CHAPTER 74

Dynamic Wave Setup

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Abstract

The wave setup for a given wave spectrum was re-evaluated with the radiation stress term including the low-frequency terms. This setup which is referred to here as "dynamic wave setup", was compared with the steady setup, which is generated by including only the non-periodic radiation stress terms.

The results of the study showed that the dynamic wave setup is greater than the steady wave setup, sometimes almost double its value. Therefore, the dynamic wave setup is important for engineering applications, particularly in the study of storm surges and coastal engineering problems.

Introduction

Longuet-Higgins and Stewart (1963) showed theoretically from their concept of radiation stress, and Bowen et al. (1968) verified it in the laboratory, that there is a water level setdown outside the surf zone followed by a water level setup within the surf zone because of the presence of the wave motion.

Longuet-Higgins and Stewart (1963) considered only the monochromatic waves, but Collins (1972), Battjes (1974), and Wu et al. (1978) re-evaluated wave setup by considering the radiation stress for a given spectrum.

The radiation stress derived for a wave spectrum should include the interaction between any wave component and itself as well as the interaction between any wave component and other wave components. However, the studies carried out by Collins (1972), Battjes (1974), and Wu et al. (1978) considered only the self-interaction terms.

Lo (1981) derived the radiation stress term for any given wave spectrum. In this derivation both non-periodic and long-period oscillation (low frequency) terms were included. The low-frequency terms were generated from the non-linear transfer of energy of the phase difference between different wave components.

The purpose of this study is to re-evaluate wave setup for a given wave spectrum with the radiation stress term including the low-frequency terms. This setup which is referred to here as "dynamic wave setup" was compared with the

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steady wave setup, which was generated by including only the non-periodic radiation stress terms.

Governing Equations for the Surf Beat Wave on a Sloping Bottom

The motion is considered irrotational and the fluid incompressible. Integrating the continuity and momentum equations from the bottom, $Z = -h$, to the free surface, $Z = \eta$ (water surface displacement), imposing the Leibnitz rule, and substituting the no-flow bottom boundary condition (BBC), the kinematic free surface boundary condition (KFSBC), and applying the time average over one incoming wave period one can obtain the governing equations for the sloping bottom surf beat wave. For convenience, only a uniform bottom slope will be considered.

Leibnitz Rule. The Leibnitz rule of differentiation will be used to obtain a derivative from within an integral. It is stated as follows

$$\int_{\alpha(x)}^{\beta(x)} \frac{\partial f(x,z)}{\partial x}\, dz = \frac{\partial}{\partial x}\int_{\alpha(x)}^{\beta(x)} f(x,z)dz - f(\beta,z)\,\frac{\partial \beta(x)}{\partial x} + f(\alpha,z)\frac{\partial \alpha(x)}{\partial x} \qquad (1)$$

where f (x, z) is any function; x is a variable of integration; z is a dummy variable of integration; α and β are limits of integration.

Boundary Conditions. At the bottom, $Z = -h$, a no-flow boundary condition exists. In vector form

$$\vec{V} \,.\, \vec{n} = 0 \qquad (2)$$

where

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k} \qquad (3)$$

$$\vec{n} = \frac{\left(\frac{\partial h}{\partial x}\vec{i} + \frac{\partial h}{\partial y}\vec{j} + \vec{k}\right)}{\sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + 1}} \qquad (4)$$

(Unit vector normal to the bottom)

where $\vec{i}, \vec{j}, \vec{k}$ are unit vectors in the x, y, z directions; $\vec{V}$ is the velocity vector and u, v, w are the x, y and z components of the velocity vector.
BBC can be rewritten as

$$u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + w = 0, \quad \text{at } z = -h(x,y)$$

At the free surface, the vertical velocity (w) must account for the changes in the instantaneous water surface (η)

$$w = \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}, \quad \text{at } z = \eta \tag{5}$$

This is the KFSBC.

Continuity Equation. The general form of the continuity equation in three dimensions is

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \tag{6}$$

Integrating each term with respect to depth between Z = -h and Z = η, and imposing the Leibnitz Rule

$$\frac{\partial}{\partial t}\int_{-h}^{\eta} \rho dz - \rho_{\eta}\frac{\partial \eta}{\partial t} - \rho_{-h}\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_{-h}^{\eta} \rho u dz - (\rho u)_{\eta}\frac{\partial \eta}{\partial x} - (\rho u)_{-h}\frac{\partial h}{\partial x} +$$

$$\frac{\partial}{\partial y}\int_{-h}^{\eta} \rho v dz - (\rho v)_{\eta}\frac{\partial \eta}{\partial y} - (\rho v)_{-h}\frac{\partial h}{\partial y} + (\rho w)_{\eta} - (\rho w)_{-h} = 0 \tag{7}$$

Invoking the boundary conditions BBC and KFSBC, the above equation simplifies to

$$\frac{\partial}{\partial t}\int_{-h}^{\eta} \rho dz - \rho_{-h}\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_{-h}^{\eta} \rho u dz + \frac{\partial}{\partial y}\int_{-h}^{\eta} \rho v dz = 0 \tag{8}$$

Considering the depth (relative to still water level) as constant in time, and the problem as two-dimensional (no dependency on the y - direction), the above equation becomes:

$$\frac{\partial}{\partial t}\int_{-h}^{\eta} \rho dz + \frac{\partial}{\partial x}\int_{-h}^{\eta} \rho u dz = 0 \tag{9}$$

Consider the horizontal water particle velocity as the superposition of two components,

$$u = u_I + u_s \tag{10}$$

where

u_I = the horizontal water particle velocity of the incoming wave

u_s = the horizontal water particle velocity of the surf beat wave which is uniform over depth since the water depth is relatively shallow in comparison to the surf beat wave length.

Thus, for a fluid of constant density ρ, the continuity equation becomes

$$\frac{\partial}{\partial t}(h + \eta) + \frac{\partial}{\partial x}u_s(h + \eta) + \frac{\partial}{\partial x}\int_{-h}^{\eta} u_I \; dz = 0 \tag{11}$$

The time averages are defined

$$\overline{F} = \frac{1}{T}\int_{o}^{T} F dt \tag{12}$$

then

$$F_s = \overline{F}_s = \frac{1}{T}\int_{0}^{T} F_s dt$$

where T is the incoming wave period, and F_s represents a general surf beat variable. Time averaging the continuity equation, we obtain

$$\frac{\partial \overline{\eta}}{\partial t} + \frac{\partial}{\partial x}u_s(h + \overline{\eta}) + \overline{(\frac{\partial}{\partial x}\int_{-h}^{\eta} u_I)} \; dz = 0 \tag{13}$$

where $\overline{\eta} = \eta_s$, is the water surface displacement of the surf beat wave. The time-averaged water surface displacement of the incoming wave, $\overline{\eta}_I$, is zero.

Following the definitions of Phillips (1966) for the total flux

$$M_x = \overline{\int_{-h}^{\eta} \rho u_I dz} \tag{14}$$

and for the mean transport velocity

$$u_I = \frac{M_x}{\rho(h + \overline{\eta})} \tag{15}$$

The continuity equation can be written as

$$\frac{\partial \eta_s}{\partial t} + \frac{\partial}{\partial x}(u_s + u_I) \quad (h + \eta_s) + \frac{\partial}{\partial x}(\overline{u_I \eta_I})_{z=0} = 0 \tag{16}$$

where

$$\frac{\partial}{\partial x}(u_s + u_I)\eta_s \quad << \frac{\partial \eta_s}{\partial t}$$

The final result is then

$$\frac{\partial \eta_s}{\partial t} + \frac{\partial}{\partial x} \quad (u_s + u_I)h + \frac{\partial}{\partial x}(\overline{u_I \eta_I})_{z=0} = 0 \tag{17}$$

Equation of Motion. Without considering the shear stresses, the general form of the equations of motion are:

x Direction.

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} = \frac{1}{\rho}\frac{\partial P}{\partial x} \tag{18}$$

z Direction.

$$\frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial vw}{\partial y} + \frac{\partial w^2}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g \tag{19}$$

where p = pressure, and g = gravitational constant.

Following the same procedure as outlined before the final equation of motion for the case of two dimensional sloping bottom surf beat wave becomes

$$\frac{\partial(u_s + \bar{u}_I)h}{\partial t} = -gh\frac{\partial \eta_s}{\partial x} - \frac{1}{\rho}\frac{\partial S_{xx}}{\partial x} - \frac{\partial}{\partial t}(\overline{u_I \eta_I})_{z=0} \tag{20}$$

where

$$S_{xx} = \rho \int_{-h}^{0} (\overline{u_I^2 - w_I^2 dz}) + \frac{1}{2} \quad \rho g \overline{\eta}^2 \tag{21}$$

It should be noted that the sloping bottom surf beat wave problem is governed by equation of motion (Eq. 20), and the continuity equation (Eq. 17).

Irregular Wave Conditions

The irregular sequence of linear waves can be represented as an infinite sum of simple harmonic waves

$$\eta(x,\ t) = \sum_n a_n \cos\ (k_n x - \sigma_n t + \epsilon_n) \tag{22}$$

where

$$a_n = \sqrt{2\rho_\eta(\sigma)\Delta\sigma}$$

$P_\eta(\sigma)$ = the energy density spectrum of the irregular sea, varying with wave angular frequency σ, and

ϵ_n = the phase angle (radians) for the nth wave component.

When small amplitude wave theory is used to estimate the flow regime in a wave system from the surface profile, the linear relationship between the surface profile and the velocity potential results in

$$\phi(x,\ z,\ t) = -\sum_n \frac{a_n g}{\sigma_n} \frac{\cosh\ k_n\ (h+z)}{\cosh\ k_n h} sin\ (k_n x - \sigma_n t + \epsilon_n); \tag{23}$$

the dispersion relationship is given by

$$\sigma_n^2 = gk_n\ tanh\ k_n h \tag{24}$$

also the horizontal and vertical water particle velocity components are given by

$$u\ (x,\ z,\ t) = \sum_n \frac{a_n k_n g}{\sigma_n} \frac{\cosh\ k_n(h+z)}{\cosh\ k_n h} \cos\ (k_n x - \sigma_n t + \epsilon_n) \tag{25}$$

and

$$w\ (x,\ z,\ t) = \sum_n \frac{a_n k_n g}{\sigma_n} \frac{\sinh\ k_n(h+z)}{\cosh\ k_n h} \sin\ (k_n x - \sigma_n t + \epsilon_n) \tag{26}$$

After some algebraic manipulation the following equations result:

$$\eta^2 = \sum_{n=1}^{N} a_n^2 \cos^2\theta_n + \sum_{m=1}^{N-1} \sum_{n=m+1}^{N} a_m a_n \left[\cos\ (\theta_m - \theta_n) + \cos\ (\theta_m + \theta_n)\right], \tag{27}$$

$$u^2 = \sum_{n=1}^{N} A_n^2 \cos^2\theta_n + \sum_{m=1}^{N-1} \sum_{n=m+1}^{N} A_m A_n \left[\cos\ (\theta_m - \theta_n) + \cos\ (\theta_m + \theta_n)\right], \tag{28}$$

$$w^2 = \sum_{n=1}^{N} B_n^2 \sin^2\theta_n + \sum_{m=1}^{N-1} \sum_{n=m+1}^{N} B_m B_n \left[\cos\ (\theta_m - \theta_n) + \cos\ (\theta_m + \theta_n)\right] \tag{29}$$

where N is the total number of the wave components, and

$$\theta_n = k_n x - \sigma_n t + \epsilon_n$$

$$A_n = -a_n\sigma_n \frac{\cosh k_n(h+z)}{\sinh k_n h}$$

$$B_n = -a_n\sigma_n \frac{\sinh k_n(h+z)}{\sinh k_n h}$$

The sum of the phases are not considered for the surf beat wave problem, because they are in the high frequency region. Then

$$S_{xx} = \frac{\rho}{z}\sum_{n=1}^{N} \frac{a_n^2\sigma_n^2}{\sinh^2 k_n h} h + \frac{1}{4}\rho g \sum_{n=1}^{N} a_n^2$$

$$+ \rho \sum_{m=1}^{N-1} \sum_{n=m+1}^{N} \frac{a_m a_n \sigma_m \sigma_n \sinh (k_m - k_n)h}{(k_m - k_n)\sinh k_n h \sinh k_m h} \cos (\theta_m - \theta_n)$$

$$+ \frac{\rho g}{z} \sum_{m=1}^{N-1} \sum_{n=m+1}^{N} a_m a_n \cos (\theta_m - \theta_n) \tag{30}$$

and

$$(\overline{u_I\eta_I})_{z=o} = \frac{1}{2}\sum_{m=1}^{N} \sum_{n=1}^{N} a_m a_n \sigma_n \frac{\cosh k_n h}{\sinh k_n h} \cos (\theta_m - \theta_n) \tag{31}$$

Method of Solutions

The governing equations are

1. Equation of Motion

$$\frac{\partial q}{\partial t} = -gh\frac{\partial \eta_s}{\partial x} - \frac{1}{\rho}\frac{\partial S_{xx}}{\partial x} - \frac{\partial}{\partial t}(\overline{u_I\eta_I})_{z=0} \tag{32}$$

2. Continuity Equation

$$\frac{\partial \eta_s}{\partial t} + \frac{\partial q}{\partial x} = -\frac{\partial}{\partial x}(\overline{u_I\eta_I})_{z=0} \tag{33}$$

where

$$q = (u_s + u_I)h$$

The complete mathematical solution of these equations is complicated. However, these equations can be solved numerically by applying finite difference techniques to the method of characteristics. This procedure accounts for bottom slope (wave reflection) and the transient nature of the problem.

Due to the depth variations and associated changes in the slopes of the characteristic lines, the characteristic lines in the x-t plane form a curved network and the results require interpolation to give values on a regular rectangular grid system. This problem can be resolved through the introduction of the following dimensionless variables, which will transform the curved characteristics to straight lines:

$$
\begin{aligned}
&1. \quad Y = \frac{\sqrt{x}}{x_2} \\
&2. \quad t' = \frac{t}{T} \\
&3. \quad Z = \frac{\eta_s C_2 \sqrt{Y}}{U^2 T} \\
&4. \quad Q = \frac{q}{U^2 T \sqrt{Y}} \\
&5. \quad r = 1 - \sqrt{\frac{h_1}{h_2}} \\
&6. \quad F_1 = \frac{1}{u^2}\left[\frac{1}{\rho}\frac{\partial S_{xx}}{\partial x} + \frac{\partial}{\partial t}(\overline{u_I \eta_I})_{z=0}\right] \\
&7. \quad F_2 = -\frac{C_2}{u^2}\left[\frac{\partial}{\partial x}(\overline{u_I \eta_I})_{z=0}\right]
\end{aligned}
\qquad (34)
$$

where

$$T = (x_2 - x_1) / \overline{C}$$

$$\overline{C} = (C_1 + C_2) / z$$

$$C_1 = \sqrt{gh_1} = \sqrt{gmx_1}$$

$$C_2 = \sqrt{gh_2} = \sqrt{gmx_2}$$

$$U = 0.22815 \ gm \ h_{b(max)}$$

x_1 is the distance from shore (x = 0) to the wall boundary; x_2 is the distance from shore to the offshore boundary, and m is the beach slope.

By using the above transformations, the equation of motion and continuity equation are rewritten in the form

$$\frac{\partial Q}{\partial t'} + \gamma\left[\frac{\partial z}{\partial Y} - \frac{z}{2Y}\right] = -\frac{F_1}{\sqrt{Y}} \tag{35}$$

$$\frac{\partial z}{\partial t'} + \gamma\left[\frac{\partial Q}{\partial Y} - \frac{Q}{2Y}\right] = -\frac{F_2}{\sqrt{Y}} \tag{36}$$

The sum of these equations gives

$$\frac{d(Q+z)}{dt'} = \frac{\gamma}{2Y}[z - Q] - \frac{F_1}{\sqrt{Y}} - F_2\sqrt{Y} \quad \text{for} \quad \gamma = \frac{dY}{dt'} \tag{37}$$

and the difference of these equations gives

$$\frac{d(Q-z)}{dt'} = \frac{\gamma}{2Y}[z + Q] - \frac{F_1}{\sqrt{Y}} + F_2\sqrt{Y} \quad \text{for} \quad \gamma = -\frac{dY}{dt'} \tag{38}$$

A path in the Y, t' plane having a constant slope $\pm\gamma$ is the characteristic line and the values of $Q \pm Z$ along such lines are in accordance with equations 37 and 38. These equations are solved by finite different methods.

1. *Boundary Condition.* At the shore, where the water depth equals zero, this is a singular point. For convenience, a wall will be assumed for the calculations, i.e., the no-flow boundary condition at the wall is q = 0.
   In deep water, the forcing function of the surf beat wave is very small. Therefore, the boundary condition at offshore is a radiative condition; the surf beat wave will propagate offshore and no surf beat wave will enter the system under consideration from offshore.

2. *Initial Condition.* The sloping bottom surf beat problem is a transient problem. The transient nature is due to the increase in the surf beat wave height during propagation toward shore under the action of the forcing function, S_{xx}. At time zero, no wave occurs on the water surface, then later an incoming wave group propagates from infinity to shore with a surf beat wave force at the speed of the incoming wave velocity due to the forcing function induced by the incoming wave.

3. *Changing Breaking Zone.* A spilling wave breaking was assumed in this model, the incoming wave breaking height and local depth are related by

$$a_b = kh$$

The value of k is constant. In this study, the constant k will be taken as 0.39 and the wave height in the surf zone will depend only on the water depth. For an irregular wave field, the location of wave breaking will vary with the water depth. Therefore, the wave breaking process occurs in an area instead of at only one location as the regular wave breaking process.

Comparison of the Steady Wave Setup with the Surf Beat Wave Simulation Model

A given deep water wave spectra and the surf beat wave simulation model is applied to compare the difference between the surf beat wave and the steady wave setup. Figure 1 shows the assumed deep water wave spectra. The width of the energy frequency band ϵ defined by Rice (1945) is equal to 0.16. The incoming wave periods range from 8 to 12 seconds, and ten wave components with equal frequency increment are selected from the given spectrum. The significant wave height is 4.76 ft. In the simulation model, the bottom slope is 0.02, the offshore boundary water depth is 80 ft and the shore boundary water depth is 1.2 ft. The time step is 1.383 s.

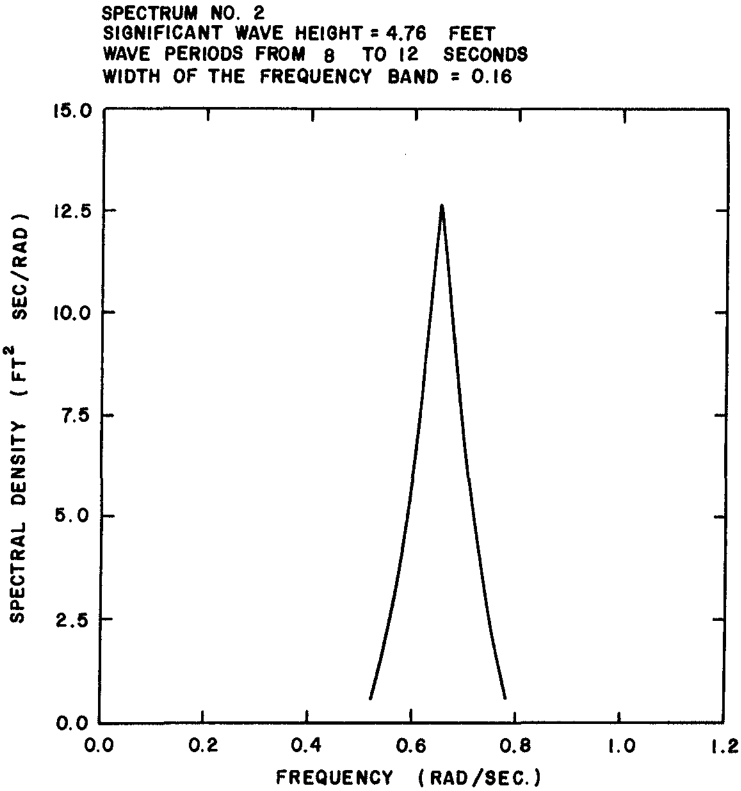

Figure 1. Deep water incoming wave spectrum for the study of the correlation between the surf beat and the wave envelope.

Figure 2 presents the comparisons of the steady wave setup with the surf beat wave on the shore boundary, where the steady wave setup is calculated from

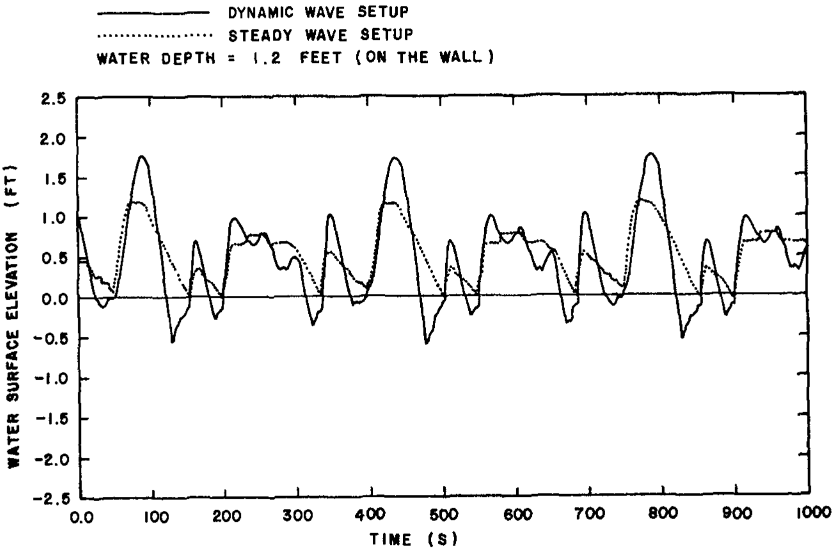

Figure 2. Comparison of the steady wave setup with the dynamic wave setup (simulated by using the wave spectrum in Figure 1).

$$\bar{\eta} = -\frac{K^2 h_b}{16} - \frac{\frac{3}{8}K^2}{1 + \frac{3}{8}K^2}(h_1 - h_b) \tag{39}$$

and h_1 is the water depth at the shore boundary, h_b is the wave breaking depth, K is a constant (here it is 0.78). It is clear that the steady setup and the surf beat wave are almost in phase; however, the crest of the surf beat wave is higher than the steady wave setup height. This setup due to the existence of the surf beat wave is referred to as the "dynamic wave setup", because it includes the contribution from both the non-periodic and low frequency radiation stress terms, and the steady wave setup is generated by including only the non-periodic radiation stress terms.

Conclusions

The results of the study showed that the dynamic wave setup is greater than the steady wave setup, sometimes almost double its value. This is due to the existence of the surf beat wave. Therefore, the dynamic wave setup is important for engineering applications, particularly in the study of storm surges and coastal engineering problems.

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