CHAPTER 35

Subgrid Modelling in Depth Integrated Flows

by

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Introduction

Hydrodynamic simulations in coastal engineering studies are still most commonly carried out using two-dimensional vertically integrated mathematical models. As yet, three-dimensional models are too expensive to be put into general use. However, the tendency with 2-D models is to use finer and finer resolution so that it becomes necessary to include approximations to some 3-D phenomena.

It has been shown by many authors that simulations of large scale eddies can be quite realistic in 2-D models (c.f. Abbott et al. 1985).

Basically there exists two different mechanisms of circulation generation. The first one is based on a balance between horizontally and grid-resolved momentum transfers and the bed resistance - i.e. a balance between the convective momentum terms and the bottom shear stress.

The second one is due to momentum transfers that are not resolved at the grid scale but appears instead as horizontally distributed shear stresses.

In many practical situations the circulations will be governed by the first mechanism.

This is the case if the diameter of the circulation and the grid size is much larger than the water depth. In this situation the eddies are friction dominated so that the effect of sub-grid eddy viscosity is limited.

In this case 2-D models are known to produce very realistic results and several comparisons with measurements have been reported in the literature.

However, when $\Delta x \leq h$ then the eddy viscosity becomes the most important parameter determining the flow pattern. In this case the modelling operation must proceed more cau-

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tiously and a proper closure of the equations must be made in order to describe the effective shear stresses in the momentum equations.

Effective Stresses

The effective stresses arise in the momentum equations via various filtering processes.

The commonly recognized filtering processes are:

Scale 1: Filter out the random molecular motion	→ molecular diffusion → viscosity
Scale 2: Filter out the turbulent motion below a given scale	→ turbulent diffusion → eddy-viscosity
Scale 3: Depth averaging to filter out the vertical velocity profile for 2-D models	→ dispersion → bed & surface shear stresses → horizontal ("effective") shear stresses

It is generally accepted that the eddy terms due to the processes at scale 1 and 2 are negligible relative to scale 3. The effect of the depth-averaging of the velocity profile was taken care of by Elder (1959) who used a logarithmic profile to derive the following expression for the eddy viscosity,

$$E = K_v \cdot h \cdot u^* \qquad , \; K_v \cong 6$$

where

$$u^* \equiv \sqrt{\frac{\tau}{\rho}} = \sqrt{g} \cdot \frac{u}{C} \qquad , \; C = \text{chezy number}$$

However, field measurements and modelling generally show that Elder's coefficient is several orders of magnitude too small. This is illustrated by table 1 in which calibrated eddy coefficients are compared to Elder's expression for 4 different model applications.

Case	h m	u m/s	u* m/s	Δx m	Δt s	E calib. m^2/s	6·hu* m^2/s
A	8	0.7	0.05	50	30	2-5	2.4
B	20	1.0	0.1	500	300	40-50	12
C	30	0.5	0.03	6000	600	≅500	5.4
D	1000	0.1	0.003	30000	900	>6000	18

Table 1 Comparison between Elder's eddy coefficient and calibrated coefficients.

In most classical texts, the development of the equations for nearly-horizontal flow stop at scale 3. However it turns out that for application to numerical modelling it is necessary to extend the filtering process to scale 4, that of the model resolution:

Scale 4: Horizontal averaging over the model grid size Δx	$\longrightarrow$ $\longrightarrow$	additional dispersion additional eddy

When Δx and h are of the same order of magnitude then the processes at both scales 3 and 4 must be considered.

Many modellers attempt to account for the processes at scale 4 by increasing the coefficient, K_v, in Elder's formular, K_v hu*, but this is a mistake.

Since the purpose of the eddy viscosity is to represent sub-grid processes it is natural to relate the length scale to Δx and the time scale to Δt. Hence the eddy viscosity at scale 4 could be considered in one of the following forms:

$$K_1 \cdot \frac{\Delta x^2}{\Delta t} \quad , \quad K_2 \cdot \Delta x \cdot u \quad , \quad K_3 \cdot \Delta t \cdot u^2$$

In table 2 the 3 different forms of the eddy viscosity for scale 4 have been compared to calibrated results in 5 different situations.

Case	h (m)	u (m/s)	Δx (m)	Δt (S)	$E_{cal.}$ (m^2/s)	K_1	K_2	K_3
A	8	0.7	50	30	1-5	0.06-0.01	0.14-0.03	0.34-0.07
B	20	1.0	500	300	40-50	0.06	0.10	0.17
C	30	0.5	6000	600	≅500	0.008	0.17	3.3
D	40	1.0	20	10	1-3	0.075-0.025	0.15-0.05	0.30-0.10
E	1000	0.1	30000	900	≅6000	0.006	2.0	667

Table 2 Eddy coefficients of scale 4.

The form $K_2 \cdot \Delta \cdot u$ appears to be promising since K_2 is almost constant in the 5 different cases.

A more advanced approach is the Smagorinski type of eddy viscosity which depends on horizontal gradients of the depth-averaged flow velocity. This approach will be introduced in the following section.

Smagorinski Eddy Viscosity

The Smagorinsky sub-grid model has been widely used and is generally believed to be correct for homogeneous, isotropic turbulence. Various authors have extended this model to inhomogeneous or anisotropic turbulence, e.g. close to a wall (Schuman, 1975) and to the viscous sub-layer in the boundary layer (Moin and Kim 1982).

The flow equation after filtering out turbulence below the scale Δ, is commonly written

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = - \frac{1}{\rho} \frac{\partial P}{\partial x_i} + F_i + \frac{\partial}{\partial x_j}(E \cdot S_{ij}) \tag{1}$$

where i, j = 1, 2, 3

u = filtered velocity vector
P = pressure
F_i = body forces
E = turbulent eddy viscosity

$$S_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2}$$

The Smagorinsky sub-grid eddy viscosity takes on the form (cf. Leslie, 1982)

$$E = l^2 (S_{ij} S_{ji})^{\frac{1}{2}} \tag{3}$$

The mixing length ℓ is determined by

$$\ell = C_s \cdot \Delta x \tag{4}$$

It has been shown by Lilly (1965) and Leonard (1974) that the resultant energy cascade i.e. the dissipation of the large scales is consistent with the Kolmogorov power spectrum and the constant C_s is dependent only on Kolmogorov's universal constant. Lilly (1965) used the value $C_s = 0.1825$.

In order to extend the Smagorinsky model to 2-D free surface flow, it is necessary to integrate the flow equations over the vertical.

By analogy to the 3-D form we get,

2-D modelling:

$$\frac{\partial U}{\partial t} - \left[\frac{\partial}{\partial x}\left(E \cdot \frac{\partial U}{\partial x}\right) + \frac{1}{2} \frac{\partial}{\partial y}\left(E\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\right)\right] \tag{5}$$

where

$$E = l^2 \cdot \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \frac{1}{2} \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)^2\right]^{\frac{1}{2}} \tag{6}$$

i.e. the local velocities have simply been replaced by the depth-averaged velocities U and V.

The mixing length is still determined from Eq. (4) but it must be expected that the value of the empirical constant C_s will differ from the values established in 3-D modelling. From the first applications of the model it appears that C_s should be found in the interval 0.4 to 0.8.

Results

Some results from the application of the model to Haraldsund in the Faroe Islands are shown in Figs. 1 and 2. Water depths in the area are up to 60 m while the grid size is only 20 m, so it is certain that bed friction is not the governing factor in the size and intensity of the eddies produced by the model. In such a case, dispersive terms such as those introduced by eddy viscosity should be necessary to produce a realistic flow pattern and have a major effect on the eddies, as is seen in Figs. 1 and 2.

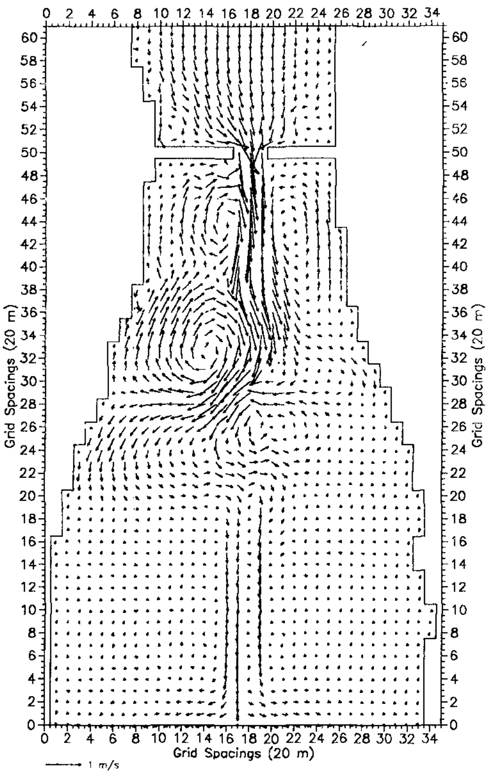

Fig. 1 No eddy viscosity.
Manning Number = 32.

With Zero eddy viscosity a quite unrealistic flow pattern is produced, Fig. 1. In Fig. 2, the results at the same time with C_s = 0.75 are much more plausable.

With C_s = 0.75, eddy viscosities of 1-2 m^2/s were computed by the Smagorinsky model in regions of maximum velocity gradient, and these are of the correct order of magnitude. Finally, it is reported that the results were very sensitive to the value of C_s. In this and other studies, it seems that C_s = 0.40 to 0.80 produces realistic results.

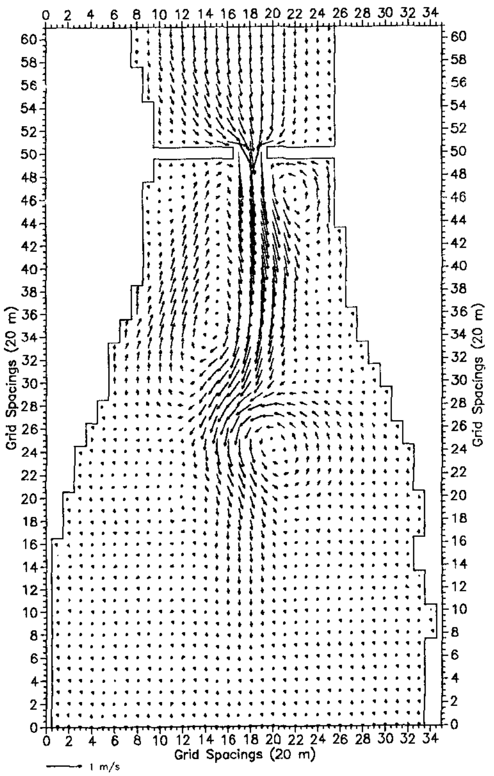

Fig. 2 Smagorinsky eddy, C_s = 0.75.
Manning Number = 32.

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