CHAPTER 48

LOCAL APPROXIMATIONS : A NEW WAY OF DEALING WITH IRREGULAR WAVES

by

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ABSTRACT

The concept of local approximations via locally defined frequencies is introduced as a tool for dealing with irregular, non linear waves. Practical testing has been performed on the problem of estimating surface elevations from measured bottom pressures. In these tests the new method even in its simplest form proves more accurate than the linear spectral method. With respect to computational effort the new method requires orders of magnitude less than spectral methods, or wave by wave analysis of similar accuracy.

INTRODUCTION

Dealing with irregular waves is one of the major tasks of coastal engineers, and doing it in an appropriate way is often a big problem.

Traditionally, two major approaches have been applied namely spectral analysis and wave-by-wave analysis. Spectral analysis relies on essentially "linear thinking" and has therefore got problems with non linear waves. Wave by wave analysis can be non linear, but there are problems of ambiguity with respect to defining the individual waves.

In order to obtain reasonable accuracy for wave-by-wave analysis of non linear waves one must apply a suitable, high order wave theory. But the complexity of such theories makes it desirable for the practicing engineer to find a different method by which reasonable results can be obtained with a simple wave theory e.g. linear wave theory.

This problem was solved by Daemrich et al (1980) who realised that the long flat trough of a non linear wave might be represented by a correspondingly long flat sine wave, and similarly, the short, strongly curved crest section could be represented by a short, steep sine wave. We may call their method "half-wave-by-half-wave-analysis".

In the following we shall take their idea a bit further and not just apply different frequencies to trough and crest, but apply a locally defined frequency to each point in the wave record.

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Once we have a frequency to work with, there are many wave theories to chose from. It seems however that something simple like linear wave theory with a simple stretching term inbuilt does a good job in most cases. Using the concept of the locally defined frequency, it is in fact possible to devise very simple transfer functions, which are essentially only non linear filters. These semi empirical transfer functions can have very good accuracy and hence be very useful for handling data by hand or by micro processor.

As an example the new idea is applied to the problem of estimating water surface elevations from measured bottom pressures under steep irregular waves. The results are encouraging.

THE CONCEPT OF LOCAL FREQUENCY

A basic assumption to most existing wave theories is that the water motion is periodic and can be described by a suitable combination of the simple harmonic functions.

$$\sum_{0}^{\infty} A_n \cos n\omega t + B_n \sin n\omega t \qquad (1)$$

where A_n and B_n are constants and $\omega = 2\pi/T$. Such combinations can (at least assymptotically) describe the shape of natural waves but in the following we shall take a different approach. We shall deal with slowly curving parts of the wave as if they were parts of long sine waves and with strongly curved parts as if they were part of short sine waves. A step in this direction was taken by Daemrich et al (1980) who applied "half wave by half wave analysis" to irregular waves and thus treated the water motion between two consecutive surface zero crossings as part of a sine wave with a period of twice the interval between the zero crossings.

In the following we shall go a step further and apply a locally defined frequency to each individual point in the time series. The local frequency f is defined as that of the sine curve which matches the wave shape locally. Figure 1 shows measured surface elevations and dynamic bottom pressure under a steep wave and the corresponding variation of the locally defined frequency. We see that the local frequency is well behaved in the crest and trough areas but becomes erratic near the zero crossings. Therefore the concept of local frequency is only really useful in relation to problems where the main interest is on the extreme elevations and/or velocities, but fortunately that is most often the case. Among the meaningful local frequencies the extremes are produced by the surface elevation. In the flat trough, $f(\eta,t)$ falls to just under 0.5/T, and at the crest it reaches 1.7/T. For $f(p,t)$ the range is from 0.6/T to 1.2/T. It is interesting to note that even for the very steep wave in Figure 1 which in the spectral sense contains many high harmonics, the upper limit of $f(\eta,t)$ in the crest area is less than 2/T i.e. lower than the frequency of the second harmonic.

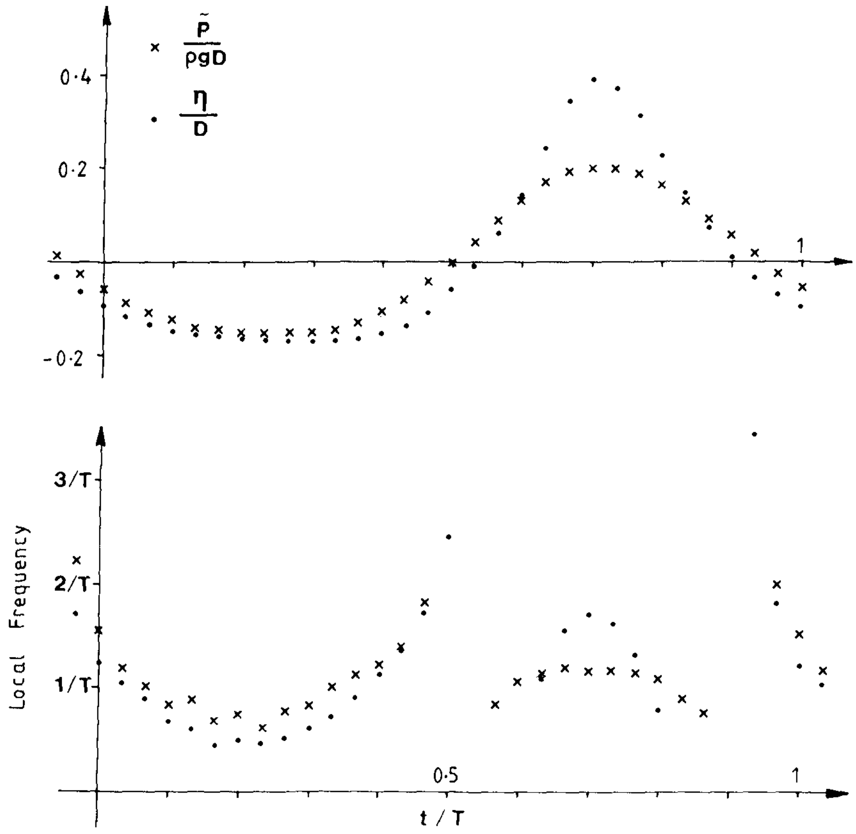

Figure 1 : Top : Measured surface elevations η and dynamic bottom pressures $\tilde{p}$ for a steep 1.8 second wave in 0.4 metres of water. Bottom : local frequencies derived from $\tilde{p}$ and η. The local frequencies $f(\tilde{p},t)$ and $f(\eta,t)$ are well behaved in the crest and trough areas but tend to be erratic near the zero crossings. The well behaved frequencies are typically between half and two times the crest to crest frequency 1/T.

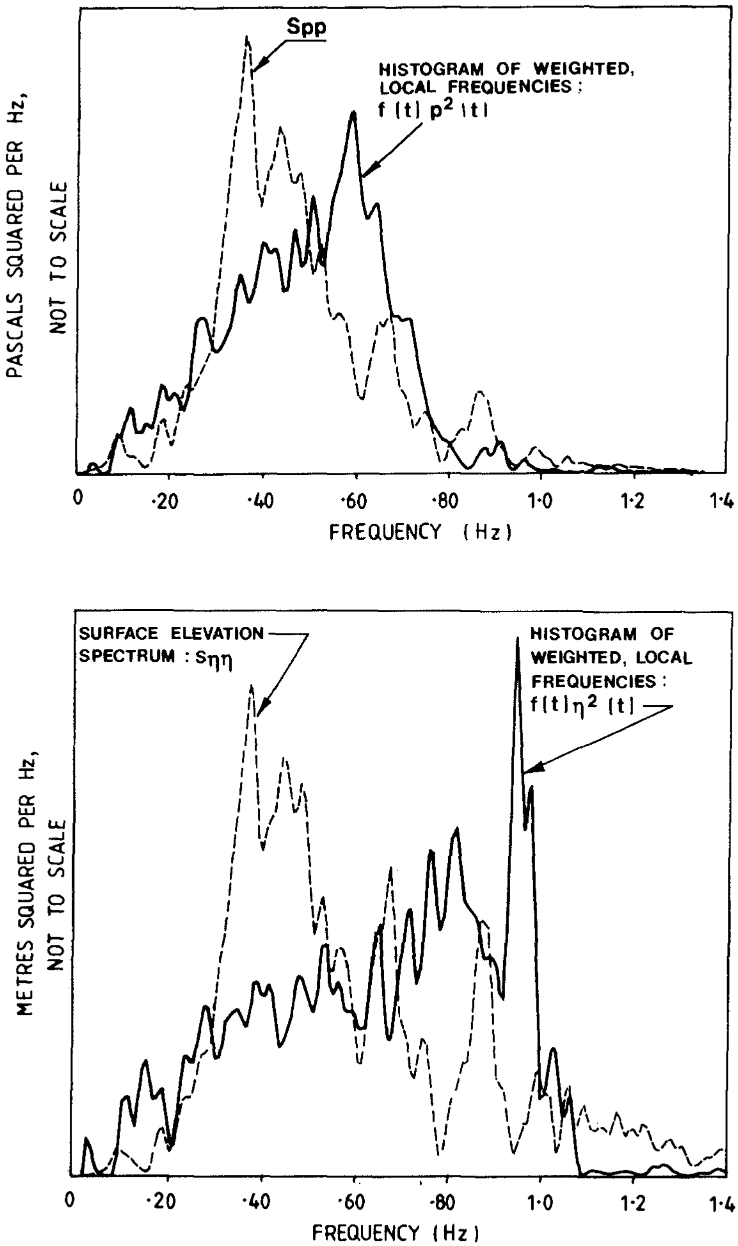

Figure 2 : Histogram of weighted local frequencies and conventional energy spectrum for both $\tilde{p}$ and η from the same irregular wave record.

A histogram of the local frequencies from a wave record weighted with the square of the local amplitude may serve like a spectrum for describing the distribution of wave energy. The meaning of the average frequency is rather obvious but the physical implications of other statistics like the coefficient of variation need further investigation. An example of the histogram of weighted local frequencies is shown in Figure 2 together with the conventional spectrum of the same record. We see that the histogram has a much simpler shape. Possible areas of application for the histogram are break water design where increased attention has been paid to wave shapes recently, see e.g. Bruun et al (1985), and remote sensing of wave climates where the distribution of surface slopes and curvatures influence the surface reflectivity.

CALCULATING THE LOCAL FREQUENCY

A frequency f or radian frequency $\omega = 2\pi f$ can be affixed to any point $x_m = x(t_m)$ in a time series by fitting a cosine curve locally around the point. The simplest fitting proceedure is to pick the uniquely defined curve

$$x(t) = A \cos(\omega t - \varphi) \tag{2}$$

which passes through the point and its neighbour on either side, while more robust estimates may be obtained by least squares fitting to more than three points.

The exact value of ω for a 3-point fit can be found in the following way. Let

$$x_m = A \cos \theta \tag{3}$$

and hence

$$x_{m\pm 1} = A \cos(\theta \pm \omega\Delta) \tag{4}$$

where Δ is the time increment between the three points. Then it follows from trigonometric identities that ω can be found from

$$\omega = \frac{1}{\Delta} \cos^{-1}\left(\frac{x_{m+1} + x_{m-1}}{2 x_m}\right) \tag{5}$$

This estimate is always defined if $x_n \neq 0$ but the imaginary values which result if

$$2|x_m| < |x_{m+1} + x_{m-1}| \tag{6}$$

are not immediately interpretable in physical terms. Note that the time interval Δ is not necessarily equal to the sampling interval. Often it is more appropriate to choose Δ a few times larger than the sampling interval.

A slightly simpler estimate, which avoids evaluation of the inverse cosine function in (5) can be obtained via the identity

$$x'' = -\omega^2 x \tag{7}$$

which holds for all functions of the form (2). The second derivative x'' is estimated by

$$x'' \approx \frac{x_{m-1} - 2x_m + x_{m+1}}{\Delta^2} \tag{8}$$

Then using (7) we have

$$\omega^2 = -\frac{x''}{x} \approx -\frac{x_{m-1} - 2x_m + x_{m+1}}{\Delta^2 x_m} \tag{9}$$

Again Δ can be a multiple of the sampling interval. The estimate

$$\hat{\omega}^2 = -\frac{x_{m-1} - 2x_m + x_{m+1}}{\Delta^2 x_m} \tag{10}$$

is a biased estimate of ω^2. It under estimates ω in accordance with

$$\hat{\omega}^2 = \omega^2\left[1 - \tfrac{1}{12}(\omega\Delta)^2\right] + O(\Delta\omega)^6 \tag{11}$$

Hence an improved estimate can be obtained from

$$\omega_1^2 = \hat{\omega}^2\left[1 + \tfrac{1}{12}(\hat{\omega}\Delta)^2\right] \tag{12}$$

The latter formula is within one percent of error for $\Delta < T/6$ or $\omega\Delta \lesssim 1.0$.

A PRACTICAL APPLICATION

Pressure transducers have several advantages over other field instruments used for wave recordings. They are the most reliable and the easiest to install. It is therefore of interest for practicing engineers to be able to derive other wave properties, such as surface elevations and velocities from measured bottom pressures. We shall now see that local approximations provide a very efficient tool for dealing with this problem.

For a sine wave, the surface elevation $\eta(t)$ is related to the dynamic bottom pressure $\tilde{p}(t)$ by

$$\eta(t) = \frac{\tilde{p}(t)}{\rho g} \cosh kD \tag{13}$$

where ρ is the fluid density, g the acceleration of gravity and D the water depth. The wave number k is related to the radian frequency through the dispersion relation

$$kD \tanh kD = \frac{\omega^2}{g} D \tag{14}$$

The traditional way of deriving $\eta(t)$ from $\tilde{p}(t)$ for irregular waves has been by using spectral analysis. The method involves three steps:

1. Find the discrete pressure specturm $S_{pp}(\omega_i)$

2. Transform each spectral estimate in accordance with (13) :

$$S_{\eta\eta}(\omega_i) = S_{pp}(\omega_i)\ \cosh(k_i D)/\rho g$$

3. Create the $\eta(t)$ time series from $S_{\eta\eta}(\omega_i)$ by the discrete, inverse Fourier transform.

This traditional method is neither fast nor accurate but fortunately we can develop alternative methods involving the use of local approximations which are both faster and more accurate.

The most straight forward approach is to apply the linear-wave formula (13) using a local wave number determined from the local frequency through (14). Such an approach does however systematicly under estimate $\eta(t)$ because linear wave theory, from which (13) is taken, is not geared to handle finite surface elevations. We are free to use any wave theory we like as soon as we have a frequency to work with, but for this purpose it is adequate just to use a modified version of Equation (13)

$$\eta(t) = \frac{\tilde{P}(t)}{\rho g} \cosh\, k\left(D + \frac{\tilde{P}(t)}{\rho g}\right) \qquad (15)$$

where k is the local wave number determined from the local frequency through (14). Replacing the depth D by D + $\tilde{P}/\rho g$ in the argument of the cosh-function accounts in a way for the fact that the instantaneous water surface can be a finite distance away from the mean water level. Equation (15) based on local frequencies is superior to the traditional spectral method with respect to both speed and accuracy. An example is shown in Figure 3 where estimates from both methods are compared to the actual, measured surface elevations. The local approximations method recovers the crest height a little better and the shape of the troughs is more accurately represented as well. In terms of the normalised deviation dev $(x,y) = \sqrt{\sum(x-y)^2/\sum y^2}$ based on the full 123 second record, the local approximation estimates gave a deviation of 0.215 while the spectral estimates gave a deviation of 0.242. The fine details of the results depend on the measures applied with each method to overcome noise problems. This aspect will be treated in detail later.

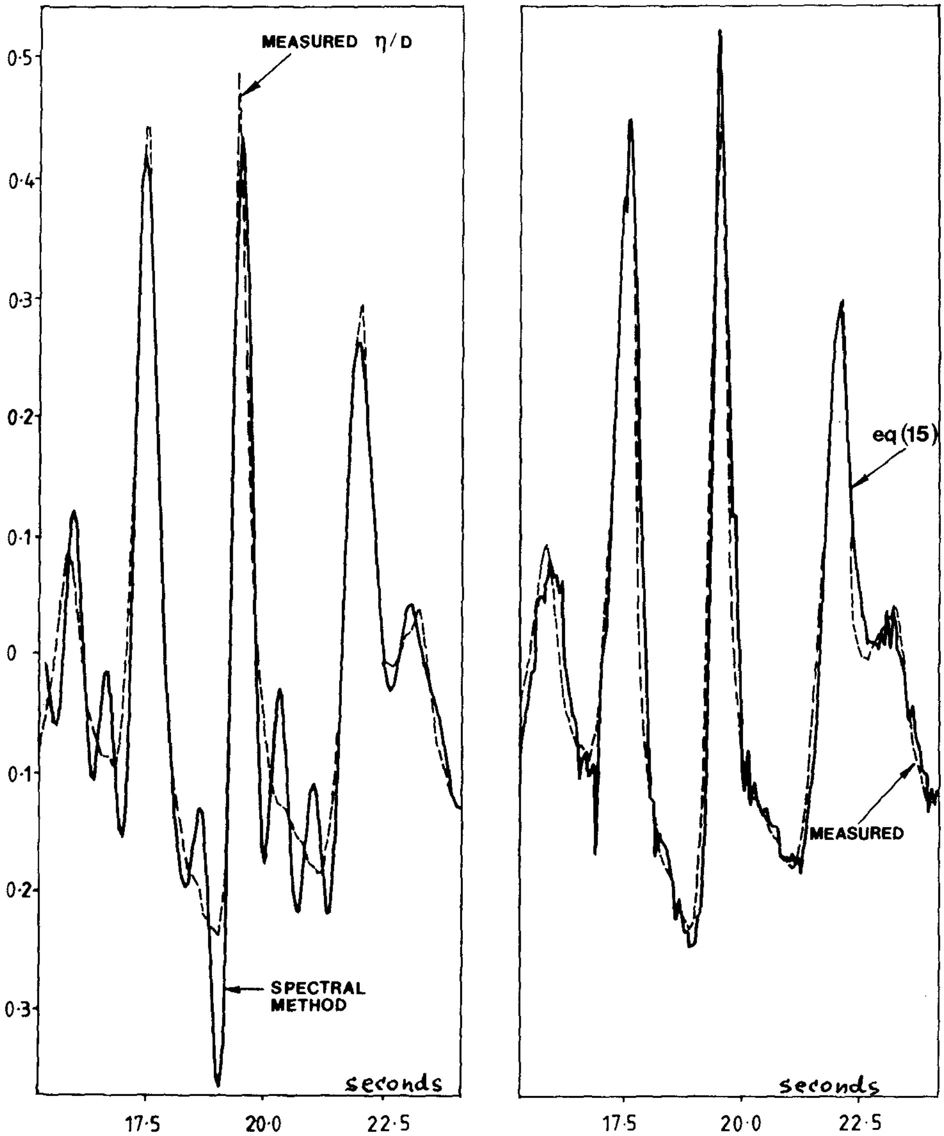

Figure 3 : Estimates of surface elevations from bottom pressures compared to measured surface elevations. The water depth D was 0.38 metres. We see that equation (15) with local frequencies is slightly better at estimating crest heights and much better at estimating depth and shape of the troughs than the spectral method. No smoothing has been applied anywhere in the processing. The cut off frequency of 1.45 Hz applied in the spectral method was deemed optimal.

SEMI EMPIRICAL TRANSFER FUNCTIONS

Semi empirical transfer functions are useful in many practical cases. Also with respect to these can the concept of local frequencies be helpful. Consider again the problem of estimating surface elevations $\eta(t)$ from dynamic bottom pressures $\tilde{p}(t)$ in irregular waves. We want a very simple algorithm for processing data by hand or hy micro processor.

From a heuristic physical consideration we may argue that the strength with which a water surface signal is transferred to the bottom depends on the ratio between the radian frequency ω of the signal and the natural frequency of the water column $\sqrt{g/D}$ i.e.

$$\frac{\eta}{\tilde{p}/\rho g} = F\left(\frac{\omega^2}{g}D\right) \qquad (16)$$

and here we may replace D by $D + \tilde{p}/\rho g$ in order to account for finite changes in depth hence we have

$$\frac{\eta}{\tilde{p}/\rho g} = F\left[\frac{\omega^2}{g}\left(D+\frac{\tilde{p}}{\rho g}\right)\right] \qquad (17)$$

For this purpose we will use the simplest possible estimate of the local frequency, namely

$$\hat{\omega}^2 = -\frac{\tilde{p}_{n-1} - 2\tilde{p}_n + \tilde{p}_{n+1}}{\tilde{p}_n \Delta^2} \qquad (18)$$

and hence we have

$$\frac{\eta_n}{\tilde{p}_n/\rho g} = F\left[\frac{-\tilde{p}_{n-1} + 2\tilde{p}_n - \tilde{p}_{n+1}}{\tilde{p}_n \Delta^2 g}\left(D+\frac{\tilde{p}_n}{\rho g}\right)\right] \qquad (19)$$

To determine the empirical function F we now plot the left hand side of (19) versus the argument of F for a few accurately known data points, for example taken from Dean's stream function tables (Dean, 1974). That has been done in Figure 4 and we see that F can be adequately represented by a simple exponential, $F(x) \approx \exp(2x/3)$. Hence the full expression for our transfer function is

$$\eta_n = \frac{\tilde{P}_n}{\rho g} \exp\left[\frac{2}{3}\,\frac{-\tilde{P}_{n-1} + 2\tilde{P}_n - \tilde{P}_{n+1}}{\tilde{P}_n \Delta^2 g}\left(D + \frac{\tilde{P}_n}{\rho g}\right)\right] \qquad (20)$$

a simple one line expression which compares favourably with the spectral method with respect to accuracy, see Figure 5. The normalised deviation for the eq(20)-estimates was 0.208 while the corresponding value for the spectral estimates was 0.242.

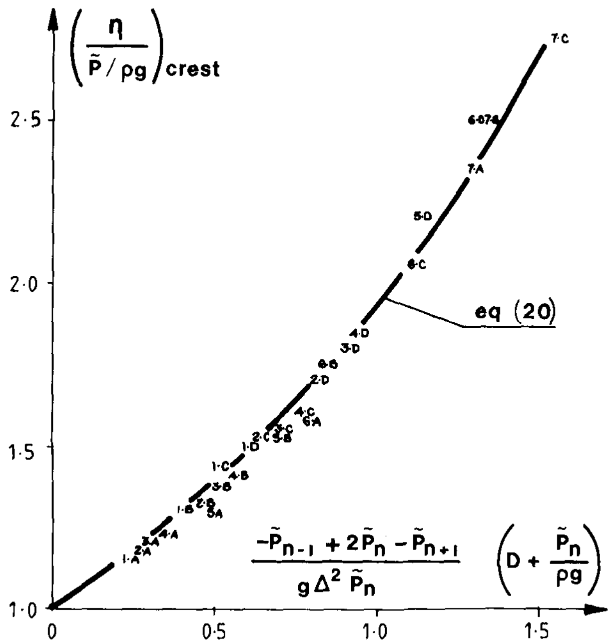

Figure 4 : Data from Dean's stream function tables from which the empirical function in (19) can be determined. We see that equation (20) provides an easy and reliable fit to the data.

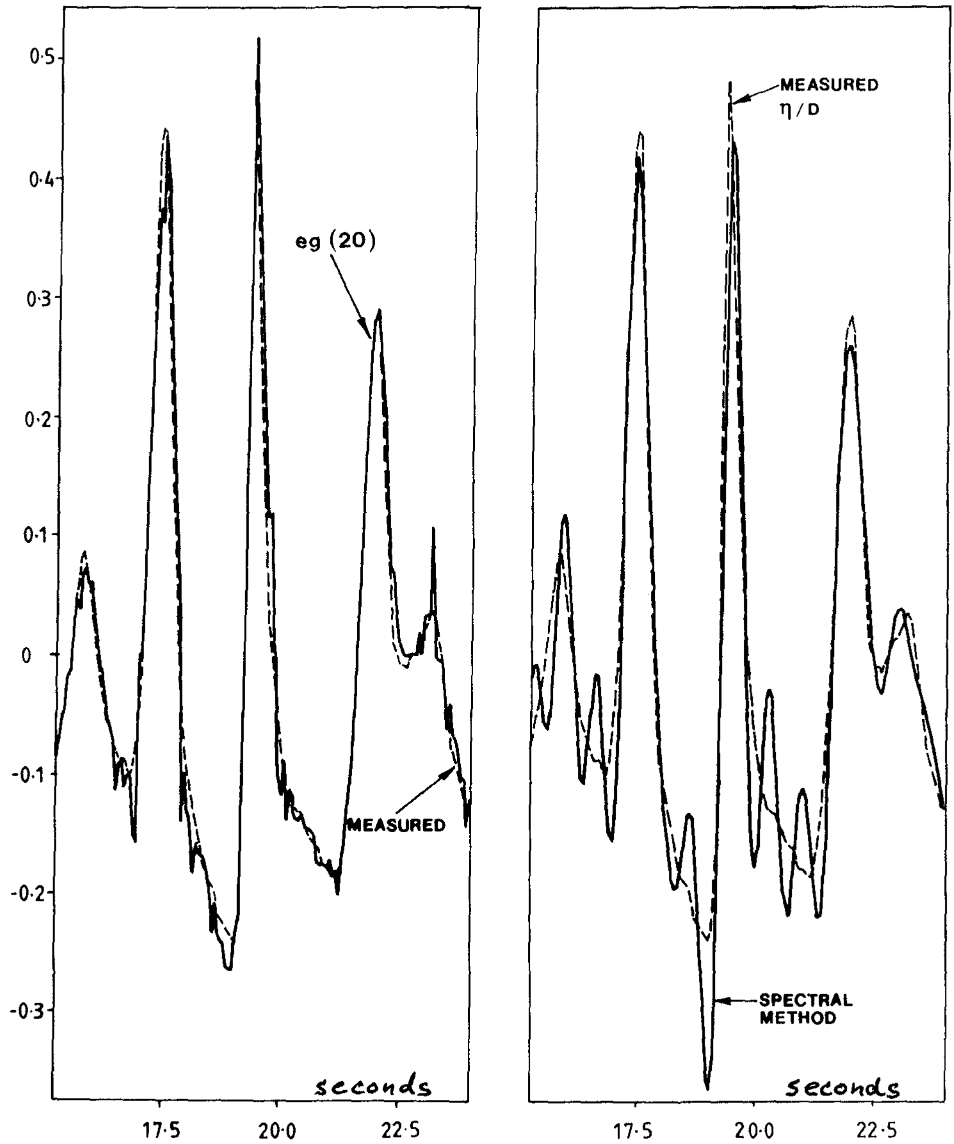

Figure 5 : Surface evaluation estimates by equation (20) and the spectral method compared to actual, measured surface elevations. Equation (20) matches both crests and troughs better than the spectral method and of course it requires much less computational effort.

RESOLUTION VERSUS NOISE

In dealing with real data, there is always a problem of balancing resolution against noise, especially when high frequency signals have to be amplified strongly as is the case when deriving surface elevations from bottom pressures.

The resolution is determined by the time increment Δ. A small Δ gives high resolution while a large Δ overlooks high frequency variations. There is no general rule for the choice of Δ since the optimum must obviously depend on the relative strength and the difference in frequency between signal and noise. However, the following value will generally provide a reasonable first guess.

$$\Delta = \sqrt{D/g} \tag{21}$$

The fact that the meaningful, locally defined frequencies generally occupy a narrower band than all the harmonics needed for a spectral description (see Figure 2) means that it is generally possible to apply a lower cut off frequency when using a local approximations method than with a spectral method.

While the above is true, it must be remembered that there are special problems with defining the local frequency near zero crossings. Unreasonably large or imaginary values of ω may occur. A reasonable upper limit for acceptable local frequencies is $\omega^2 = 1.5$ g/D, which corresponds to cosh kh = 2.6

When imaginary or unacceptably large values of ω occur it is generally adequate to assign the value zero to ω, at least for those practical problems that have been dealt with so far.

Because the details of the water motion close to zero crossings is generally unimportant, the problem of the local frequency being ill conditioned in this area is of little practical consequence.

DISCUSSION

The use of local approximations via locally defined frequencies is recommended for practical analysis of irregular waves. Firstly because it requires far less computational effort than any other available method, e.g. wave by wave analysis, Fenton's local polinomial method (Fenton 1986) or the previously most popular spectral method. Secondly because it seems to be superior to the spectral method with respect to accuracy.

The strength of local frequency methods such as (15) or (20) relative to the linear spectral method lies first of all in the fact that it is very easy to apply stretching simply by replacing the depth D with D + $\tilde{p}/\rho$ g. Secondly, in the fact that the spectral method over amplifies both signal and noise at high frequencies. This happens because it wrongly assumes that "all waves are free waves" in order to be able to calculate $k(\omega)$ from the disperson relation (14). Talking about the dispersion relation it must be admitted that applying it to a locally defined frequency probably makes very little physical sense. To this one can only reply that the justification for using equation (15) with its reliance on the dispersion relation (14) lies in its proven efficiency.

Semi empirical transfer functions like (20) do not rely on any wave theory but only on a bit of physical intuition and with this somewhat more "unassuming" nature, and their good performance they may turn out to be the most useful area of application for the concept of locally defined frequencies. The conclusions stated in this paper are essentially based on experience with a single practical problem, namely that of deriving water surface elevations from dynamic pressures measured at the bottom. This problem is however a very tough one so it is highly likely that local approximations using locally defined frequencies will prove a useful tool for solving many problems involving irregular, non linear waves.

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