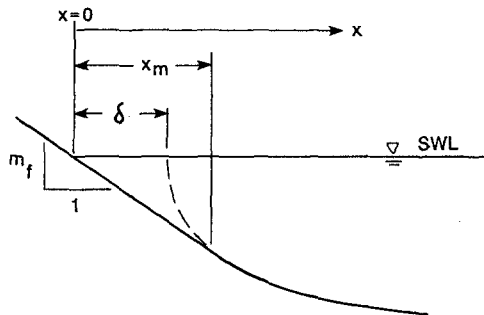
Figure 1:   The non-singular equilibrium beach profile.

described by the foreshore slope and the shape factor of the concave-up part of the profile.

WAVE-INDUCED SET-UP

The variation in mean water surface associated with change in onshore radiation stress, or wave-induced set-up $\bar{\eta}$, has been shown to be equal to

$$\bar{\eta} = \begin{cases} \nu(h_b - h) - \dfrac{\kappa^2}{16} h_b & x \leq x_b \\[2ex] \dfrac{-H^2 \, k}{8 \sinh 2kh} & x \geq x_b \end{cases} \qquad (4)$$

after Longuet-Higgins and Stewart (1964), and Dean and Dalrymple (1984), among others, where h_b and x_b are the still-water depth and location of the wave breakpoint, respectively, H is the local wave height, and k is the wavenumber. Shoreward of the breakpoint, the wave height H is assumed proportional to the total water depth d, where

$$\kappa = H/d \qquad (5)$$

and

$$\nu = \frac{3\kappa^2}{8 + 3\kappa^2} \qquad (6)$$

Across the planar foreshore part of the profile, the slope of the set-up water surface is uniform and equal to $-\nu \, m_f$. The point of intersection of the set-up water surface and the foreshore, x_s, is

$$x_s = \frac{(\nu - \kappa^2/16) \, h_b}{(\nu - 1) \, m_f} \qquad (7)$$

This location corresponds to the "set-up still water line", or more precisely, the mean water line in the influence of static wave set-up.

The total water depth d across the profile is

$$d = h + \bar{\eta} \tag{8}$$

Combining Eqs. (1), (4), and (8), the total water depth across the non-signular equilibrium profile is

$$d = \begin{cases} (1-\nu)\; m_f\; x + (\nu - \frac{\kappa^2}{16})\; h_b & x_s \leq x \leq x_m \\[2em] (1-\nu)\; a\; (x-\delta)^{2/3} + (\nu - \frac{\kappa^2}{16})\; h_b & x \geq x_m \end{cases} \tag{9}$$

For purposes of notation, observe that the set-up water surface slope across the planar foreshore part of the profile is

$$s_f = (1-\nu)\; m_f \tag{10}$$

McDougal and Hudspeth (1981, 1984) showed that the to-tal water depth across the concave-up $h=ax^{2/3}$ profile can be well-approximated by the form

$$d = \tilde{a}\; x^{1/2} \tag{11}$$

This expression models the set-up water surface well except as x approaches 0 (since d does not go to zero as x→0). However, the poorer fit near x=0 is essen-tially inconsequential in the present case if the exact form of the set-up water surface is used across the planar foreshore part of the profile; i.e., in the re-gion x=0. Accordingly, the following expressions are adopted for the total water depth:

$$d = \begin{cases} s_f\; x + (\nu - \frac{\kappa^2}{16})\; h_b & x_s \leq x \leq x_m \\[1.5em] \hat{a}\; (x-\hat{\delta})^{1/2} & x \geq x_m \end{cases} \tag{12}$$

Requiring continuity of total water depth and water surface slope at the matchpoint x_m yields

$$\hat{a} = s_f\; \sqrt{2(x_m - x_s)} \tag{13a}$$

$$\hat{\delta} = \frac{1}{2}\; (x_m + x_s) \tag{13b}$$

Non-dimensionalizing all length scales by the distance to the breakpoint x_b yields

$$D = \begin{cases} s_f\; X + (\nu - \frac{\kappa^2}{16})\; s_b & X_s \leq X \leq X_m \\[1.5em] \hat{A}\; (X - \hat{\Delta})^{1/2} & X \geq X_m \end{cases} \tag{14}$$

$$\hat{A} = s_f\; \sqrt{2(X_m - X_s)} \tag{15a}$$

$$\hat{\Delta} = \frac{1}{2} (X_m + X_s) \tag{15b}$$

where upper-case denotes non-dimensional terms such as

$$X = x / x_b \tag{16}$$

and where the term s_b has been introduced as

$$s_b = h_b / x_b \tag{17}$$

LONGSHORE CURRENT

In dimensional form, wave-induced longshore current can be described through the depth- and time-averaged equation of motion in the alongshore direction (Bowen, 1969; Longuet-Higgins 1970),

$$-\frac{\partial}{\partial x} S_{xy} + \tau_{by} + \frac{\partial}{\partial x}(\mu_e d \frac{\partial}{\partial x} v) = 0 \tag{18}$$

where S_{xy} is the onshore-directed flux of the alongshore component of momentum, τ_{by} is the alongshore component of bottom stress, v is the depth- and time-averaged longshore current, and μ_e is the turbulent eddy viscosity. Following the classical approximations of Longuet-Higgins (1970) and others,

$$\tau_{by} = -\frac{\kappa}{\pi} C_f \rho \sqrt{gd} \ v \tag{19}$$

where C_f is an empirically determined friction factor of order 0.01. The turbulent viscosity is assumed to vary linearly with distance offshore

$$\mu_e = N \rho x \sqrt{gd} \tag{20}$$

where N is a numerical constant.

Non-dimensionalizing all length-scales by the breaker distance x_b and velocity by the classical Longuet-Higgins solution of longshore current at the breaker line with no mixing, i.e.,

$$v_o = \frac{5}{16} \frac{\kappa\pi}{C_f} \sin \theta_b \ \sqrt{gd_b} \ s_b \tag{21}$$

where θ is the wave angle and the "b" subscript refers to breaking conditions, the alongshore equation of motion (Eq. 18) becomes

$$\frac{N \pi}{\kappa C_f} \frac{\partial}{\partial X}(XD^{3/2} \frac{\partial V}{\partial X}) - D^{1/2}V = -D^{3/2} \frac{\partial D}{\partial X} \frac{1}{s_b{}^2} \tag{22}$$

where upper-case denotes non-dimensional terms once again such as

$$V = v/v_o \quad \text{and} \quad X = x/x_b \tag{23}$$

To determine the longshore current, Eq. (22) must be solved across each of three regions across the non-singular equilibrium profile: (i) the planar foreshore, (ii) the concave-up segment landward of the breakpoint, and (iii) the concave-up segment seaward of the break-point. Both the magnitude and gradient of the velocity must be matched between each region. McDougal and Hudspeth (1984) demonstrated these solutions but the resultant expressions are somewhat cumbersome and tedious to apply.

In the present study, Eq. (22) is solved for the long-shore current across each of the three regions; both the magnitude and gradient are matched at the planar/concave-up matchpoint x_m but only the magnitude is matched at the breakpoint x_b. The resulting discontinuity in the velocity gradient at the breakpoint is small for the physically-typical case where x_m is small compared to x_b (i.e., where the toe of the foreshore is located close to shore relative to the breakpoint).

Incorporating the non-dimensional expressions for total water depth D (Eq. 19) to Eq. (22), and solving for velocity subject to the above-described matching conditions, the non-dimensional longshore current is given by

$$V = \begin{cases} A_1 \ (X-X_s)^{P_1} + (\dfrac{s_f}{s_b})^2 \ \dfrac{1}{1-\frac{5}{2}P} \ (X-X_s) & X_s \leq X \leq X_m \\[4mm] C_1 \ [\dfrac{\cosh(\lambda \tilde{X}^{1/4})}{\lambda \ \tilde{X}^{1/2}} - \dfrac{\sinh(\lambda \tilde{X}^{1/4})}{\lambda^2 \ \tilde{X}^{3/4}}] + \dfrac{\hat{A}^2}{2s_b^2} & X_m \leq X \leq 1 \quad (24) \\[4mm] C_2 \ [\dfrac{1}{\tilde{X}^{1/2}} + \dfrac{1}{\lambda \ \tilde{X}^{3/4}}] \ \exp(-\lambda \tilde{X}^{1/4}) & X \geq 1 \end{cases}$$

where

$$P_1 = -\frac{3}{4} + (\frac{9}{16} + \frac{1}{P})^{1/2} \tag{25}$$

$$P = \frac{N \pi \ s_f}{\kappa \ C_f} \ ; \quad P \neq 2/5 \tag{26}$$

$$\lambda = 4/\sqrt{P_2} \tag{27}$$

$$P_2 = \frac{N \pi \hat{A}}{C_f} \ = \ P \ \sqrt{2(X_m - X_s)} \tag{28}$$

$$\tilde{X} = (X - \hat{\Delta}) \tag{29}$$

The coefficients are given by

$$A_1 = (X_m - X_s)^{-P_1} \ [C_1 \ \Phi - \alpha(X_m - X_s) + \frac{\hat{A}^2}{2s_b^2} \ ] \tag{30}$$

$$C_1 = -\frac{\hat{A}^2}{2s_b^2} \ (3Q + \lambda Q^2 + \frac{3}{\lambda}) \ \exp(-\lambda Q) \tag{31}$$

$$C_2 = \frac{\hat{A}^2}{2s_b^2} \left[-\frac{3Q}{\lambda} \cosh \lambda Q + (Q^2 + \frac{3}{\lambda^2}) \sinh \lambda Q \right] \qquad (32)$$

and where

$$\alpha = (\frac{s_f}{s_b})^2 \frac{1}{1-\frac{5}{2}P} \quad ; \quad P \neq 2/5 \qquad (33)$$

$$\Phi = \frac{1}{\lambda R^2} \cosh \lambda R - \frac{1}{\lambda^2 R^3} \sinh \lambda R \qquad (34)$$

$$R = (X_m - \hat{\Delta})^{1/4} \qquad (35)$$

$$Q = (1 - \hat{\Delta})^{1/4} \qquad (36)$$

Application of Eqs. (24) through (36) can be cumbersome. However, each of the terms in Eq. (24) which describe the longshore current can be expressed solely as a function of the non-dimensional matchpoint location X_m and the mixing parameter P. Recall that X_m is simply described by

$$X_m = x_m / x_b \qquad (37)$$

where

$$x_m = \frac{4}{9} (\frac{a}{m_f})^3 \qquad (38)$$

and

$$x_b = (\frac{H_b}{\kappa a})^{3/2} + \frac{4}{27}(\frac{a}{m_f})^3 \qquad (39)$$

Hence, the non-dimensional longshore current profile can be readily calculated from the (i) breaking wave height H_b, (ii) breaker index κ, (iii) concave-up beach shape parameter a (commonly denoted by "A" in the literature), (iv) foreshore slope m_f, and (v) selection of a mixing parameter P, where $P \neq 0.4$.

Figures 2 through 9 depict each of the terms necessary to evaluate the longshore current distribution from Eq. (24) for a breaker index $\kappa = 0.8$, for various chosen values of the mixing parameter P and for calculated values of the non-dimensional planar-foreshore/concave-up matchpoint location X_m.

Figure 10 illustrates the non-dimensional longshore current across the non-singular equilibrium beach profile for various locations of the profile matchpoint X_m. Note that the current maxima approximately corresponds to the location of the profile matchpoint.

Figure 11 illustrates longshore current distributions observed across beaches with non-singular equilibrium shaped profiles -- compared to the distributions predicted for each case by Eq. (24) for several values of the mixing parameter. The first five cases represent

dye observations from a moveable-bed laboratory model
for five different breaking/surf conditions; the last
case is from current meter data collected at Duck,
N.C., (Bodge and Dean, 1987). The agreement is fairly
good, although the longshore current distributions ob-
served in the laboratory appear displaced landward by
about 2/10'ths of a surf zone width relative to the
predicted distributions.
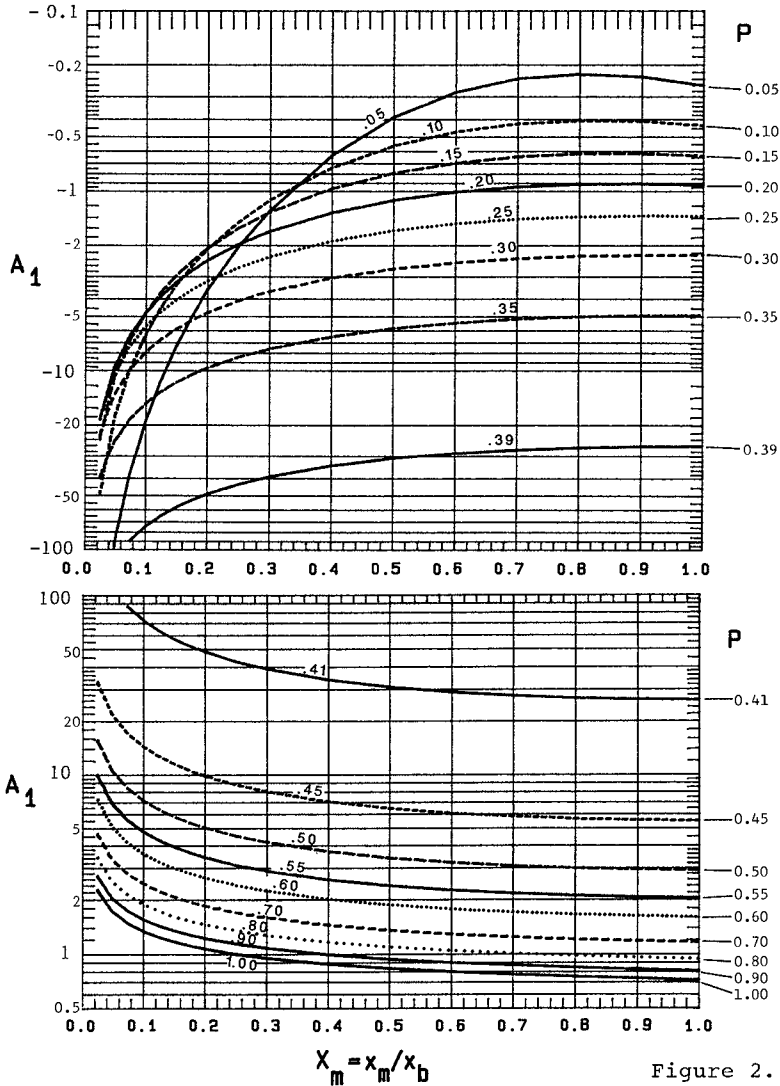
Figure 2.

Figure 3

Figure 4

$$X_m = x_m/x_b$$

Figure 5

$$X_m = x_m/x_b$$
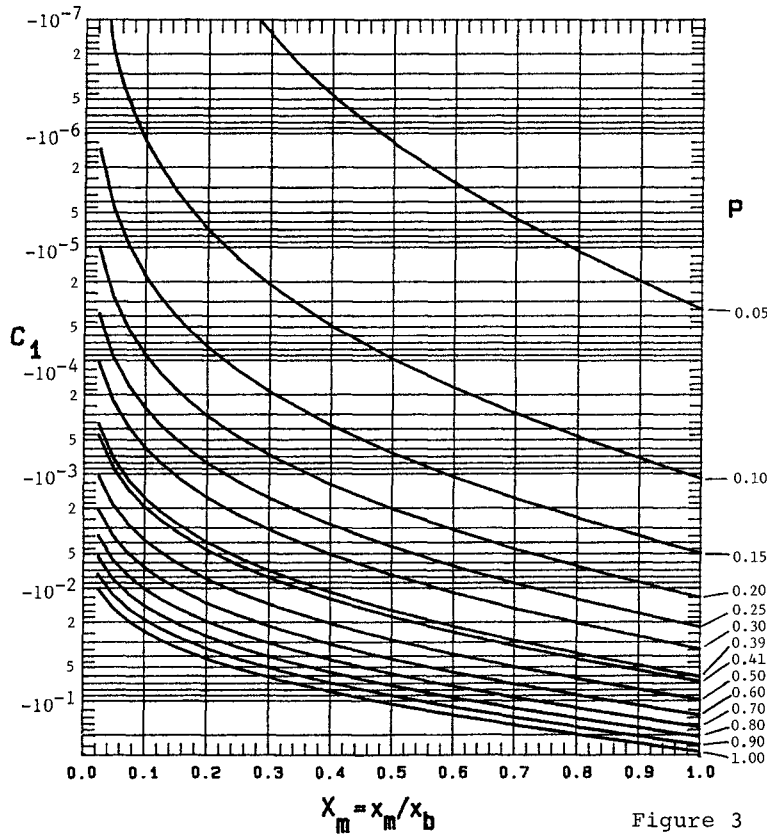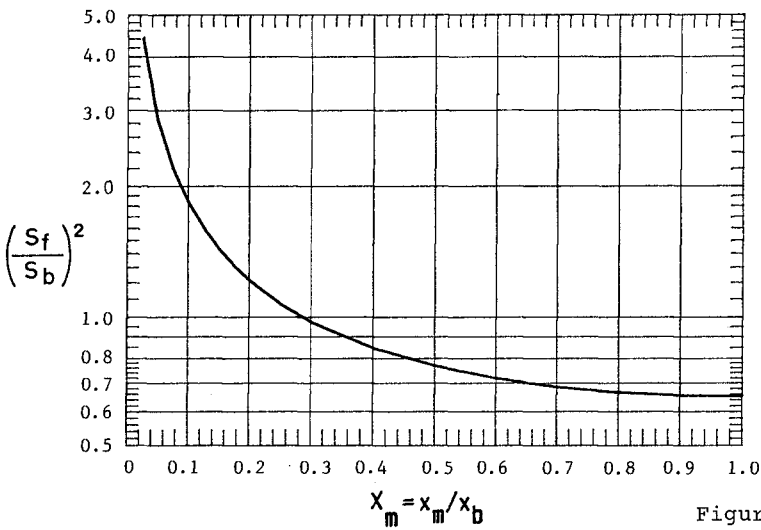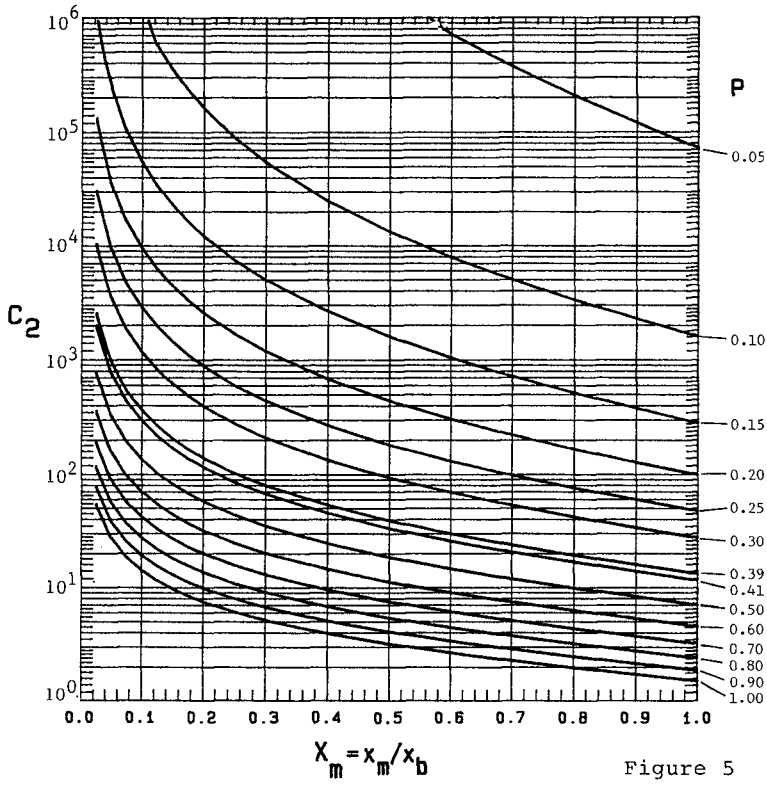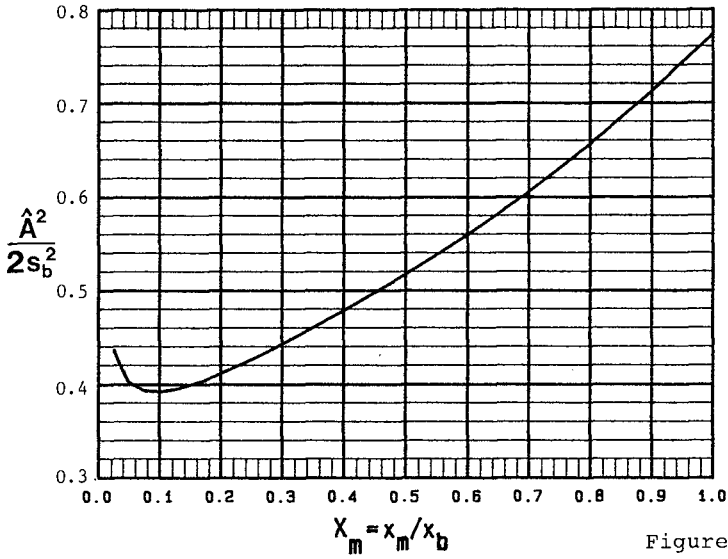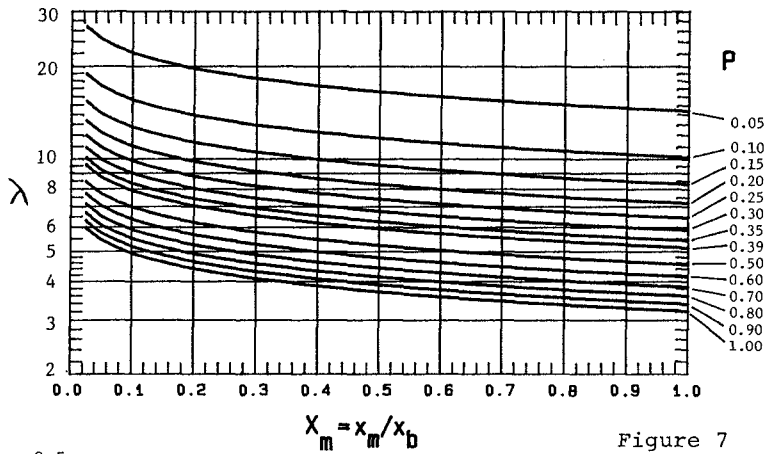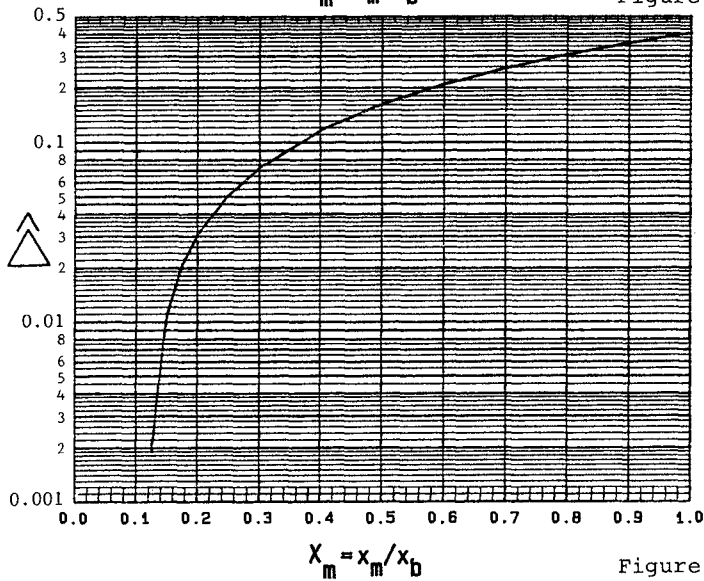
Figure 6

Figure 7


Figure 8
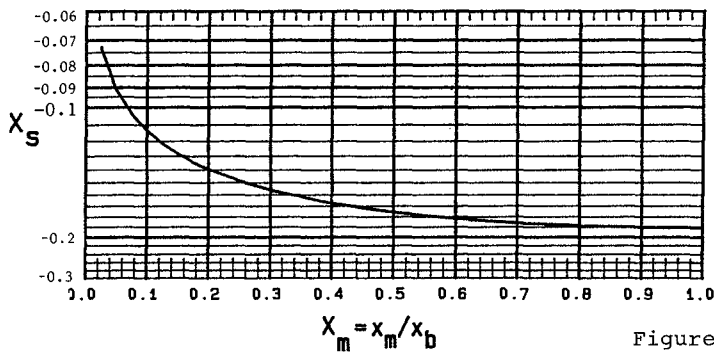
Figure 9
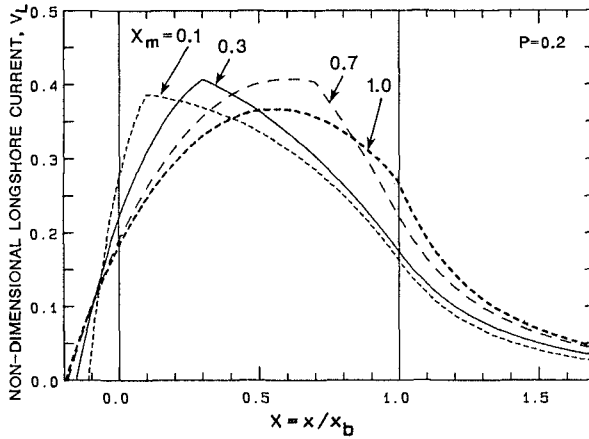
Figure 10: Longshore current predicted across a non-singular equilibrium profile for various profile matchpoints.
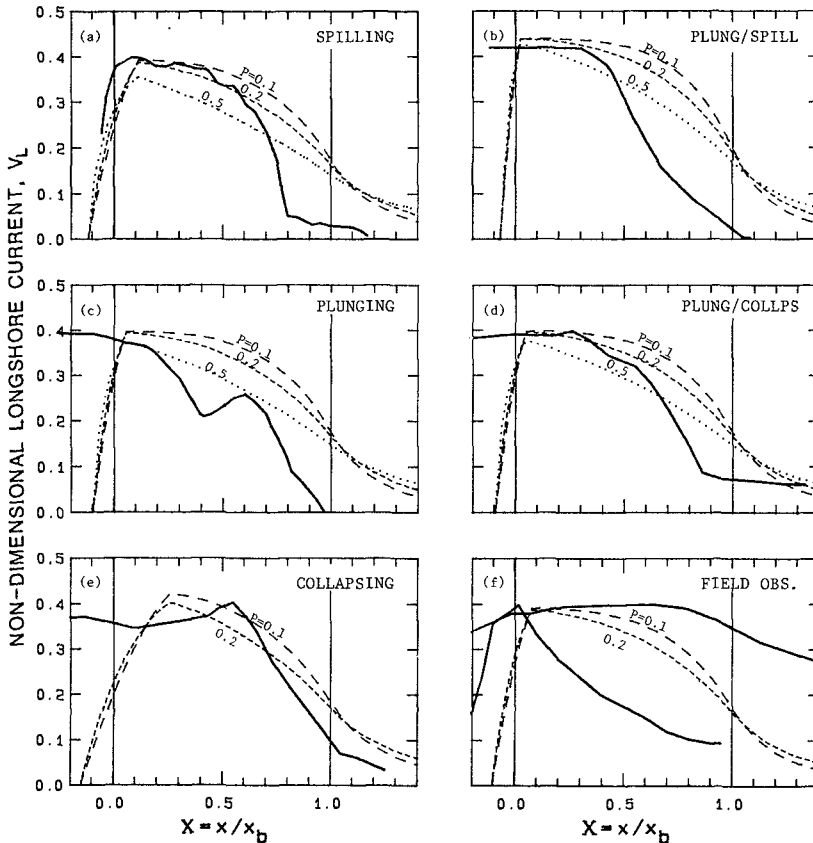

Figure 11: Observed (bold) and predicted (dashed) longshore current distribution across non-planar beaches.

LONGSHORE SEDIMENT TRANSPORT

Bodge and Dean (1987) presented a distributed longshore transport model based upon results of short-term impoundment of longshore sediment transport across rapidly-deployed, low-profile groins in the field and laboratory environments. Their expression suggests that sediment is mobilized in proportion to the rate of local wave energy dissipation per unit surf zone volume and is transported alongshore by local longshore current; viz.,

$$q_\ell(x) = k_\ell \frac{1}{h} \frac{\partial}{\partial x}(e\, c_g)\, v \tag{40}$$

where $q_\ell(x)$ is the local longshore sediment transport rate per unit offshore distance, k_ℓ is a dimensional coefficient of proportionality, and ec_g is the local wave energy flux.

Again assuming that local wave height is proportional to the local water depth, and non-dimensionalizing lengths by the breaker distance and velocities by the planar-beach no-mixing longshore current at the breakpoint, Eq. (40) becomes

$$Q_\ell = D^{1/2}\, s_b^{-3/2}\, V\, \frac{dD}{dX} \tag{41}$$

where Q_ℓ is the local (not total) longshore transport rate per unit distance offshore normalized by the local rate at the breakpoint; i.e.,

$$Q_\ell = q_\ell\, /\, q_{\ell b} \tag{42}$$

where

$$q_{\ell b} = k_\ell\, \frac{5}{16}\, \rho\, g^{3/2}\, \kappa^2\, d_b^{1/2}\, v_o\, s_b \tag{43}$$

Equation (41) can be evaluated to yield the longshore sediment transport distribution across the surf zone (landward to the location of the set-up still water line x_s) via Eq. (24) for the longshore current and Eq. (14) for the total water depth. Figure 12 illustrates the transport distributions for various locations of the profile matchpoint X_m. Note that the maxima closely corresponds to the location of the profile matchpoint.

Figure 13 compares the longshore transport distributions across a non-singular equilibrium profile as predicted by the "energetics" model (Bagnold, 1963), the "stress" model (Komar, 1977), and the present model (Bodge and Dean, 1987).
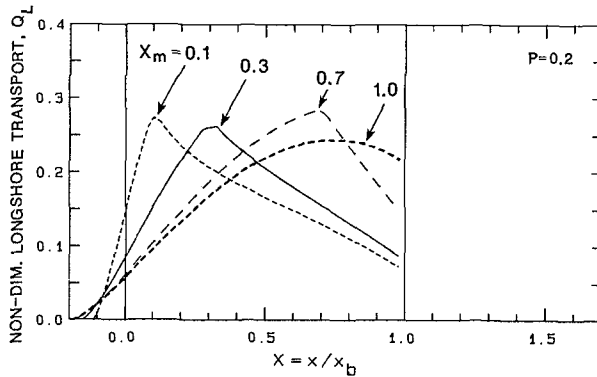
Figure 12: Longshore transport across a non-singular equi-
   librium profile for various profile matchpoint locations.
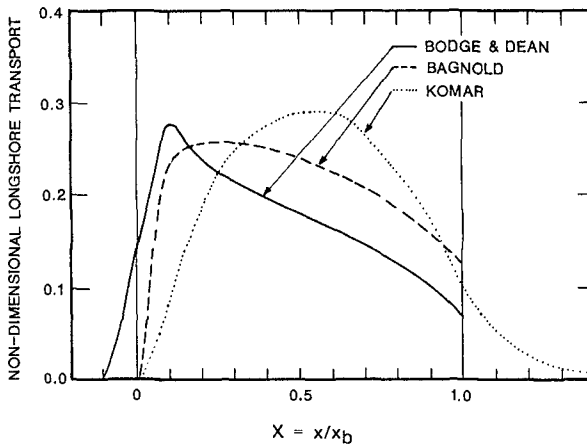
Figure 13: Comparison of longshore transport distribution
   models evaluated for a non-singular equilibrium profile
   with matchpoint $X_m=0.1$ and mixing parameter $P=0.1$.

SUMMARY AND CONCLUSIONS

The non-singular equilibrium beach profile consists of
a planar foreshore "matched" to a concave-up profile
across the surf zone.   The longshore current and sedi-
ment transport distributions across such a profile can
be readily developed through knowledge of the foreshore
slope, concave-up profile shape parameter, breaker
height, and selection of a Longuet-Higgins type mixing
parameter.   The current and transport maxima are pre-
dicted at or about the location of the planar-fore-
shore/concave-up profile matchpoint.   Since this point
is typically located fairly close to shore relative to
the breakpoint, this suggests that the peak current and
transport are more closely located to shore than to the

breakpoint.   This result agrees with similar solutions
for non-planar beaches, such as McDougal and Hudspeth
(1981, 1984), but is in contrast to classical predic-
tions for planar beaches -- where current and transport
is predicted as maximum at the outer half of the surf
zone and vanishing at the shoreline.   Moreover, the
results for the non-planar beach, as in the present
study, agree more closely with recent field and labora-
tory observations.

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