CHAPTER 12

Generalized Wave Theory for a Sloping Bed

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ABSTRACT This paper discusses the development from first principles of a first-order solution for non-breaking waves on a gently sloping bottom. The theory is derived in a similar fashion as was done by Swart and Loubser (1978) for vocoidal waves on a horizontal bottom. The resulting covocoidal theory was compared to an extensive data set for waves over a sloping bottom (Nelson, 1981) and is tested for analytical validity. It adheres exactly to continuity and the kinematic free surface boundary condition, and shows comparable errors in the dynamic free surface boundary condition to that found for the better, general horizontal bed wave theories.

1. BACKGROUND

In the past decade increasing emphasis has been placed in nearshore and coastal dynamics studies on the measurement in the field of wave and current motions, infragravity activity, suspended sediment loads and associated shoreline changes. These studies have demonstrated that periodic motions in the nearshore, non-breaking wave area are extremely non-linear. It has been shown that various time-independent wave properties such as the maximum horizontal orbital velocity at the bed and the transfer function between incident energy flux and longshore currents, are predicted reasonably well by using linear wave theory (e.g. Le Mehaute et al, 1968 and various recent references). However, time-dependent properties such as wave shape and orbital excursion are non-linear due to the shallow relative depth and asymmetrical due to the sloping bottom.

Elegant computer models are being developed to predict cross-shore wave-driven movement (Baillard, 1982, Stive, 1986). These models rely on an accurate prediction of wave-driven mass transport, and for this purpose use higher moments derived from orbital velocities predicted with Stokesian wave theories. Swart and Loubser (1979) already showed that this mass transport is a function of the non-adherence of these theories to the continuity

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equation. In this respect the cross-shore transport theories would be improved by using a better higher-order wave theory, although this would admittedly increase the computational effort involved. Swart (1978) derived a variable-order cosinusoidal wave theory, the so-called vocoidal wave theory, from first principles. Swart and Loubser (1979) showed that as far as both experimental and analytical validity are concerned, vocoidal theory is comparable to the best other commonly used wave theories. Swart (1982) developed a higher-order Fourier analysis which allows the extraction of vocoidal component waves from a wave spectrum instead of sinusoidal or cosinusoidal waves as in a usual Fourier analysis. Swart (1987) reported on nearshore wave data, gathered in shallow water just outside the breaker zone. In as yet unreported analyses of the data, Swart found deviations from the observed spectra, which can be shown to be consistent with asymmetries associated with the waves shoaling on a sloping bed.

The purpose of the study reported on herein was therefore to investigate the possibility to generalise vocoidal theory to make it applicable for waves shoaling over mildly sloping bottoms. If this can be achieved, it would be possible to without excessive computer effort establish a technique to extract the non-symmetrical, higher-order wave shapes in shallow water and to predict orbital motions and higher moments thereof with greater accuracy than with the Stokesian theories. The potential to improve wave force calculations with asymmetrical waves would also be worth pursuing.

Section 2 reviews the method of derivation of the new theory, after which the analytical and experimental validity of the theory is reviewed in Sections 3 and 4. Section 5 contains a summary of conclusions.

2. DERIVATION

A very similar approach was followed for the development of the generalised vocoidal theory than was followed for the original derivation of vocoidal water wave theory on a horizontal bed. In the following the general assumptions, the governing equations, the boundary conditions and the derivation are given with regular reference to the horizontal bed derivation.

General assumptions

For the derivation of a generalised water wave theory the following general assumptions are made:

* only non-breaking waves are considered;
* the water movement is two-dimensional;
* the flow is frictionless;
* the fluid density is invariant in time and space,

that is, the fluid is incompressible and homogeneous;
* surface tension effects are neglected;
* the sea bed is mildly sloping, that is, it exhibits a slow variation in water depth in the direction of wave propagation;
* the wave motion is periodic and the slope is mild enough for relationship (1) below to be approximtely valid.

$$\frac{d\theta(x,t)}{dt} = \frac{\partial\theta(x,t)}{\partial t} + c\,\frac{\partial\theta(x,t)}{\partial x} \qquad \ldots(1)$$

In expression (1) $\theta(x,t)$ is any time or space-dependent parameter, t denotes time, x is the distance in the direction of wave travel, and c denotes wave celerity. The second-last assumption above, namely, that of a mildly-sloping bed, implies in reality that it is assumed that the bed slope is mild enough to assume that the governing equations are not affected by the sloping bed. The only way in which the formulation used for the derivation of the theory is affected is via the boundary conditions at the sea bed which will be discusssed in more detail below.

Governing equations

The three basic equations which govern the water wave problem are the equation for the conservation of fluid mass, the equations of motion and the expression for the rotationality of a fluid particle.

Continuity equation

In the case of an incompressible, homogeneous fluid in the two-dimensional situation, the continuity equation can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \qquad \ldots(2)$$

where z = vertical dimension positive upwards from the bed and u and w are velocity components in the x and z directions.

Equations of motion

For a frictionless fluid the two-dimensional equations of motion are:

x - direction:

$$\frac{1}{\rho}\frac{\partial\Phi}{\partial x} + \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = 0 \qquad \ldots(3)$$

z - direction:

$$g + \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + \frac{\partial w}{\partial z} = 0 \qquad ...(4)$$

where p = pressure and ρ is fluid density.

Angular velocity

$$R = \frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) \qquad ...(5)$$

It should be noted that if equation (5) should point to a net rotation, this is purely the result of the mathematical manipulation. It can be used to derive rotational wave theories without considering how much such a type of flow i.e. frictionless, "rotatioinal" flow could originate. It was shown, however, in e.g. the vocoidal theory (Swart 1978) that such an approach is advantageous in the development of the theory whilst the net rotation which results is in fact negligible (see Bleach, 1982).

Boundary conditions
The following three boundary conditions are used:

Kinematic bed boundary condition (KBBC)

$$w(x;\ z=0) = u(x;\ z=0)\tan\alpha \qquad ...(6)$$
(see Figure 1 for axis-notation)

Kinematic free-surface boundary condition (KFSBC):

$$w(x;\ z=d+\eta) - u(x;\ z=d+\eta)\frac{\partial \eta}{\partial x} = \frac{\partial \eta}{\partial t} \qquad ...(7)$$

where η = is the surface elevation, measured relative to the mean water level.

Dynamic free-surface boundary condition (DFSBC)

$$p(x);\ z=d+\eta) = 0 \qquad ...(8a)$$

Owing to the fact that the free surface is a streamline, one can use Bernoulli's equation for irrotational flow to transform equation (8) to:

$$Q = \frac{\eta}{d} + \frac{1}{2gd}\left[\{u(x;\ z=d+\eta) - c\}^2 + \{w(x;\ z=d+\eta)\}^2\right] - \frac{c^2}{2gd} \qquad ...(8b)$$

where Q is the constant total energy along the free surface.

Further qualitative assumptions

Nelson (1981) specifically found that the various parameters related to what could be termed bulk properties such as wave celerity and potential energy are virtually unaffected by the slope, whereas time - (or space -) dependent parameters are strongly affected. On the basis of the vocoidal horizontal bed solution and the various results obtained by Nelson (1981) specific, qualitative assumptions were made in addition to those already outlined above.

Swart (1979) assumed that the wave profile on a horizontal bed is qualitatively given by:

$$\frac{\eta}{H} = \mathrm{voc}(P,X) - \eta_{*t} \qquad \ldots(9)$$

where voc(P,X) = the vocoidal function = variable-order-cosinusoidal function

$$= \{\cos^2(\pi X)\}^{P} \qquad \ldots(10)$$

X = dimensionless time = t/T; η_{*t} = dimensionless (with respect to wave height) trough depth and P is the wave profile parameter. The only assumption made regarding P is that it is equal to or greater than unity.

Analogously, it is assumed that the following qualitative definition is representative of **surface elevation** for a sloping bed:

$$\frac{\eta}{H} = h_\alpha\{\mathrm{voc}(P,X)-\eta_{*t}\}+0.5r_\alpha\{\mathrm{cov}(P,X)(1+\cos(\pi X))-(1-\eta_{*t})\} \qquad \ldots(11)$$

where cov (P,X) is the newly defined "covocoidal" function given by

$$\mathrm{cov}(P,X) = 1 - \mathrm{voc}\ (P,X) \qquad \ldots(12)$$

and η_{*t} is defined in equation (9).

The parameters h_α and r_α are unknowns to be determined from the solution to the continuity equation, equations of motion and the surface and sea bed boundary conditions. **It is assumed that P, the vocoidal wave profile parameter, has the same value as in the horizontal bed solution.**

It can be seen that this equation for wave shape (equation 11) has the horizontal bed solution as a special case when $r_\alpha = 0$ and $h_\alpha = 1$.

The horizontal orbital velocity **u** is given by:

$$u/c = \frac{\eta\ M(X)k\ \cosh\{M(X)kz\}}{\sinh\{M(X)k(d+\eta)\}} + \frac{q_\alpha}{c(d+\eta)} \qquad \ldots(13)$$

where M(X) is the vocoidal orbital velocity parameter and q_α is the mass flow due to a sloping sea bed. When $q_\alpha = 0$ this equation is the same as that used by Swart (1979) in the derivation of vocoidal theory.

It will be shown below that the effect of slope on the variation of M(X) with X is such that the values of this parameter as obtained for a horizontal bed solution are not valid on a sloping bed.

It is assumed that the wave celerity c is defined by the same equation as for the horizontal bed vocoidal solution.

$$\frac{c^2}{gd} = \frac{1}{kd} \tanh (Nkd) \qquad \ldots(14)$$

This "assumption" was checked as part of finding a solution to the wave boundary value problem and was found to be valid as a first approximation (see Figure 2).

In solving for the wave boundary value problem as defined by equations (2) to (8b), the unknowns to be quantified are h_α and r_α (in equation 11) and M(X) and q_α(in equation 13).

Solution for wave profile parameters r_α and h_α

The DFSBC is used to solve for the optimum ratio r_α/h_α. The rms value of the deviation over the wave profile ($0 \leqslant X \leqslant 1$) from the mean value of the Bernoulli constant Q (equation 8b) at the free surface, determines to what extent the theory adheres to the DFSBC, i.e.

$$E(DFSBC) = \frac{1}{41}\{\sum_{j=1}^{41} (Q - \bar{Q})^2\} \qquad \ldots(15)$$

where Q is given by equation 8b and $\bar{Q}$ is the mean value of Q in the area $0 \leqslant X \leqslant 1$.

The value of q_α in the horizontal orbital velocity component is obtained from the continuity equation and the kinematic boundary conditions, namely,

$$\frac{q_\alpha}{c(d+\eta)} = \frac{\lambda}{d} \int_0^1 \left(\frac{u_s}{c} - \frac{u_b}{c}\right) \tan \alpha \, dX \qquad \ldots(16)$$

which also yields an expression for w, namely:

$$w = w(X) + w_{mass} + u \tan \alpha \qquad \ldots(17)$$

In the above subscripts "s" and "b" refer to "surface" and "bed" respectively. In equation (17) $w(X)$ represents the cyclic vertical orbital velocity as given in the Appendix, with the qualification that the derivatives of η and $M(X)$ are derivatives of these values for waves on a slope. Furthermore w_{mass} is the component of the vertical orbital velocity which is due to the mass transport q_α. It can from the continuity equation be shown to be equal to

$$w_{mass} = \frac{q_\alpha}{d} \left(\frac{H}{\lambda}\right) (z/d) \frac{\partial(\eta/H)}{\partial X}(1+\eta/d)^{-2} \qquad \ldots(18)$$

The method of solution now involves finding that ratio of r_α/h_α for which the variation of E(DFSBC) is minimized. The following initial conditions were used:

T_C = 1, 2, 3, 4, 5, 10, 20, 30, 40, 50, 60.

H/d = 0.01, 0.02. 0.05, 0.10, 0.20, 0.50, 1.0.

$\tan\alpha$ = 0.001, 0.002, 0.005, 0.01, 0.02, 0.05, 0.10, 0.20.

The numerical results for r_α/h_α were shown by inspection to be functions of H/d, T_C and $\tan\alpha$.

$$r_\alpha/h_\alpha = G(H/d,\ T_C,\ H/d) \qquad \ldots(19)$$

The detailed curve-fitted expressions for r_α/h_α are given in Table I.

The absolute values of r_α and h_α are determined by the fact that

$$\eta_c - \eta_t = H \qquad \ldots(20)$$

where subscripts c and t refer to "crest" and "trough" respectively. η_c and η_t are known in terms of r_α and h_α. Tables IIa and IIb show typical values of r_α and h_α.

Solution for orbital velocity parameters M(X) and q_α

The orbital velocity parameter $M(X)$ is found by rewriting the particle rotation R (equation 5) through substitution of the appropriate expressions for u and w. In the same way as was done by Swart (1979) the orbital velocity parameter was found by defining

$$M(X) = R_M\, M_1(X) \qquad \ldots(21)$$

where $M_1(X)$ is the first-order, i.e. small amplitude, approximation of $M(X)$ and R_M is the correction which has to be made to negate the small-amplitude assumption. **It is assumed that the expression for R_M in the Appendix is also valid for the covocoidal theory.** The value of $M_1(X)$ is found from the small-amplitude, covocoidal version of equation 5, that is

$$\frac{\partial^2(\eta/H)}{\partial X^2} + 4\pi^2(\eta/H)\, M_1^2\,(X) = 0 \qquad \ldots(22)$$

It can be shown (Swart, 1988) that at the wave crest (X=0) the values of M_1 for covocoidal and vocoidal theory are related, namely

$$M_{1c}(\text{covoc}) = (h_\alpha - r_\alpha)^{0.5}\, M_{1c}(\text{voc}) \qquad \ldots(23)$$

M_1(covoc) is called $M_{c1\alpha}$ herein.

Swart (1988) solved for $M_1(X)$ from equation (22) for a range of initial conditions, namely

T_c = 1,2,3,4,5,10,20,30,40,50,60
h/d = 0,01,0.02,0.05,0.10,0.2,0.5,1.0
r_α/h_α = 0.01,0.02,0.05,0.10,0.20,0.40

Figure 3 shows a typical variation of $M_1(X)$ for covocoidal theory in comparison with $M_1(X)$ for vocoidal theory on a horizontal bed. The value of $M_1(X)$ at x = 0.5 was called $M_{t1\alpha}$ and was established from the numerical results for the whole range of initial conditions to be

$$M_{t1\alpha} = g_1(T_c)\; g_2(r_\alpha/h_\alpha) \qquad \ldots(24)$$

Curve-fitted expressions $M_{t1\alpha}$ given in Table III.

It is straightforward to find a solution for q_α. Equation (16) was integrated numerically using the Rombert integration method for a range of initial conditions:

T_c = 5, 10, 20, 30, 40, 50, 60
H/d = 0.1, 0.2, 0.5, 1.0
$\tan\alpha$ = 0.01, 0.02, 0.05, 0.10

By means of curve-fitting an expression was found from the numerical results for the mass transport velocity:

$$\frac{q_\alpha}{c(d+\eta)} = R(H/d,\ T_c,\ \tan_\alpha) \qquad \ldots(25)$$

The nature of the function R is given in Table IV.

This concludes the derivation of covocoidal theory.

3. ANALYTICAL VALIDITY

As part of the validation of vocoidal water wave theory for a horizontal bed Swart and Loubser (1979) performed an analytical validity study in which the adherence of vocoidal theory to the original governing equations and boundary conditions was compared with that for twelve other commonly used water wave theories, namely, linear Airy theory, Stokes II and V, three different gravity cnoidal theories, Dean's stream function theory, two different rotational cnoidal theories and three different approximated wave theories. The two free surface boundary conditions and the adherence to the continuity equation were computed for each of the theories for a wide range of initial conditions.

Vocoidal theory adheres exactly to the kinematic free surface boundary condition and the continuity equation. Swart and Loubser (1979) showed that the error in the DFSBC was for a range of non-linearities covering all non-breaking waves either the lowest of all theories tested or not statistically different from the theory with the lowest error at the 95 per cent confidence level, for non-linearity values exceeding 260, which cover intermediate and shallow water conditions.

In exactly the same manner as was done in the Swart and Loubser study the error in the DFSBC was computed at 41 equidistant points over one wave period for covocoidal theory and a mean error calculated in subsets which were chosen to represent waves of roughly equal non-linearity, i.e. which had approximately equal wave shapes. The results were compared with those for vocoidal theory in the earlier study and it was concluded that although covocoidal theory exhibits errors in the DFSBC which grow marginally bigger as slope increases (see Figure 4), the covocoidal theory still had such small errors in the DFSBC that it did not differ significantly at the 95 per cent confidence level from vocoidal theory, which as stated above was shown to be amongst the best theories in the Swart and Loubser (1979) study as far as adherence to the DFSBC is concerned (see Figure 4). It is therefore concluded that the theory has a sufficiently sound basis to serve as a good first approximation of waves on a sloping bed.

4. EXPERIMENTAL VALIDITY

Experimental validation studies for wave theories have traditionally used laboratory data (Le Mehaute **et al**, 1968; Swart and Loubser, 1979 and Hattori 1986). Good full-scale data are nowadays being gathered in the field and in mega-flumes. Due to the fact that such data are usually for random waves, it is not simple to establish the exact cause(s) of correspondence or non-correspondences of theory to data.

Nelson (1981) performed an extensive set of experiments for slopes from horizontal to 1 in 10, to test the effect of bed slope on wave profile parameters and wave celerity. His celerity data yield results totally compatible with the theoretical predictions in Figure 2. Figure 5 defines the wave profile parameters tested. Using the same range of initial conditions (H/d, T_c, $\tan\alpha$) as in the Nelson study, theoretical predictions of the parameters defined in Figure 5 were made. The resulting correspondences between Nelson's measured data and the theoretical predictions in this study are shown in Figure 6. It can be concluded that good, qualitative correspondence is achieved, although there are some quantitative divergences. Nevertheless, wave shape changes on beds of different slope are predicted well as a first approximation.

Swart and Loubser (1979) showed that vocoidal theory was consistently the best predictor of the thirteen theories tested as far as the prediction of horizontal orbital velocity is concerned. No good data exists to test systematically the effect of increasing bottom slope on the orbital velocity. Four orbital velocity data sets were selected from the Swart and Loubser study which have roughly similar initial wave conditions (H/d, T_c) and for which the bed slope varies between 0 and 1 in 20. Although the covocoidal orbital velocity shows some improvement over vocoidal theory, particularly as far as the variation of orbital velocity under the wave trough with distance from the bed is concerned, the improvement expected in the absolute value of the orbital velocity did not materialise (see Figure 7). Nevertheless, the overall correspondence with the data is satisfactory. To improve the inconclusiveness as far as orbital velocity prediction is concerned, emphasis should be placed on gathering high quality orbital velocity data, preferably in the field to eliminate the introduction of spurious mass transport, for a range of bed slopes.

5. CONCLUSIONS

- A **new wave theory,** the so-called covocoidal wave theory, was derived from first principles and is valid for all non-breaking waves on a mildly sloping bottom.

- **No simplifying assumptions** were made and the finite amplitude condition was preserved. Some approximation was, however, introduced because the numerical results were represented by means of curve-fitted expressions to facilitate easier application.

- The **analytical validity** of the theory was established to be as good as or better than the better, commonly-used horizontal bed wave theories, by virtue of the fact that the theory adheres exactly to the KFSBC and to continuity and has a minimised error in the DFSBC which was shown to not differ statistically from vocoidal theory (for $\tan\alpha=0$) at the 95 per cent confidence level.

- **Experimental validity** was established primarily by comparison of theoretical predictions with an extensive data set collected by Nelson (1981) to establish the effect of bottom slope on wave profile parameters and wave celerity. Indications on the prediction of orbital velocity are inconclusive due to the lack of appropriate data.

- The new **covocoidal theory** can serve as basis for the study of a variety of wave processes on a sloping bed outside the breaker zone.

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TABLE I: Equations for determination of r_a and h_a

$G_a = 0.16\,(H/d)^{0.77}$		...(I. 1)
$G_b = 5\tan\alpha$		...(I. 2)
$G_c = 0.15(H/d)^{0.5}$		...(I. 3)
$G_d = 0.28(H/d)^{-0.42}+1.6$		...(I. 4)
$G_e = 1 + [\ln\{2.04(H/d)^{-1/16}\}]^{10}$		...(I. 5)
$G_f = G_e \exp(-G_d \exp(-G_c Tc))$		...(I. 6)
$G_g = 0.48(1-\exp(-48H/d))$		...(I. 7)
$G_h = G_f G_g$		...(I. 8)
$G_i = 1 - \exp(-22(\tan\alpha)^{0.5})$		...(I. 9)
$G_j = (G_h - G_a)G_i$		...(I.10)
$G_k = G_j + G_a G_b$		...(I.11)
$G_l = 0.03\ln(\tan\alpha) + 0.275$		...(I.12)
$G_m = 0.424 + 0.0128(\tan\alpha)^{-0.6}$		...(I.13)
$G_n = 1$	for $H/d < 0.5$	
$G_n = G_l(H/d - 0.5)^{G_m} - 0.1$	for $H/d > 0.5$	
If $G_n > 1$ then $G_n = 1$.		...(I.14)
$r_a/h_a = G_k G_n$		...(I.15)
$F_a = 0.66 + 0.065(\ln P)^{0.5}$		...(I.16)
$F_b = 1 + 1.375P^{-F_a}$		...(I.17)
$F_e = -0.92(\ln P)^{0.25} + 1.82$		...(I.18)
$F_d = 1 + 0.2584\exp(-0.353(P-1)^{F_e})$		...(I.19)
$r_a + h_a = 1 + F_b\,(r_a/h_a)^{F_d}$		...(I.20)

r_a and h_a are then found from equations (I.15) and (I.20)

TABLE IIa: Typical values for r_α

		.01	.02	.05	.10	.20	.50	1.00
1.0	.01	.0061	.0153	.0296	.0325			
	.02	.0068	.0168	.0335	.0362			
	.05	.0077	.0186	.0362	.0417			
	.10	.0089	.0207	.0404	.0488			
2.0	.01	.0067	.0167	.0333	.0386	.0345		
	.02	.0074	.0183	.0365	.0428	.0392		
	.05	.0083	.0202	.0403	.0485	.0477		
	.10	.0095	.0223	.0446	.0557	.0598		
5.0	.01	.0084	.0213	.0456	.0589	.0650	.0634	
	.02	.0092	.0233	.0498	.0647	.0722	.0727	
	.05	.0102	.0253	.0541	.0714	.0821	.0899	
	.10	.0115	.0275	.0585	.0789	.0940	.1153	
10.0	.01	.0117	.0302	.0697	.0991	.1244	.1531	.0169
	.02	.0127	.0329	.0758	.1082	.1364	.1695	.0227
	.05	.0139	.0353	.0813	.1169	.1494	.1909	.0329
	.10	.0151	.0375	.0860	.1250	.1630	.2172	.0456
20.0	.01	.0198	.0526	.1294	.1904	.2406	.2810	.0290
	.02	.0215	.0570	.1403	.2065	.2613	.3066	.0383
	.05	.0231	.0604	.1485	.2192	.2789	.3327	.0529
	.10	.0243	.0628	.1538	.2280	.2929	.3586	.0688
50.0	.01	.0579	.1470	.3161	.3954	.4097	.3697	.0332
	.02	.0625	.1586	.3410	.4266	.4426	.4017	.0437
	.05	.0656	.1661	.3571	.4475	.4669	.4312	.0598
	.10	.0672	.1693	.3636	.4574	.4816	.4573	.0768

TABLE IIb: Typical values for h_α

		.01	.02	.05	.10	.20	.50	1.00
1.0	.01	.9978	.9971	.9987	.9994			
	.02	.9977	.9971	.9994	1.0003			
	.05	.9975	.9972	1.0002	1.0018			
	.10	.9974	.9974	1.0014	1.0040			
2.0	.01	.9977	.9971	.9995	1.0008	.9997		
	.02	.9976	.9972	1.0003	1.0020	1.0010		
	.05	.9974	.9974	1.0013	1.0038	1.0035		
	.10	.9973	.9976	1.0026	1.0064	1.0079		
5.0	.01	.9974	.9974	1.0023	1.0063	1.0077	1.0066	
	.02	.9973	.9976	1.0037	1.0084	1.0104	1.0096	
	.05	.9972	.9979	1.0051	1.0111	1.0144	1.0159	
	.10	.9971	.9982	1.0067	1.0142	1.0198	1.0261	
10.0	.01	.9970	.9983	1.0092	1.0196	1.0256	1.0216	1.0000
	.02	.9970	.9988	1.0116	1.0233	1.0298	1.0247	1.0004
	.05	.9969	.9993	1.0137	1.0269	1.0343	1.0289	1.0011
	.10	.9969	.9998	1.0157	1.0304	1.0391	1.0339	1.0022
20.0	.01	.9970	1.0033	1.0250	1.0291	1.0225	1.0081	1.0002
	.02	.9971	1.0046	1.0282	1.0323	1.0248	1.0090	1.0005
	.05	.9972	1.0057	1.0308	1.0348	1.0267	1.0099	1.0010
	.10	.9973	1.0065	1.0324	1.0365	1.0282	1.0108	1.0016
50.0	.01	1.0053	1.0175	1.0210	1.0132	1.0059	.9999	.9996
	.02	1.0062	1.0193	1.0228	1.0144	1.0064	1.0001	.9995
	.05	1.0069	1.0205	1.0239	1.0151	1.0069	1.0003	.9995
	.10	1.0072	1.0210	1.0244	1.0154	1.0071	1.0005	.9995

TABLE III: Equations for determination of K_{t1e}

q_a	$= K_{t1a}$ from equation (24) for $H/d = 0.1$	...(III.1)
q_b	$= 1.67\ q_a Tc^{-1.44}$	...(III.2)
q_c	$= 0.21\ q_a$	...(III.3)
If $q_b > q_c$ then $q_b = q_c$		...(III.4)
q_d	$= 4\ (\frac{r_a}{h_c})^{0.5}$	...(III.5)
q_e	$= \exp[-4.3(P_t - 1)]$	..(III.6)
where P_t is defined in the Appendix		
q_f	$= q_b\ q_d$	...(III.7)
If $q_f < q_e$ then $q_f = q_e$		...(III.8)
$K_{t1e} = q_f$		...(III.9)

TABLE IV: Equations for determination of $q_s/(c(d+a))$

R_a	$= 0.9(1 - \exp(-0.538\ Tc^{R_b}))$		...(IV.1)
R_b	$= 0.26 - 0.14 \cos(0.05 \cdot Tc)$	for $Tc < 20$	...(IV.2)
	0.40	for $Tc > 20$	
R_c	$= R_d\ (H/H_b) + R_e\ (H/H_b)^2 + R_f\ (H/H_b)^3$		...(IV.3)
R_d	$= 0.795\ (1 - \exp(-0.044 Tc))$		...(IV.4)
R_e	$= 0.45 + 0.75\ (\cos^2(0.005 \cdot Tc))^2 - 0.0043 Tc$		...(IV.5)
R_f	$= -0.75\ (\cos^2(0.005 \cdot Tc/200))^2 - 0.13$		
K_g	$= 1 - \exp(-0.046\ Tc)$		...(IV.7)
K_h	$= 13 \times 10^{-3}\ (H/H_b) - 10 \times 10^{-3}\ (H/H_b)^{3.5}$		...(IV.8)
R_i	$= R_h R_g$		...(IV.9)
R_j	$= 16.12 + \ln(\tan\alpha + 10^{-7})$		...(IV.10)
R_k	$= R_i R_j$		...(IV.11)
R_l	$= R_c R_a/(R_d + R_e + R_f)$		...(IV.12)
R_m	$= R_l - R_k$		...(IV.13)
$\frac{q_s}{c(d+a)}$	$= R_m \tan\alpha$		...(IV.14)

APPENDIX A: Expressions for wave profile, orbital velocity and celerity parameters in Vocoidal theory (Swart, 1978 and Swart, 1988)

Wave profile parameter

$$\eta/H = \mathrm{voc}(P,X) - \eta_{*t} \qquad \ldots(A.1)$$

$$\mathrm{voc}(P,X) = H\ \left\{\cos^2\left(\pi\frac{X}{\lambda}\right)\right\}^P \qquad \ldots(A.2)$$

$$\eta_{*t} = (\pi P)^{-0.5}\left\{1 - (8P)^{-1} + (8P)^{-2} - (8P)^{-3}\right\} \qquad \ldots(A.3)$$

$$P = R_P P_l \qquad \ldots(A.4)$$

$$R_P = R_{Pl} - (R_{Pl} - 1)\exp\left(b\frac{\lambda}{d}\right) \qquad \ldots(A.5)$$

$$R_{Pl} = 1 + 0.0021(P_{1l} - 1) + 6.09 \times 10^{-7}(P_{1l} - 1)^{2.56} \qquad \ldots(A.6)$$

$$b = -0.0916 + 2.718 \times 10^{-4} P_{1l} \qquad \ldots(A.7)$$

$$P_l = \begin{cases} P_{l3} & \text{for } F_s < -1.1 \\ P_{l3} + 5\,(P_{l2} - P_{l3})\,(F_s + 1.1) & \text{for } -1.1 < F_s < -0.9 \\ P_{l2} & \text{for } F_s > -0.9 \end{cases} \qquad \ldots(A.8)$$

$$P_{l3} = \begin{cases} P_{10} & \text{for } P_{11} < P_{10} \\ P_{11} & \text{for } P_{11} > P_{10} \end{cases} \qquad \ldots(A.9)$$

$$P_{10} = 1 + 0.05\, U_r \qquad \ldots(A.10)$$

$$U_r = \left(\frac{H}{d}\right)\left(\frac{\lambda}{d}\right)^2 \qquad \ldots(A.11)$$

$$P_{11} = a_l + b_l\, U_r \qquad \ldots(A.12)$$

$$a_l = -0.133 - 1.767\left(\frac{H}{d}\right) + 0.00248\exp\left(8.33\left(\frac{H}{d}\right)\right) \qquad \ldots(A.13)$$

$$b_l = 0.1672 - 0.112\left(\frac{H}{d}\right) \qquad \ldots(A.14)$$

$$P_{12} = 1.03\, F_s + 9 + 3.33\exp\left\{-0.109(11 + F_s)\right\} \qquad \ldots(A.15)$$

P_l = 1 when $F_s < -11$ or when $P_l < 1$ according to the above.

$$F_s = (U_r - U_{ro}) / \theta \qquad \ldots(A.16)$$

$$U_{ro} = 63 + 90\left(\frac{H}{d}\right)^{1.48} \qquad \ldots(A.17)$$

$$\theta = \begin{cases} 1.01\exp\left\{3.31\left(\frac{H}{d}\right)\right\} & \text{for } H/d \geq 0.505 \\ 5.38 & \text{for } H/d < 0.505 \end{cases} \qquad \ldots(A.18)$$

$$P_{1l} = P_l \text{ where } \frac{\lambda}{d} = 50 \text{ and } H/d = H/d \qquad \ldots(A.19)$$

Orbital velocity parameter

$$\frac{u}{c} = \frac{\pi\, M(X)\, k \cosh\left[M(X)\, k\, z\right]}{\sinh\left[M(X)\, k\, (d+\eta)\right]} \qquad \ldots(A.20)$$

$$M(X) = \begin{cases} (M_c - M_t)\left|\cos\left(\pi \frac{X}{\lambda}\right)\right| + M_t & \text{for } r_1 P_1 < 0.5 \\ M_c\, voc\,(r_1 P_1, X) & \text{for } r_1 P_1 \geq 0.5 \\ M_c & \text{for } P = 1 \end{cases} \qquad \ldots(A.21)$$

$$r_1 = 0.4\,(1 - P_1^{-0.9}) \qquad \ldots(A.22)$$

$$M_c = R_M M_{c1} \qquad \ldots(A.23)$$

$$M_{c1} = \begin{cases} \left[P/(2\,\eta_{*c1})\right]^{0.5} & \text{for } H/d \leq 0.7 \\ \left[P_1/(2\,\eta_{*c1})\right]^{0.5}_{H/d=0.7} & \text{for } H/d > 0.7 \end{cases} \qquad \ldots(A.24)$$

$$R_M = \begin{cases} \dfrac{0.54}{1 + 6.7\,(\frac{H}{d})} + 0.46 & \text{for } \frac{H}{d} \leq 0.7 \\ 0.667 & \text{for } \frac{H}{d} > 0.7 \end{cases} \qquad \ldots(A.25)$$

$$M_t = R_M \exp\left(-4.2\,(P_1 - 1)\right) \qquad \ldots(A.26)$$

Wave celerity parameter

$$\frac{c^2}{gd} = \left(\frac{1}{Nkd}\right) \tanh Nkd \qquad \ldots(A.27)$$

$$N = \begin{cases} R_n N_1 & \text{for } R_n N_1 < 1 + \eta_c/d \\ 1 + \eta_c/d & \text{for } R_n N_1 > 1 + \eta_c/d \end{cases} \qquad \ldots(A.28)$$

$$R_n = \begin{cases} 0.67\, f_a + 0.938 & \text{for } f_a > 0.093 \\ 1 & \text{for } f_a < 0.093 \end{cases} \qquad \ldots(A.29)$$

$$f_a = \left(\frac{H}{d}\right)\left(\frac{\lambda}{d}\right)^{-0.25} \qquad \ldots(A.30)$$

$$N_1 = \begin{cases} 1 + 0.19\, f_b^{1.5} & \text{for } f_b < 0.72 \\ 0.72\, f_b + 0.6 & \text{for } f_b > 0.72 \end{cases} \qquad \ldots(A.31)$$

$$f_b = \left(\frac{H}{d}\right)\left(\frac{\lambda}{d}\right)^{0.1} \qquad \ldots(A.32)$$

$$\eta_c = H + \eta_t \qquad \ldots(A.33)$$

$$\eta_t = -\eta_{*t}\, H \qquad \ldots(A.34)$$

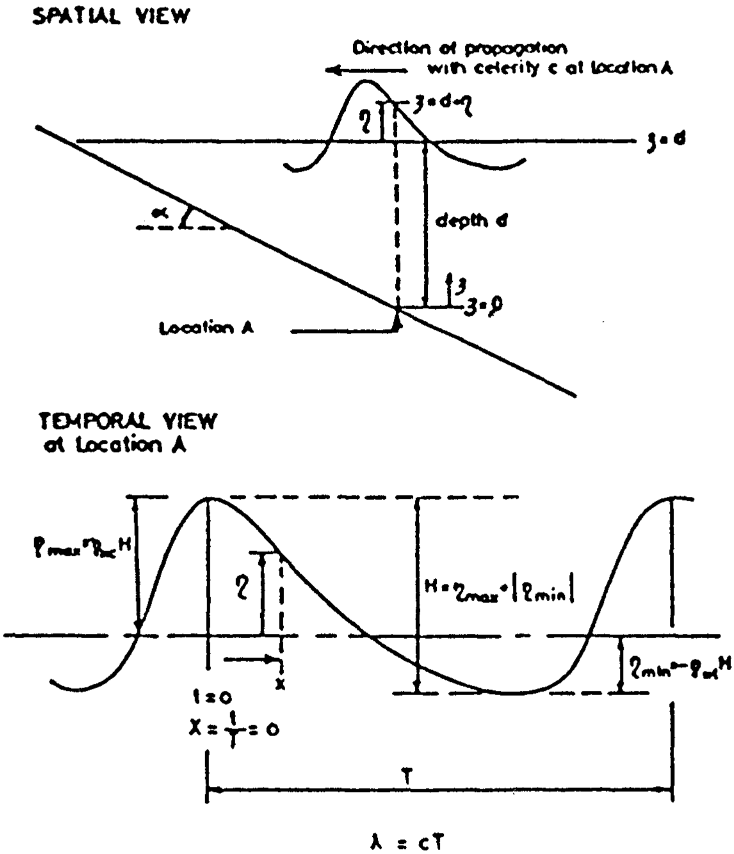


Figure 1: Axis-notation for development of generalised vocoidal theory on a sloping bottom.

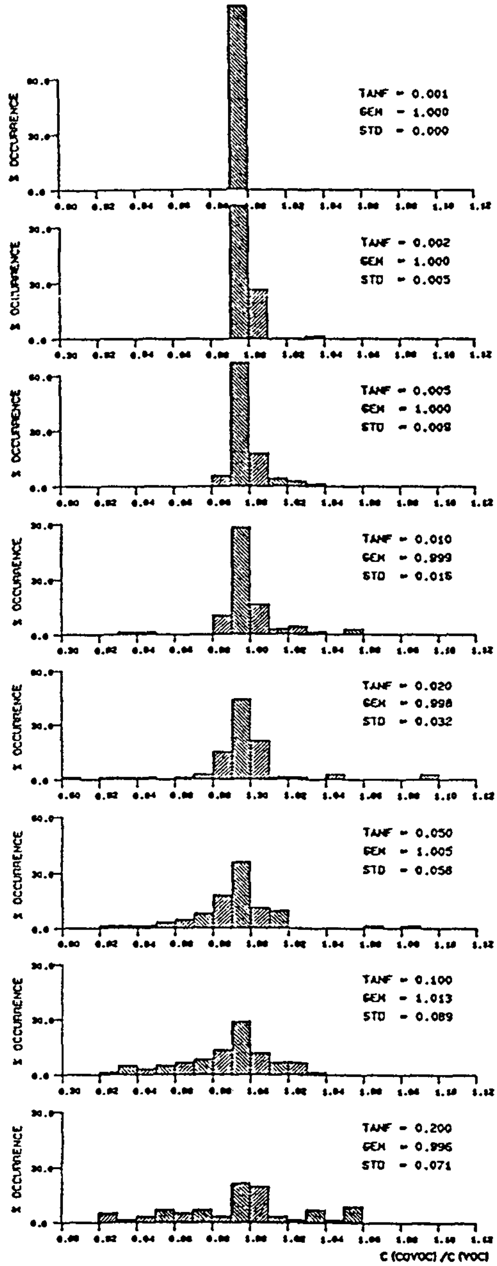


Figure 2: Distribution of the ratio of c^2/gd for a sloping bed to that for a horizontal bed for a slope of 1 in 5 and a range of initial conditions.

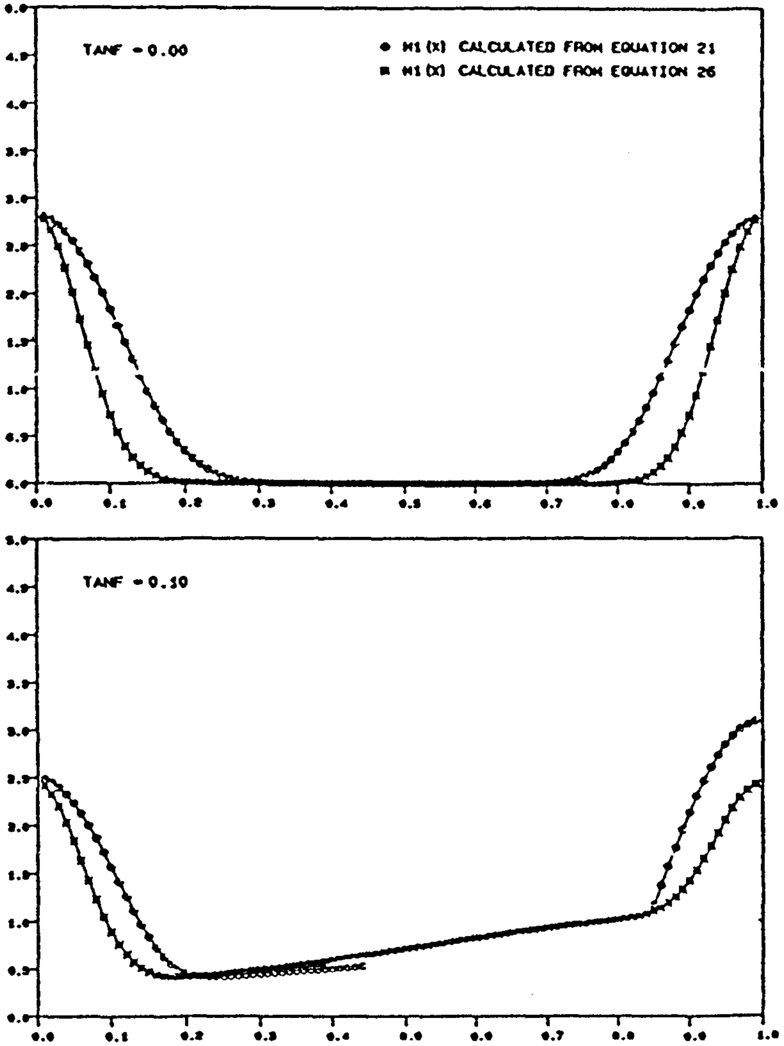


Figure 3: Typical, predicted variation of the orbital velocity parameter $M_1(X)$ for a horizontal bed and for a slope of 1 in 10 for waves with $H/d = 0.5$ and $T_c = 20$.

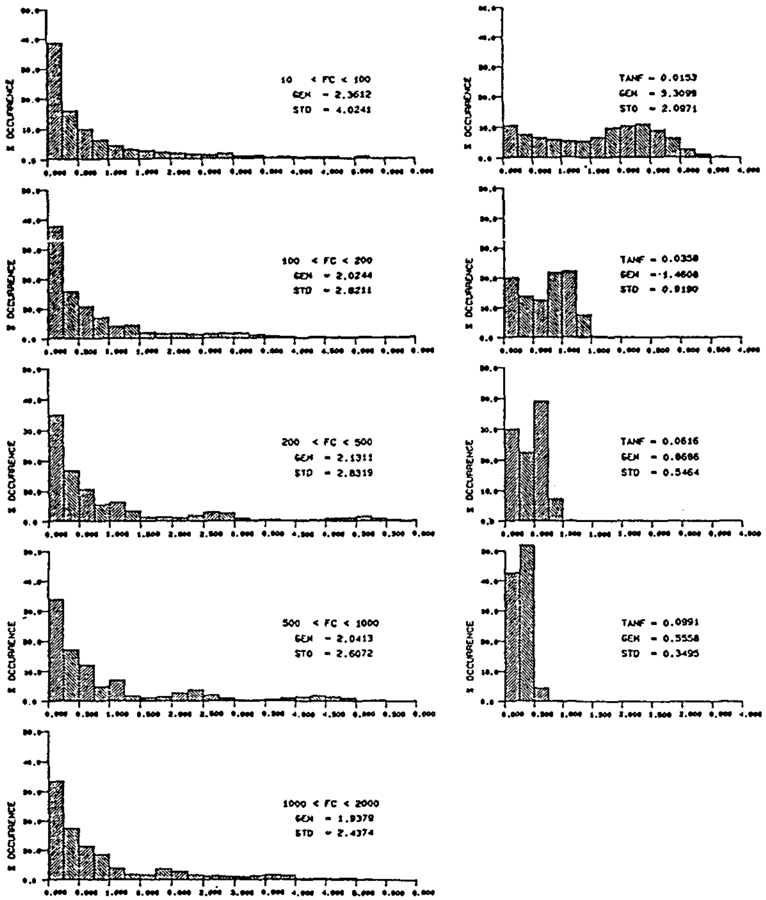
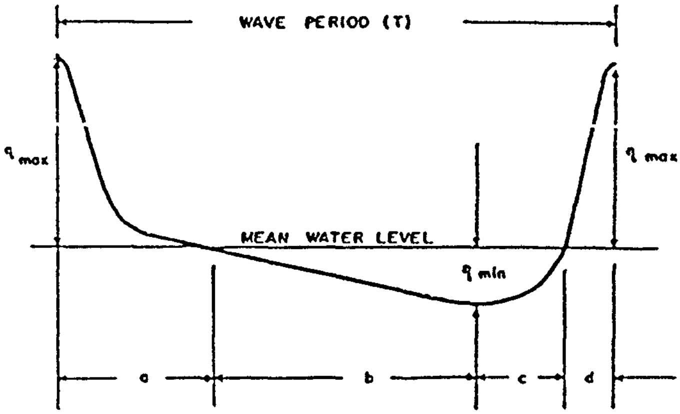


Figure 4: The error in the DFSBC for covocoidal theory for $\tan\alpha = 0.1$, and for vocoidal theory on a horizontal bed, for a range of nonlinearity parameter values.



RATIO CREST TIME / WAVE PERIOD	$(a + d)/T$
RATIO CREST HEIGHT / WAVE HEIGHT	$\eta_{max}/(\eta_{min} + \eta_{max})$
CREST SKEWNESS	$d/(a + d)$
TROUGH SKEWNESS	$b/(b + c)$
RATIO WAVE SLOPE / BED SLOPE	$\eta_{min}/(\lambda\, b \tan\alpha)$

Figure 5: Definition sketch for wave parameters.

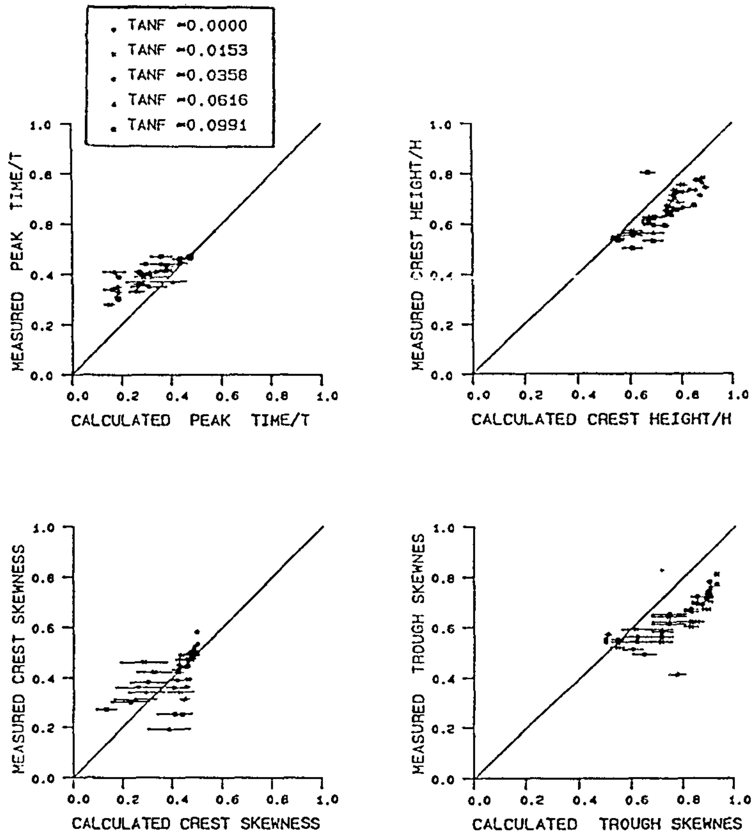


Figure 6: Comparison of predicted and observed wave parameters according to definitions in Figure 5 for a range of nonlinearity parameter values and bed slopes.

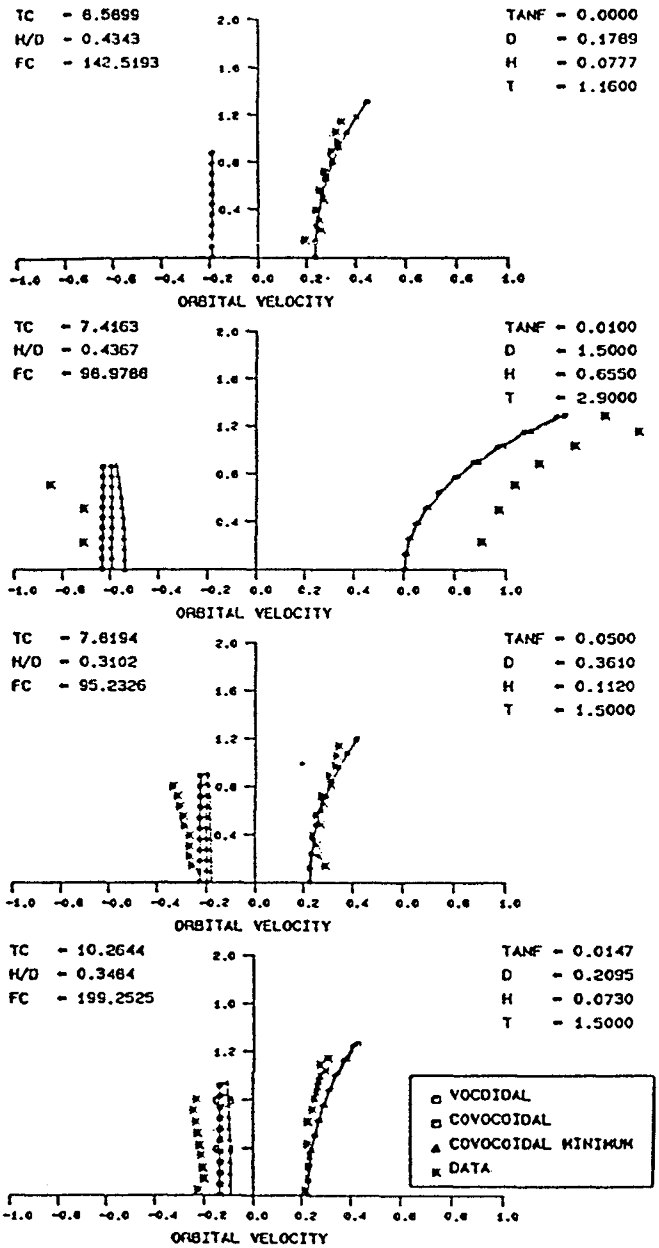


Figure 7: Comparison of predicted and observed horizontal orbital velocities under the wave crest for a range of bottom slopes and $F_c \sim 100$; $T_c \sim 8$, $H/d \sim 0.4$.