CHAPTER 51

Approximate descriptions of the focussing of water waves

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Abstract

Underwater shoals and spurs focus water waves that propagate over them. The normal theoretical approach to finding a more accurate solution of the linear equations is to interpret the envelope of crossing rays as a cusp of caustics (see Figure 1) and to use Pearcey's function (Pearcey 1946). In practical cases the ray pattern is rarely sufficiently well defined to enable the cusp parameters to be deduced. An alternative approach is presented in which a length of wave crest heading towards the focussing region is approximated by an arc of a circle or parabola (Figure 2). Corresponding approximate solutions for linear and weakly nonlinear waves are described.

1. Introduction

For calculating the refraction of waves the usual method is to use ray theory. It is a common occurrence to find points in wave ray diagrams where rays cross. The point where rays start to cross is usually a focus at the cusp of two caustics. This is illustrated in figure 1. It is unusual to obtain very clear examples in practice and if only a few rays cross a higher density of rays is needed to clarify the ray structure.

Focussing of rays is an indication that ray theory has become invalid. An improved method of solution including at least some diffraction effects is necessary. One approach is to determine the positions of caustics and use Pearcey's (1946) solution for the cusps of caustics; another approach is to use a parabolic approximation; a third method is to solve a fully elliptic form of the wave equation.

Here two simple approximations for finding the wave amplitude at the focus are presented. Both represent the focus as a single point where an angular spread of waves meet as sketched in figure 2. It is much easier to estimate such an angle than to fit an appropriate caustic cusp. The boundary of the focussing wave is also important. Here we suppose that an initial wave crest is made up of a circular arc of angle 2α, radius R, smoothly joining straight crests representing plane waves as sketched in figure 2. Uniform initial amplitude is assumed.

One approximation is based on an exact linear wave solution and includes diffraction effects. The other is based on an exact solution of a weakly nonlinear parabolic equation for refraction, the nonlinear Schrödinger equation. The approximation includes the major nonlinear

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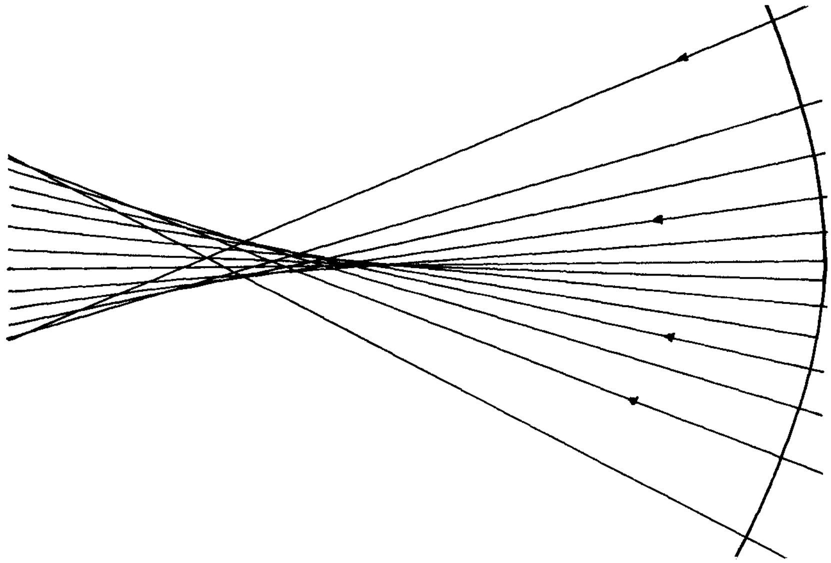

Figure 1. A typical focus.

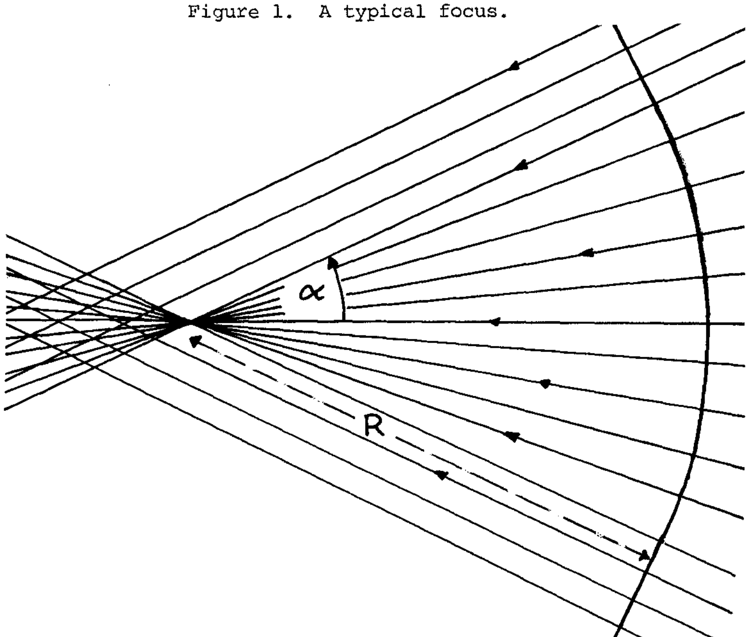

Figure 2. A circular wave focussing at a point, bounded by plane waves.

diffraction effects but neglects some of the linear diffraction. The results are different. It appears that both are likely to give upper bounds on wave amplitude. Both are easy to evaluate once α, R and the initial wave amplitude A_1 are known. The smaller value can be taken as an estimate of wave amplitude at the focus.

2. A linear solution

We consider only water of constant depth, h, and a single Fourier component in time. That is a time variation $e^{-i\omega t}$ is implicit. The wave equation to be solved is then

$$\nabla^2\zeta + k^2\zeta = 0 \tag{1}$$

where k is the wavenumber given by

$$\omega^2 = gk\tanh kh \tag{2}$$

and ζ is the, complex, surface elevation. Introduction of polar coordinates (r,θ) permits expression of the general solution of equation (1) in the form

$$\zeta = \sum_{n=0}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) J_n(kr), \tag{3}$$

where J_n are Bessel functions.

Now take the origin at the focal point of a wave system, and consider a circle of large radius R, such that $kR >> 1$. Then for all values of n such that $n << kR$ the terms of the series (3) can be divided into incoming and outgoing waves by using an asymptotic formula for J_n, for example see Abramowitz and Stegun (1964) equation (9.2.1). In particular:

$$\begin{aligned} J_o(kR) &\approx (2/\pi kR)^{\frac{1}{2}} \cos(kR - \tfrac{1}{4}\pi) \\ &\approx (2\pi kR)^{-\frac{1}{2}}\{\exp i(kR - \tfrac{1}{4}\pi) + \exp[-i(kR - \tfrac{1}{4}\pi)]\}. \end{aligned} \tag{4}$$

When combined with $e^{-i\omega t}$ the first exponential term in (4) gives an incoming wave, the second term gives an outgoing wave.

At the origin the only non zero term in the Fourier-Bessel series (3) is the J_o term, thus we only need to evaluate a_o to find the wave amplitude there. To evaluate the Fourier coefficients we suppose that on the circle radius R the amplitude and phase of *incoming* waves is known as a complex function of θ, say $A(\theta)$. Then the usual Fourier series evaluation of coefficients and the expression (4) give

$$\int_{-\pi}^{\pi} A(\theta)\,d\theta = 2\pi a_o (2\pi kR)^{-\frac{1}{2}} \exp i(kR - \tfrac{1}{4}\pi). \tag{5}$$

As an example consider the plane wave

$$\zeta = A_1 e^{ikx} = A_1 e^{ikr\cos\theta}$$

This gives

$$A(\theta) = A_1 e^{ikR\cos\theta} \quad \text{for} \quad -\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi$$
$$= 0 \quad \text{for} \quad |\theta| \geq \tfrac{1}{2}\pi. \tag{6}$$

The integral of equation (5) becomes

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{ikR\cos\theta} d\theta = (2\pi/kR)^{\frac{1}{2}} \exp i(kR - \tfrac{1}{4}\pi), \tag{7}$$

to the same level of approximation as the asymptotic formula (4) for J_0, and thus equation (5) gives $a_0 = A_1$ as is expected.

Now consider the wave field sketched in figure 2: that is an arc of a circle of angle 2α of precisely focussed waves bounded smoothly by semi-infinite plane waves. For simplicity let their amplitude at radius R have constant modulus A_1 and continuous phase. The contribution to the integral of equation (5) of the two semi-infinite plane waves is the same as that of a single plane wave, though spread over a different range of angles. The contribution of the focussing arc is simply $2\alpha A_1 \exp(ikR)$ leading to

$$a_0 = A_1\{1 + \alpha(2kR/\pi)^{\frac{1}{2}} e^{\frac{1}{4}i\pi}\}. \tag{8}$$

Thus $$|a_0| = A_1\{1 + 2\alpha(kR/\pi)^{\frac{1}{2}}\} \tag{9}$$

after neglecting terms of an order, k^2R^2, already neglected in the asymptotic result (4).

3. Nonlinear Schrodinger equation

Parabolic approximations are well suited to modelling a focus. A brief derivation for the simple two-dimensional wave equation,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \tag{10}$$

illustrates the nature of the approximation made. Equation (10) can be rewritten:

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u = c^2 \frac{\partial u}{\partial y^2} \tag{11}$$

The first operator on the left-hand side expresses the fact that waves can propagate in the $+x$ direction, and the second operator relates to propagation in the $-x$ direction.

Near a focus most waves are propagating close to one direction, say the $+x$ direction. Thus

$$u(x,y,t) = a(x,y)e^{i(kx-\omega t)}, \tag{12}$$

where $$k = \omega/c \tag{13}$$

should be a good first approximation. The exponential describes almost all of the "wavy" behaviour in the x direction. Hence one expects

$$ka \gg \frac{\partial a}{\partial x} . \tag{14}$$

Substitution of (12) into (11) gives

$$\left(-i\omega + ick + c\frac{\partial}{\partial x}\right)\left(-i\omega - ick - c\frac{\partial}{\partial x}\right) a = c^2\frac{\partial^2 a}{\partial y^2} . \tag{15}$$

The inequality (14) implies that the x derivative in the second operator of (15) may be neglected compared with the sum of the other two terms, which is $-2ick$ after using (13). The equation then reduces to

$$\frac{2i}{k}\frac{\partial a}{\partial x} + \frac{1}{k^2}\frac{\partial^2 a}{\partial y^2} = 0 \tag{16}$$

which is the parabolic wave equation (also known as the one-dimensional Schrodinger wave equation). For a derivation directly related to water waves see Radder (1979).

Using the full equations for irrotational water waves Yue and Mei (1980) showed that the appropriate extension of equation (16) for weakly nonlinear water waves (equivalent to third-order Stokes waves) is

$$\frac{2i}{k}\frac{\partial a}{\partial x} + \frac{1}{k^2}\frac{\partial^2 a}{\partial y^2} - Kk^2|a|^2 a = 0. \tag{17}$$

This is a nonlinear Schrödinger equation (NLS equation) in which

$$K = \frac{c}{c_g}.\frac{9 - 12p^2 + 13p^4 - 2p^6}{p - kh(1 - p^2)} \tag{18}$$

where $p = \tanh kh$ and $\dfrac{c}{c_g} = \dfrac{2p}{1 + kh(1 - p^2)}$.

Note, $a(x,y)$ is still the complex amplitude of the first approximation

$$a(x,y)e^{i(kx-\omega t)} .$$

A further approximation aids interpretation of solutions of the NLS equation. Substitute

$$a = Ae^{iS} \tag{19}$$

in the NLS equation (17), where A and S are real functions of (x,y). Separate real and imaginary parts. Introduce

$$D = k^2A^2 \quad \text{and} \quad v = \frac{1}{k}\frac{\partial S}{\partial y} . \tag{20}$$

A little algebra and differentiation then gives

$$D_x + (Dv)_y = 0, \tag{21}$$

$$v_x + vv_y + \tfrac{1}{2}KD_y = (A_{yy}/k^2A)_y. \tag{22}$$

The final term in equation (22) includes three y derivatives and is thus only important in rapidly varying parts of a solution. If it is neglected equation (22) becomes

$$v_x + vv_y + \tfrac{1}{2}KD = 0. \tag{23}$$

By substituting a time variation for the x variation in equations (21) and (23), they can be recognized as the nonlinear shallow-water equations with the role of gravity being taken by $\tfrac{1}{2}K$, and D and v equivalent to the depth and the velocity respectively.

Equations (21) and (23) have two sets of characteristics

$$\frac{dy}{dx} = \pm(\tfrac{1}{2}KD)^{\frac{1}{2}} + v \tag{24}$$

which corresponds to a splitting of the rays in linear theory, see Peregrine (1983).

4. A nonlinear focussing solution

Following the qualitative description of focussing given by Peregrine (1983) numerical solutions of the NLS equation (17) with initial conditions corresponding to the focussing in figure 2, were studied. These solutions all showed that where focussing waves were of a uniform amplitude at the initial value of x the amplitude remained uniform in y varying only with x in a substantial region approaching the focus. An example may be found in figure 3 of Stamnes et al (1983).

A solution corresponding to amplitude being independent of y is easily found. For waves of initial amplitude A_1 at $x = R$, focussing at $x = 0$, it is

$$a(x,y) = A_1(R/x)^{\frac{1}{2}}\exp i\{y^2/2kx - Kk^3A_1^2R\,\log(x/R)\}. \tag{25}$$

This solution is singular and unrealistic at $x = 0$. However, one effect of nonlinearity is that there is "defocussing", and the effective focus may occur before the geometric focus. This feature can be deduced from the characteristics corresponding to solution (25) which are shown in figure 3. The characteristics also show that the singularity at $x = 0$ is due to wave energy coming in from waves originally at unrealistically large values of $|y|$.

For any initial region of focussing at $x = R$ the corresponding domain of dependence is limited to a region with $x > x_0 > 0$. Within this domain of dependence solution (25) is likely to be a good approximation. The characteristics of (25) are given by

$$y = \pm A_1k(2KRx)^{\frac{1}{2}}\{(x/x_0)^{\frac{1}{2}} - 1\}, \tag{26}$$

where x_0, the position at which $y = 0$, identifies pairs of characteristics. The particular characteristics which bound a domain of depend-

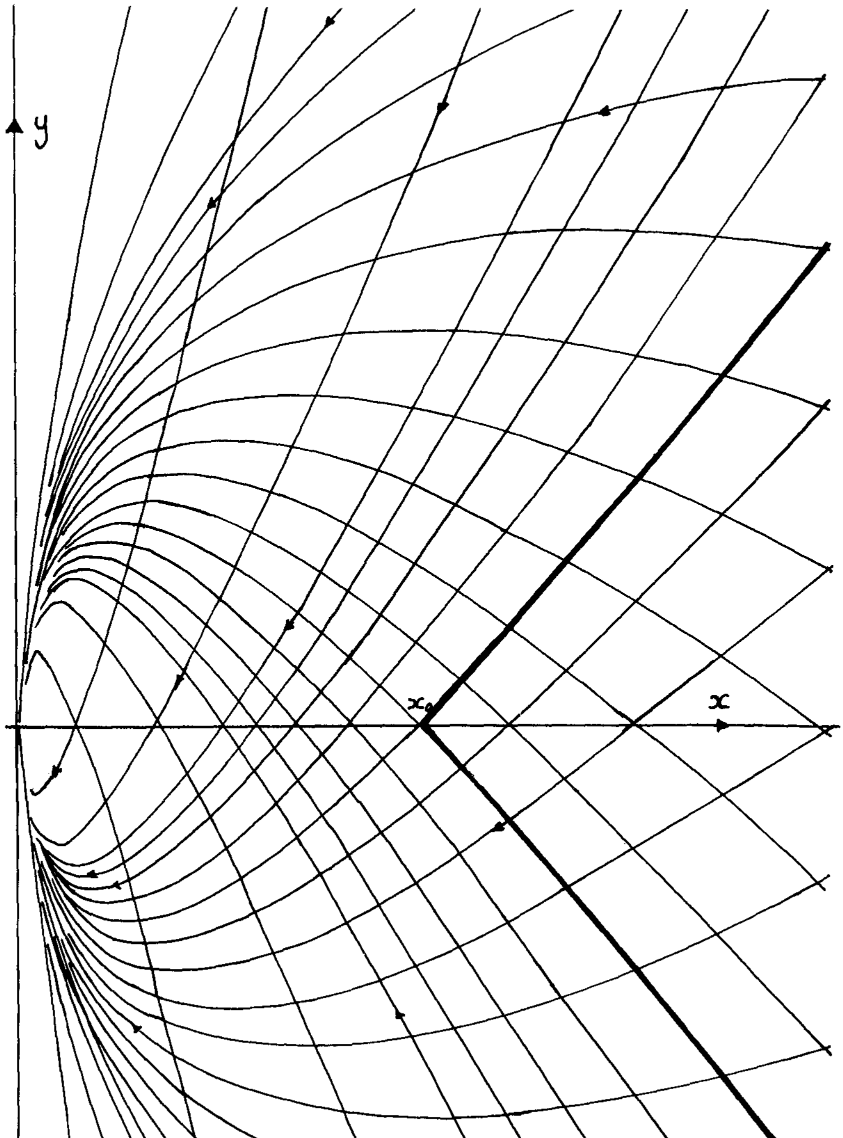

Figure 3. Some characteristics of the approximation (21) and (23) to the nonlinear Schrödinger equation for the solution (25). The boundary of a domain of dependence on initial conditions is outlined.

ence of waves from an angle of 2α at $x = R$, are found by substituting $(x,y) = (R, \alpha R)$ into (26) which gives

$$x_o = \frac{R}{\left\{1 + \frac{\alpha}{kA\ (2K)^{\frac{1}{2}}}\right\}^2} . \tag{27}$$

Unless there are more waves focussing to the same point, $x = x_o$ becomes the effective focussing position, and the corresponding value of amplitude at that point is

$$|a(x_o, O)| = A_1\left\{1 + \frac{\alpha}{kA_1\,(2K)^{\frac{1}{2}}}\right\} . \tag{28}$$

The NLS equation (17) can be scaled to give an identical equation by using the transformation

$$a^* = a/a_o, \quad x^* = a_o^2 x, \quad y^* = a_o y. \tag{29}$$

Thus any one numerical solution can be interpreted in a number of different ways. In particular, some results of computation are shown in figure 4. The amplitude of numerical solutions along $y = O$ is given for the scaled values $A^* = 1$, $R = 5$, and $\alpha R = 3, 10$ and 20, where k has been set equal to one. Examples of alternative values for these three cases are given in table 1, where L is wavelength and H is wave height.

Table 1.

kA_1	$(H/L)_1$	kR	R/L	α		
0.1	.032	500	80	3.4^o	11.5^o	22.9^o
0.05	.016	2000	318	1.7^o	5.7^o	11.5^o

5. Discussion

Two simple formulae for the maximum amplitude are given. From section 2

$$|a_o| = A\ \{1 + 2\alpha(kR/\pi)^{\frac{1}{2}}\} \tag{30}$$

and from section 4.

$$|a_o| = A_1\left\{1 + \frac{\alpha}{kA_1\,(2K)^{\frac{1}{2}}}\right\} \tag{31}$$

These are different and this difference is due to the different approximations. Expression (30) arises from attempting to include all the linear diffraction effects and is expected to be useful for waves of small steepness, even at the focus. On the other hand expression (31) arises from an approximation which neglects some linear diffraction effects, i.e. the third-order terms in equation (22), but does include

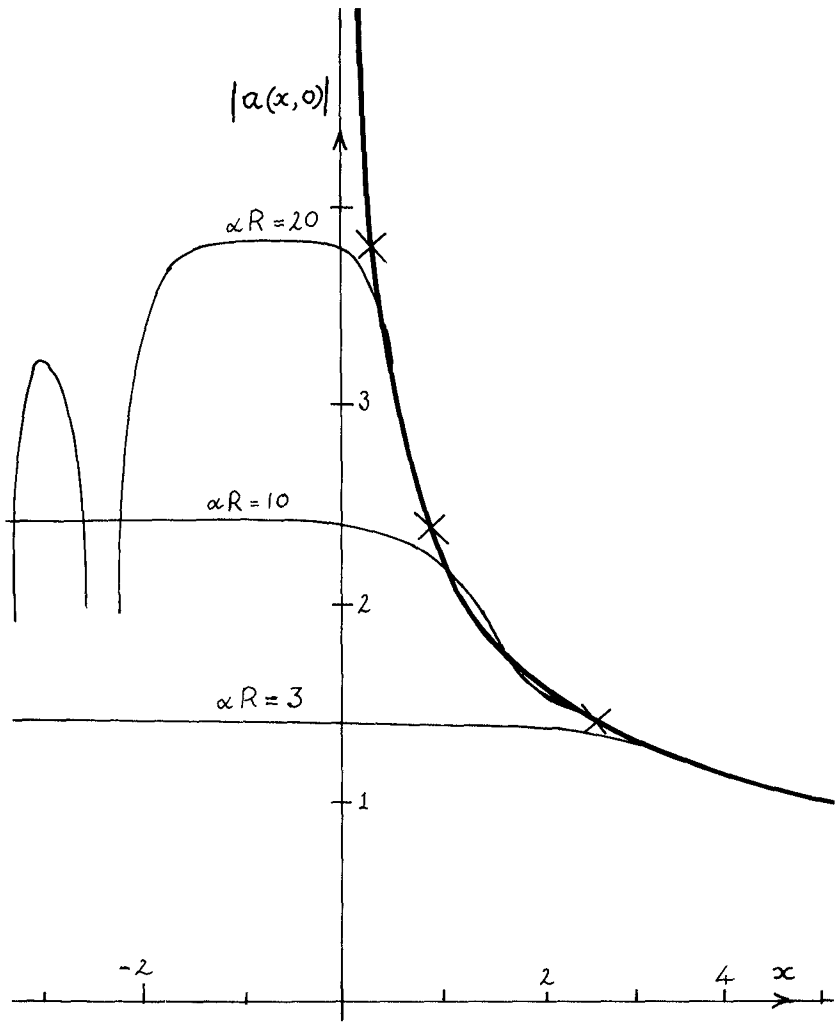

Figure 4. Variation of amplitude of waves along the axis of a focus. Numerical solutions of the NLS equation (17) compared with the solution (25) shown in a heavy curve, with crosses marking the values of x_0, equation (27), corresponding to the numerical examples.

the defocussing effects of nonlinearity.

Both the linear diffraction and nonlinear defocussing spread wave energy so that the singularity at a focus that arises from the ray approximation does not happen in practice. Both expressions (30) and (31) neglect some of the energy spreading so that both of them are likely to be upper bounds on wave energy. To see how they may be used consider a situation where the wave geometry, kR and α, is fixed and the initial steepness is allowed to vary, that is A_1 increases from zero. Initially the linear expression (30) is the lesser value; for zero amplitude waves the nonlinear effects are not relevant, but as steepness increases the two expressions become equal and for steeper waves nonlinear effects are more important and hence expression (31) is the least. This is sketched in figure 5. The initial steepness at which the two expressions are equal is

$$kA_1 = (2\pi/kRK)^{\frac{1}{2}}. \tag{32}$$

Interestingly, but perhaps in retrospect not surprisingly, this critical steepness is independent of α. As an example, if R is 100 wavelengths and $K = 3$, i.e. $kh = 1.0$, then the critical wave steepness kA_1 is 0.06 or $(H/L)_1 = 0.018$.

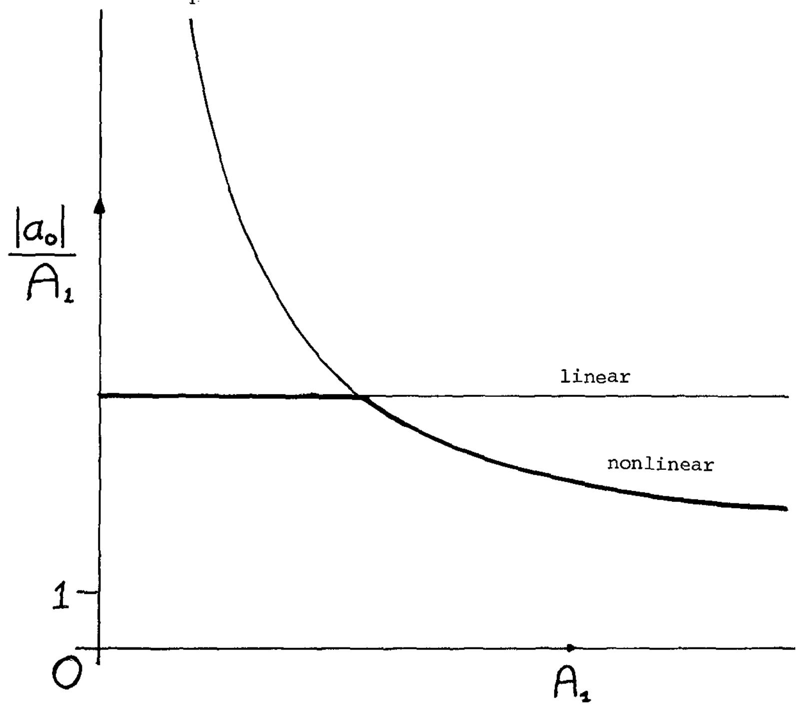

Figure 5. Variation of the linear and nonlinear expressions for amplitude at a focus with the initial wave steepness.

There are limitations in these expressions. They are derived for water of constant depth of initially uniform amplitude. Nonetheless, the author is not aware of any other simple estimates for wave amplitude at a focus and hopes that they may still provide a rough guide in practical situations for assessing the importance, or unimportance, of regions where ray theory shows that waves focus.

6. References

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