CHAPTER 37

ON JOINT DISTRIBUTION OF WAVE HEIGHTS AND DIRECTIONS

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In the individual wave analysis of short-crested irregular waves, the wave direction of an individual wave is an important quantity as well as the wave height and period. In this paper, the joint probability density of the wave height and direction is derived theoretically on the assumption of a narrow-banded frequency spectrum. A field experiment was carried out to examine the validity of the theory. The measured joint distribution agreed well with that predicted by the theory.

1. INTRODUCTION

It is of great importance to describe random sea states with sufficient accuracy. Two major methodologies in the analysis of irregular waves are the spectral analysis and individual wave analysis.

In the spectral analysis, irregular waves are regarded as a superposition of component regular waves with various frequencies and directions. From the expected values of the component wave energies, an energy spectrum is defined. Based on the concept of superposition, this analysis method is especially valid as applied to linear phenomena.

On the other hand, in the individual wave analysis, an irregular wave train is decomposed into successive individual waves defined by a zero-crossing method or others. An individual wave is usually regarded as a part of a regular wave train with the same height, period and direction. Hence, various properties of the individual wave such as the surface profile, water particle velocity and pressure can be calculated by applying a nonlinear regular wave theory. This method may not be appropriate in deep water because waves are dispersive and hence an individual wave cannot be independent of the adjoining individual waves. However, in shallow water where wave dispersion is not so rapid, we can even make a nonlinear analysis

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employing this analysis procedure.

Irregular waves are either uni-directional or multi-directional, in other terms, long-crested or short-crested. In the spectral analysis of long-crested irregular waves, the frequency spectrum is defined as the energy density with respect to frequency, and used to describe the characteristics of the wave field. On the other hand, in the individual wave analysis of long-crested irregular waves, the distribution of wave heights or the joint distribution of wave heights and periods is used. For short-crested irregular waves, the spectral analysis is made by introducing the directional spectrum which represents the energy density with respect to frequency and direction. Much effort has been devoted through theoretical and experimental works to extend basic and practical knowledge in these categories. However, the individual wave analysis of short-crested irregular waves, in which the distribution of directions should aiso be discussed, remains to be developed. If a theory is established in this category, it will be of practical use because it will aiiow to evaluate the nonlinear actions of short-crested irregular waves such as wave forces and sediment transport.

The main purpose of this study is to derive a joint probability density of wave height and direction. The envelope functions of the water surface elevation and the two components of horizontal water particle velocity are introduced to define the wave height and direction, on the assumption that the frequency spectrum is narrow-banded. Among possible definitions of wave direction of an individual wave, a definition similar to the mean direction is employed. Examples of the joint density are depicted. The present theory is examined through comparison with the result of a field experiment.

2. THEORY

2.1 Envelope Functions

Wave direction may be defined in many ways. Among them, the propagation direction of the wave crest, the direction of the surface slope, and the direction of the water particle velocity are promising candidates from a practical point of view. In the present study, the two components of horizontal water particle velocity as well as the water surface fluctuation are used for the definition. This will make it more easy to apply the result to the evaluation of wave force, sediment transport, breaking criterion, and others. In addition, the theory can easily be examined by field experiments.

Short-crested irregular waves consist of an infinite number of component waves with various periods and directions. Let the i'th component of frequency be f_i and the j'th component of direction θ_j, then the time series of the water surface elevation, η, at the location, (x, y), and the two components, u and v, of horizontal water particle velocity in the x- and y- direction can be expressed as

$$\left.\begin{aligned}\eta(t)&=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}c_{ij}\cos\phi_{ij}\\ u(t)&=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\hat{H}_i\cos\theta_j\cdot c_{ij}\cos\phi_{ij}\\ v(t)&=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\hat{H}_i\sin\theta_j\cdot c_{ij}\cos\phi_{ij}\end{aligned}\right\} \tag{1}$$

$$\phi_{ij}=k_i(x\cos\theta_j+y\sin\theta_j)-2\pi f_i t-\varepsilon_{ij} \tag{2}$$

$$\hat{H}_i=2\pi f_i\frac{\cosh k_i z}{\sinh k_i h} \tag{3}$$

where c_{ij} and ε_{ij}, respectively, denote the amplitude and phase angle of component waves which represent the frequency range Δf_i and direction range $\Delta\theta_j$. The quantity h is the water depth, z the elevation from the bottom, k_i the wave number determined from f_i by the dispersion relation, and t the time. From the definition, the amplitude is related to the directional spectrum, $S(f,\theta)$, as

$$\langle c_{ij}^2\rangle/2=S(f_i,\theta_j)\Delta f_i\Delta\theta_j \tag{4}$$

where $\langle\ \rangle$ denotes the expected value. The phase angle, ε_{ij}, has a uniform distribution between 0 to 2π.

Here, we assume that the frequency spectrum is narrow-banded. Then f_i can be expressed as the sum of the mean frequency, $\bar{f}$, and a small deviation, f_i':

$$f_i=\bar{f}+f_i' \tag{5}$$

On substituting Eqs. (2) and (5) into Eq. (1) and with use of the additional formulas of the trigonometric functions, the following expressions can be obtained:

$$\left.\begin{aligned}\eta(t)&=\eta_c\cos 2\pi\bar{f}t+\eta_s\sin 2\pi\bar{f}t\\ u(t)&=u_c\cos 2\pi\bar{f}t+u_s\sin 2\pi\bar{f}t\\ v(t)&=v_c\cos 2\pi\bar{f}t+v_s\sin 2\pi\bar{f}t\end{aligned}\right\} \tag{6}$$

where η_c, u_c, v_c, etc. are slowly-varying envelope functions defined by

$$\left.\begin{aligned}\eta_c&=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}c_{ij}\cos\phi'_{ij}\\ u_c&=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\hat{H}_i\cos\theta_j\cdot c_{ij}\cos\phi'_{ij}\\ v_c&=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\hat{H}_i\sin\theta_j\cdot c_{ij}\cos\phi'_{ij}\end{aligned}\right\} \tag{7}$$

$$\phi'_{ij}=k_i(x\cos\theta_j+y\sin\theta_j)-2\pi f_i' t-\varepsilon_{ij} \tag{8}$$

The envelope functions, η_s, u_s and v_s, are defined by substituting $\cos\phi'_{ij}$ by $\sin\phi'_{ij}$ on the right hand side of Eq. (7).

In Longuet-Higgins (1952), the probability density of wave height was determined from the envelope function of the water surface elevation. In the present study, the six envelope functions, η_c, η_s, u_c, u_s, v_c and v_s, determine the wave height and phase lags among the water surface fluctuation and the two components of horizontal water particle velocity of an individual wave. Hence, the joint density of these quantities determines the joint density of the wave height and direction of an individual wave.

The quantity, ε_{ij}, and hence ϕ'_{ij} are uniformly distributed random quantities. By applying the central limit theorem to Eq. (7) and the corresponding equations for η_s, u_s and v_s, all the six quantities are found to have normal distributions. Therefore, the joint density can be determined from the moments up to the second order. The moments of the first order, which are the mean values, vanish from the definition. The moments of the second order, which are variances and covariances, are expressed from Eqs. (1) and (7) as follows:

$$\left.\begin{aligned} &\langle\eta_c^2\rangle=\langle\eta_s^2\rangle=\langle\eta^2\rangle\equiv m_{00}, \quad && \langle\eta_c u_c\rangle=\langle\eta_s u_s\rangle=\langle\eta u\rangle\equiv m_{10}\\ &\langle u_c^2\rangle=\langle u_s^2\rangle=\langle u^2\rangle\equiv m_{20}, && \langle\eta_c v_c\rangle=\langle\eta_s v_s\rangle=\langle\eta v\rangle\equiv m_{01}\\ &\langle v_c^2\rangle=\langle v_s^2\rangle=\langle v^2\rangle\equiv m_{02}, && \langle u_c v_c\rangle=\langle u_s v_s\rangle=\langle uv\rangle\equiv m_{11} \end{aligned}\right\} \tag{9}$$

The covariances for the combination of the subscripts, c and s, such as $\langle\eta_c\eta_s\rangle$ vanish, which means they are independent of each other. With a definition, $(\eta_c, u_c, v_c, \eta_s, u_s, v_s) = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)$, the covariance matrix, M, is summarized as

$$M=[\langle\xi_i\xi_j\rangle]=\begin{bmatrix} M_3 & 0 \\ 0 & M_3 \end{bmatrix} \tag{10}$$

$$M_3=\begin{bmatrix} m_{00} & m_{10} & m_{01} \\ m_{10} & m_{20} & m_{11} \\ m_{01} & m_{11} & m_{02} \end{bmatrix} \tag{11}$$

In general, the multi-dimensional normal distribution is written as

$$p(\xi_1, \xi_2, \cdots, \xi_n) = \frac{1}{(2\pi)^{n/2}\sqrt{|M|}} \exp\left[-\frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i M_{ij}^{-1}\xi_j\right)\right] \tag{12}$$

(Rice, 1944). From Eqs. (10) and (11)

$$|M|=(m_{00}m_{20}m_{02}\Delta)^2 \tag{13}$$

$$M^{-1}=\frac{1}{\Delta}\begin{bmatrix} M_3' & 0 \\ 0 & M_3' \end{bmatrix} \tag{14}$$

$$M_3'=\begin{bmatrix} \frac{1-r_{11}^2}{m_{00}} & -\frac{r_{10}-r_{01}r_{11}}{\sqrt{m_{00}m_{20}}} & -\frac{r_{01}-r_{10}r_{11}}{\sqrt{m_{00}m_{02}}} \\ -\frac{r_{10}-r_{01}r_{11}}{\sqrt{m_{00}m_{20}}} & \frac{1-r_{01}^2}{m_{20}} & -\frac{r_{11}-r_{10}r_{01}}{\sqrt{m_{20}m_{02}}} \\ -\frac{r_{01}-r_{10}r_{11}}{\sqrt{m_{00}m_{02}}} & -\frac{r_{11}-r_{10}r_{01}}{\sqrt{m_{20}m_{02}}} & \frac{1-r_{10}^2}{m_{02}} \end{bmatrix} \tag{15}$$

where

$$\Delta=1+2r_{10}r_{01}r_{11}-r_{10}^2-r_{01}^2-r_{11}^2 \tag{16}$$

$$r_{10}=m_{10}/\sqrt{m_{00}m_{20}} \tag{17}$$

$$r_{01}=m_{01}/\sqrt{m_{00}m_{02}} \tag{18}$$

$$r_{11}=m_{11}/\sqrt{m_{20}m_{02}} \tag{19}$$

and hence Eq. (12) becomes

$$\begin{aligned} &p(\eta_c, u_c, v_c, \eta_s, u_s v_s) \\ &=\frac{1}{(2\pi)^3 m_{00}m_{20}m_{02}\Delta} \\ &\quad\times\exp\left[-\frac{1}{2\Delta}\left\{(1-r_{11}^2)\frac{\eta_c^2+\eta_s^2}{m_{00}}+(1-r_{01}^2)\frac{u_c^2+u_s^2}{m_{20}}\right.\right. \\ &\quad\quad+(1-r_{10}^2)\frac{v_c^2+v_s^2}{m_{02}}-2(r_{10}-r_{01}r_{11})\frac{\eta_c u_c+\eta_s u_s}{\sqrt{m_{00}m_{20}}} \\ &\quad\quad\left.\left.-2(r_{01}-r_{10}r_{11})\frac{\eta_c v_c+\eta_s v_s}{\sqrt{m_{00}m_{02}}}-2(r_{11}-r_{10}r_{01})\frac{u_c v_c+u_s v_s}{\sqrt{m_{20}m_{02}}}\right\}\right] \end{aligned} \tag{20}$$

Here, we take the x-axis in the principal wave direction. Then $m_{11}=0$, and hence $r_{11}=0$. Furthermore, non-dimensional variables are introduced by

$$\begin{aligned} &N_c=\eta_c/\sqrt{m_{00}}\,, \quad U_c=u_c/\sqrt{m_{20}}\,, \quad V_c=v_c/\sqrt{m_{02}} \\ &N_s=\eta_s/\sqrt{m_{00}}\,, \quad U_s=u_s/\sqrt{m_{20}}\,, \quad V_s=v_s/\sqrt{m_{02}} \end{aligned} \tag{21}$$

Since

$$\begin{aligned} &d\eta_c du_c dv_c d\eta_s du_s dv_s \\ &=m_{00}m_{20}m_{02}dN_c dU_c dV_c dN_s dU_s dV_s \end{aligned} \tag{22}$$

Eq. (20) becomes

$$\begin{aligned} &p(N_c, U_c, V_c, N_s, U_s, V_s) \\ &=\frac{1}{(2\pi)^3\Delta}\exp\left[-\frac{1}{2\Delta}\{N_c^2+N_s^2+(1-r_{01}^2)(U_c^2+U_s^2)+(1-r_{10}^2)(V_c^2+V_s^2)\right. \\ &\quad\left.-2r_{10}(N_cU_c+N_sU_s)-2r_{01}(N_cV_c+N_sV_s)+2r_{10}r_{01}(U_cV_c+U_sV_s)\}\right] \end{aligned} \tag{23}$$

where from Eq. (16) with $r_{11}=0$

$$\Delta = 1 - r_{10}^2 - r_{01}^2 \tag{24}$$

2.2 Change of Reference Phase

From the cos and sin components of the envelope functions, the amplitudes and phase angles of the water surface fluctuation and particle velocity of an individual wave can be determined. To take the phase angle, δ, of the water surface elevation as a reference phase, we shift the phases of the three wave properties by δ:

$$\begin{aligned} N_c &= N_p \cos\delta\,, \quad U_c = U_p \cos\delta - U_q \sin\delta\,, \quad V_c = V_p \cos\delta - V_q \sin\delta \\ N_s &= N_p \sin\delta\,, \quad U_s = U_p \sin\delta + U_q \cos\delta\,, \quad V_s = V_p \sin\delta + V_q \cos\delta \end{aligned} \tag{25}$$

where the subscript, p, denotes the component which is in phase with the water surface fluctuation, and the subscript, q, the component which is 90° out of the phase. The above transformation of variables implies

$$dN_c dU_c dV_c dN_s dU_s dV_s = N_p dN_p d\delta dU_p dU_q dV_p dV_q \tag{26}$$

Hence, Eq. (23) is transformed as

$$\begin{aligned} &p(N_p, \delta, U_p, U_q, V_p, V_q) \\ &= \frac{N_p}{(2\pi)^3 \Delta} \exp\left[-\frac{1}{2\Delta} \{ N_p^2 + (1 - r_{01}^2)(U_p^2 + U_q^2) + (1 - r_{10}^2)(V_p^2 + V_q^2) \right. \\ &\qquad \left. - 2r_{10} N_p U_p - 2r_{01} N_p V_p + 2r_{10} r_{01} (U_p V_p + U_q V_q) \} \right] \end{aligned} \tag{27}$$

Since the right hand side of the above equation is independent of δ, $p(N_p, U_p, U_q, V_p, V_q)$ can be obtained by simply multiplying it by 2π.

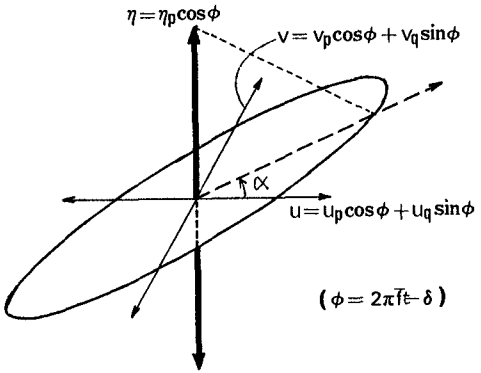

Fig. 1 The path lines of the water surface elevation and the vector of the horizontal water particle velocity.

Figure 1 depicts the physical meaning of the above parameters. The parameters are shown in dimensional form as η_p, u_p, u_q, v_p and v_q. In general, the water surface elevation oscillates sinusoidally and the velocity vector, (u, v), moves along an ellipse. The principal axis of the ellipse represents the velocity vector at the time of its maximum magnitude. Its direction does not necessarily coincide with the direction of the water particle velocity at the time of the maximum water surface elevation. The former one corresponds to the principal wave direction since it is the direction of the principal axis. The latter one corresponds to the mean wave direction since it is defined from the ratio of the two components of water particle velocity which are in phase with the water surface fluctuation. These two directions are candidates of the definition of the individual wave direction. The appropriate definition depends on the problem in which the distribution is used. In this study, the definition similar to the mean direction is employed for the sake of simplicity.

Since the wave direction is independent of U_q and V_q in this definition, Eq. (27) is integrated with respect to U_q and V_q to yield

$$p(N_p, U_p, V_p) = \frac{N_p}{2\pi\sqrt{\Delta}} \exp\left[-\frac{1}{2\Delta}\{N_p^2+(1-r_{01}^2)U_p^2+(1-r_{10}^2)V_p^2 - 2r_{10}N_pU_p-2r_{01}N_pV_p+2r_{10}r_{01}U_pV_p\}\right] \quad (28)$$

2.3 Joint Density of Wave Height and Direction

Since the wave direction is defined by $\alpha=\tan^{-1}(v_p/u_p)$ in the present study, (U_p, V_p) are transformed to the polar coordinates, (W, α):

$$U_p=W\cos\alpha, \qquad V_p=(W/\gamma)\sin\alpha \quad (29)$$

where γ is the correction factor which compensates the distortion resulted through the non-dimensionalization. This is expressed from Eq. (21) by

$$\gamma=\sqrt{m_{02}/m_{20}} \quad (30)$$

Thus γ is found to be the long-crestedness parameter. From Eq. (29)

$$dU_pdV_p=(W/\gamma)dWd\alpha \quad (31)$$

Hence, Eq. (28) becomes

$$p(N_p, W, \alpha)$$
$$= \frac{N_p W}{2\pi\sqrt{\Delta}\gamma} \exp\left[-\frac{1}{2\Delta}\left\{N_p^2 + (1-r_{01}^2)W^2\cos^2\alpha + (1-r_{10}^2)W^2\frac{\sin^2\alpha}{\gamma^2}\right.\right.$$
$$\left.\left. -2r_{10}N_pW\cos\alpha - 2r_{01}N_pW\frac{\sin\alpha}{\gamma} + 2r_{10}r_{01}W^2\cos\alpha\frac{\sin\alpha}{\gamma}\right\}\right] \quad (32)$$

Since N_p and α are the non-dimensional amplitude (half wave height) and direction of an individual wave, the joint probability density of wave height and direction can be obtained by integrating the above equation from 0 to ∞ with respect to W. This can be done by using the following relationship:

$$\int_0^\infty W\exp\left[-\frac{1}{2\Delta}(aW^2 - 2bN_pW)\right]dW$$
$$= \frac{\Delta}{a} + \frac{\sqrt{\Delta}\,bN_p}{a^{3/2}}\exp\left[\frac{1}{2\Delta}\frac{b^2N_p^2}{a}\right]\sqrt{\frac{\pi}{2}}\left(1+\mathrm{erf}\left(\frac{bN_p}{\sqrt{\Delta a}}\right)\right) \quad (33)$$

where erf denotes the error function defined by

$$\mathrm{erf}(\zeta) = \sqrt{\frac{2}{\pi}}\int_0^\zeta \exp\left(-\frac{\zeta'^2}{2}\right)d\zeta' \quad (34)$$

The result of the integration is as follows:

$$p(N_p, \alpha) = \frac{1}{2\pi\gamma}\left\{\frac{\sqrt{\Delta}}{a}N_p\exp\left[-\frac{N_p^2}{2\Delta}\right]\right.$$
$$\left. + \frac{b}{a^{3/2}}N_p^2\exp\left[-\frac{1}{2}\frac{c}{a}N_p^2\right]\sqrt{\frac{\pi}{2}}\left(1+\mathrm{erf}\left(\frac{bN_p}{\sqrt{\Delta a}}\right)\right)\right\} \quad (35)$$

where

$$\left.\begin{aligned} a &= (1-r_{01}^2)\cos^2\alpha + 2r_{10}r_{01}\cos\alpha\frac{\sin\alpha}{\gamma} + (1-r_{10}^2)\frac{\sin^2\alpha}{\gamma^2} \\ b &= r_{10}\cos\alpha + r_{01}\frac{\sin\alpha}{\gamma} \\ c &= \cos^2\alpha + \frac{\sin^2\alpha}{\gamma^2} \end{aligned}\right\} \quad (36)$$

We see from the above result that the joint probability density, $p(N_p, \alpha)$, of wave amplitude and direction can completely be determined from r_{10}, r_{01} and γ. Figure 2 shows some examples of the joint density.

When the directional distribution is narrow and symmetrical, $r_{10} \to 1$ and $r_{01}=0$. Then Eq. (35) is simplified as

$$p(N_p, \alpha) = \frac{1}{\sqrt{2\pi}\gamma}\frac{N_p^2}{\cos^2\alpha}\exp\left[-\frac{1}{2}\left(1+\frac{\tan^2\alpha}{\gamma^2}\right)N_p^2\right] \quad (\cos\alpha > 0) \quad (37)$$

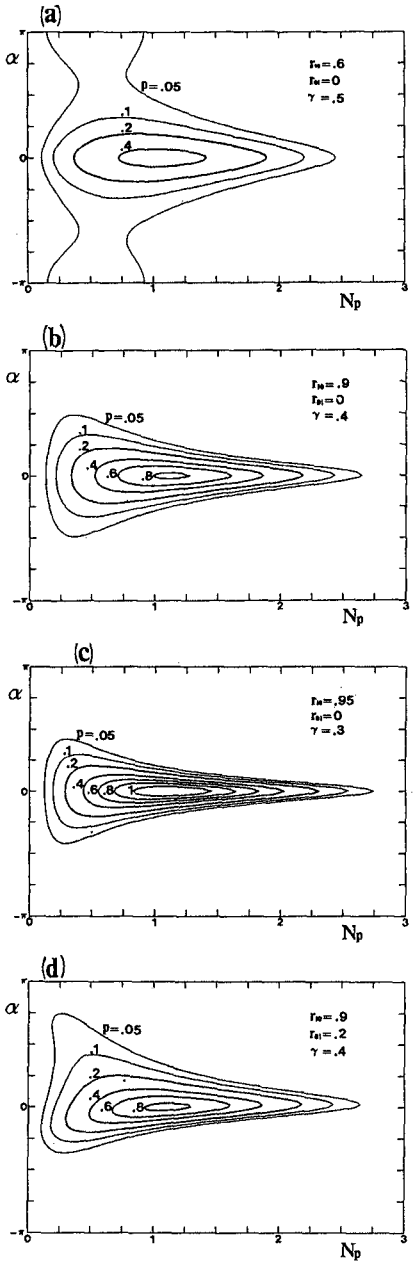

Fig. 2 Examples of the joint probability density of wave height and direction.

In this case, the mean wave height, $2\bar{N}_p(\alpha)$, for a fixed wave direction defined by

$$2\bar{N}_p(\alpha)=2\int_0^\infty N_p p(N_p, \alpha)dN_p \Big/ \int_0^\infty p(N_p, \alpha)dN_p \tag{38}$$

can be obtained analytically as

$$2\bar{N}_p(\alpha)=8\Big/\sqrt{2\pi\left(1+\frac{\tan^2\alpha}{\gamma^2}\right)} \tag{39}$$

This shows that the mean wave height decreases as the direction shifts away from the principal direction.

2.4 Joint Density of Two Components of Water Particle Velocity

In this section, we derive the joint probability density, $p(U_p, V_p)$, of the two components of water particle velocity at the time of maximum water surface elevation. For this purpose, we have only to integrate Eq. (28) from 0 to ∞ with respect to N_p. By using Eq. (33) in the derivation, the following result can be obtained:

$$\begin{aligned} p(U_p, V_p) &= \frac{1}{2\pi}\Bigg[\sqrt{\Delta}\,\exp\left[-\frac{1}{2\Delta}\{(1-r_{01}^2)U_p^2+2r_{10}r_{01}U_pV_p+(1-r_{10}^2)V_p^2\}\right] \\ &\quad +(r_{10}U_p+r_{01}V_p)\exp\left[-\frac{1}{2}(U_p^2+V_p^2)\right] \\ &\quad \times\sqrt{\frac{\pi}{2}}\left(1+\mathrm{erf}\left(\frac{r_{10}U_p+r_{01}V_p}{\sqrt{\Delta}}\right)\right)\Bigg] \end{aligned} \tag{40}$$

Hence, it can be seen that $p(U_p, V_p)$ is determined from only r_{10} and r_{01}. However, γ is necessary to compensate the distortion in the non-dimensionalization. By using the same method as that of Cartwright and Longuet-Higgins (1956), the probability density of the quantities at the time of the maximum water surface elevation can more directly be derived without introducing envelope functions. Through this derivation, Eq. (40) can be obtained without the assumption of a narrow-banded frequency spectrum.

Figure 3 depicts examples of $p(U_p, V_p)$. The probability for U_p to take a negative value decreases as r_{10} increases.

For a symmetrical wave field, in which the principal direction coincides with the mean direction, $r_{01}=0$. Then, Eq. (40) becomes

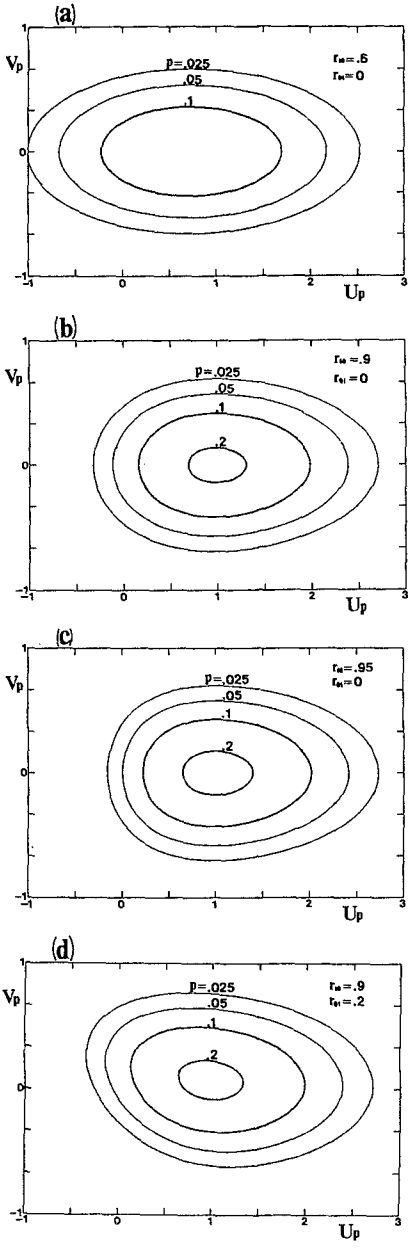

Fig. 3 Joint distribution of the two components of water particle velocity at the time of the maximum water surface elevation.

$$p(U_p, V_p) = \frac{1}{\sqrt{2\pi}}\left[\begin{array}{l}\sqrt{1-r_{10}^2}\exp\left[-\dfrac{U_p^2}{2(1-r_{10}^2)}\right] \\ +r_{10}U_p\exp\left[-\dfrac{U_p^2}{2}\right] \\ \quad\times\sqrt{\dfrac{\pi}{2}}\left(1+\mathrm{erf}\left(\dfrac{r_{10}U_p}{\sqrt{1-r_{10}^2}}\right)\right)\end{array}\right]\times\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{V_p^2}{2}\right] \quad (41)$$

This indicates that U_p and V_p are independent of each other. The root-mean-square values of these parameters are

$$(U_p)_{\mathrm{rms}}=1+r_{10}^2 \quad (42)$$

$$(V_p)_{\mathrm{rms}}=1 \quad (43)$$

Furthermore, in the limit of $r_{10}\to 1$, Eq. (41) becomes

$$p(U_p, V_p)=U_p\exp\left[-\frac{U_p^2}{2}\right]\times\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{V_p^2}{2}\right] \quad (44)$$

which shows U_p has a Rayleigh distribution and V_p has a normal distribution. The average magnitude of the water particle velocity for a fixed direction can be calculated as

$$\bar{W}(\alpha)=\int_0^\infty Wp(W,\alpha)dW\Big/\int_0^\infty p(W,\alpha)dW \quad (45)$$

With the same assumption that gives Eq. (44), the following explicit expression can be obtained:

$$\bar{W}(\alpha)=2\Big/\sqrt{\frac{\pi}{2}\left(\cos^2\alpha+\frac{\sin^2\alpha}{\gamma^2}\right)} \quad (46)$$

3. FIELD EXPERIMENT

3.1 Procedure

A field experiment was carried out near Oarai Port in Ibaraki prefecture, Japan, on August 28, 1985. The port which is shown in Fig. 4 faces to the Pacific Ocean. The primary purpose of the experiment is to measure the spatial distribution of directional spectra resulted from the diffraction due to an offshore breakwater. A set of an ultrasonic wave gage and an electro-magnetic current meter was used to measure simultaneously the water surface elevation and the two components of horizontal water particle velocity near the bottom, from which the directional spectrum at the measuring location was calculated. Among four sets of instruments used, three are fixed all day at No. 0, 4 and 8 in the figure, and the other one was moved to measure the directional spectra at No. 1, 2, 3, 5, 6 and 7. The sampling interval is 0.5s and the

duration of one record is 17min. 3s. In the figure, the distribution of significant wave heights and principal and mean wave directions is shown. The details of the result are found in Izumiya et al. (1986).

The location No. 0 at which the water depth is about 13m is for measuring the incident wave condition. This is the only location at which the recorded data is long enough to calculate two-dimensional joint distributions. Data for 22 hours, and hence 75 records are available there. Since high-frequency waves generated by fishing boats appeared in the data, a numerical band-pass filter which passes the components with the frequency between 0.056 to 0.19Hz was employed. This makes the band width of the data narrower than that of the incident waves. The average significant wave height and period are 62cm and 9.4s, respectively. The variance coefficients, i.e. the ratios of the standard deviations and average values, are 0.088 and 0.033 for the significant wave height and period, respectively, which allows to regard the wave field as steady. However, to reduce the effect of variation of wave field, the average values for each record are used for normalizing individual wave properties in the record. The total number of waves is 7343, and the average values, γ, r_{10} and r_{01}, for the 75 records are 0.31, 0.95 and 0.00, respectively.

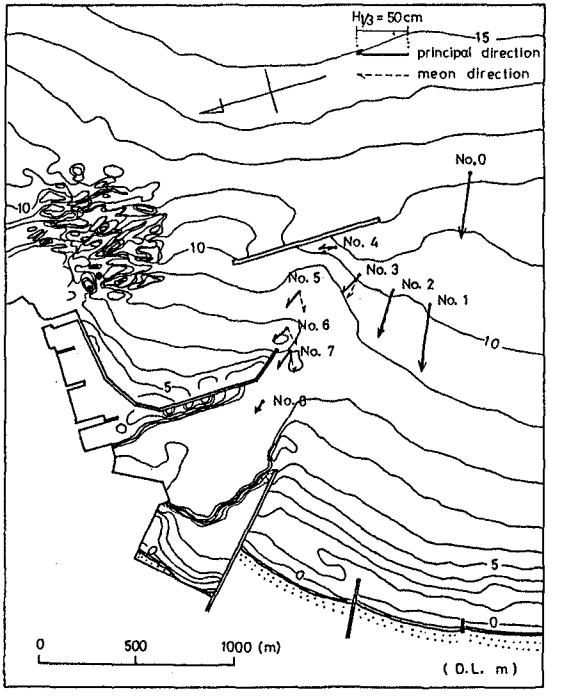

Fig. 4 Observation site.

3.2 Result

Figure 5 compares the predicted and measured joint probability densities of wave height and period. The distance from the origin and the angle from the horizontal axis represent the non-dimensional height and wave direction, respectively. The wave height is non-dimensionalized by the average wave height. Solid and dot lines denote contour lines of the joint density predicted by the present theory. The numbers in a matrix form show the measured values. The values are multiplied by 100. The agreement between the theory and experiment is quite good.

Figure 6 compares the measured and predicted joint probability densities of the two components of horizontal water particle velocity at the time of maximum water surface elevation. As well as in Fig. 5, contour lines represent the theoretical prediction and numbers represent the experimental result. They agree well with each other.

4. CONCLUSION

In the present study, the joint probability density of wave height and direction is derived from the envelope functions of the water surface elevation and the two components of horizontal water particle velocity. In the derivation, the frequency spectrum is assumed to be narrow-banded, and the wave direction of an individual wave is defined similarly to the mean direction. The result is expressed in explicit forms which contain only the long-crestedness parameter and the correlation coefficients between the water surface elevation and the two components of horizontal water particle velocity.

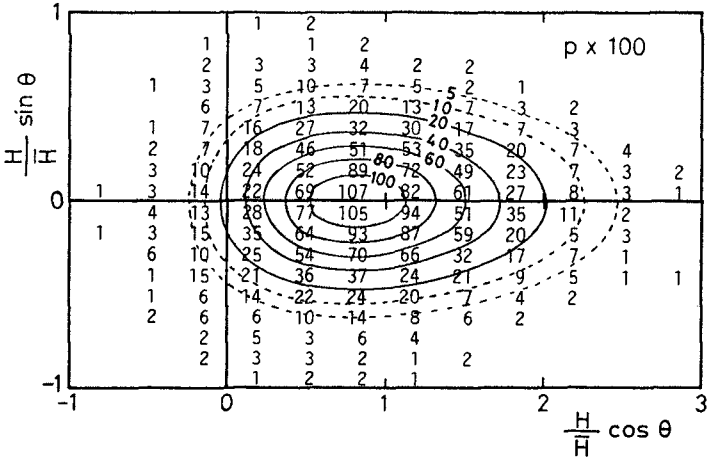

Fig. 5 Comparison between the predicted and measured joint probability densities of wave height and period.

The validity of the present theory is examined by a field experiment. The measured and predicted joint probability densities agree well with each other. Since the numerical band-pass filter used for eliminating ship waves makes the frequency spectrum narrow-banded, the theory remains to be examined for a wide range of sea conditions.

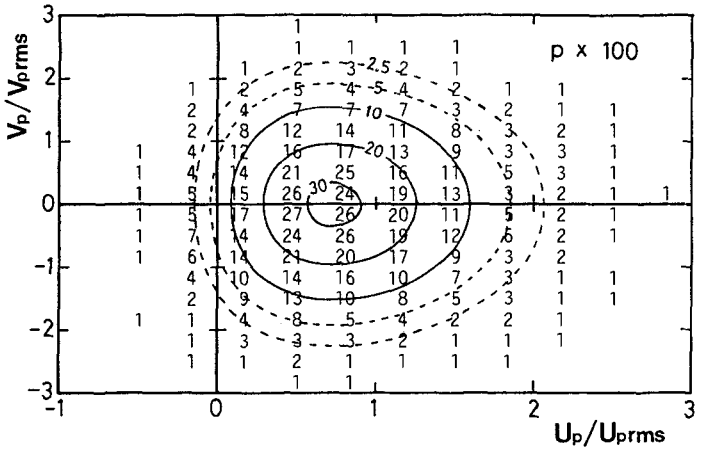

Fig. 6 Comparison between the measured and predicted joint probability densities of the two components of horizontal water particle velocity at the time of the maximum water surface elevation.

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