CHAPTER 34

Nonlinear Theory on Particle Velocity and Pressure of Random Waves

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ABSTRACT

In this paper, to the third approximation, we used the Fourier-stieltjes integral rather than Fourier coefficient to develop a weakly nonlinear theory. From the theory, the nonlinear spectral components for water particle velocity and wave pressure can be calculated diretly from the directional spectrum of water surface displacement. Computed results based on the nonlinear theory were compared with that of experiment made by Anastasiou (1982). Furthermore, in accordance with the different characteristics of wave properties, such as wave steepness, water depth and so on, the nonlinear effects on wave kinematic and pressure properties were extensively investigated by using some standard power spectra.

1. INTRODUCTION

To the first approximation, ocean waves can be described as a superposition of statistically independent free waves which have random phases and satisfy the linear wave theory. More recently, however, it has been found by many investigators that these spectral components of ocean waves do not necessarily follow the linear wave theory, especailly in the shallow water areas. Several nonlinear random wave theories were developed by, such as Phillip (1960), Hasselmann (1962), Weber (1977) and Masuda(1979) et al. But their description were mostly confined to the spectrum of water surface displacement.

The studies on random and nonlinear characteristics with respect to the wave kinematics and pressures are seldom discussed. Hudspeth (1975) developed a simulation method to determine the uni-directional second-order sea surface characteristics; however, in his report the water particle kinematics are computed by the digital linear filter technique. Following the nonlinear wave theory developed by Longuet-Higgings (1963), Sharma (1979) proposed a second order directional wave theory by using discrete frequency component to predict the wave kinematic and dynamic property. In the present study, for similar work but to the theird order approximation, we used the Fourier-stieltjes integral rather than Fourier coefficient to develop a weakly nonlinear theory for wave kinematics and wave pressures.

2. FORMULATION

For irrotational motion of an incompressible fluid, there is a velocity potential $\Phi(x,z,t)$ governed by

$$\nabla^2\Phi(\vec{x},z,t)=0 \tag{1}$$

Here $x=(\vec{x},y)$ is the horizontal co-ordinates, z is the vertical one and t

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is the time. The solution of (1) with the condition that Φ vanishes as z=-h is given by

$$\Phi(\vec{x},z,t)=\int_K \frac{\cosh|k|(h+z)}{\cosh|k|h} e^{i\chi} dA(K) \tag{2}$$

where K denotes (ω,k). ω is angular frequency, k is wave number vector, the increment dA(K) is a random variable and $\chi=k\vec{x}-\omega t$ is the phase. The integration is over the entire ω and k space. In the same way the surface displacement η can be expressed as

$$\eta(\vec{x},t)=\int_K e^{i\chi} dB(K) \tag{3}$$

Substituting with equation (2) and (3) in the kinematic boundary condition at water surface, and using the series expasion

$$\frac{\cosh\{|k|(h+z)\}}{\cosh|k|h}=1+|k|z\tanh|k|h+\frac{1}{2}|k|^2z^2+\ldots \tag{4}$$

we can find the relationship between dA(K) and dB(K) to the third order as

$$\begin{aligned} dA(K)=&\frac{-i\omega}{|k|\tanh|k|h} dB(K) \\ &+\int_{K_1} \frac{i\omega_1\cdot k\ k_1}{|k||k_1|\tanh\{|k|h\}\tanh\{|k_1|h\}}\cdot dB(K_1)dB(K-K_1) \\ &+\int_{K_1}\int_{K_2} \frac{i\omega_1}{|k|\tanh\{|k|h\}}\{\frac{1}{2}|k_1|^2+k_1\cdot k_2 \\ &-\frac{k\cdot(k_1+k_2)k_1\cdot(k_1+k_2)}{|k||k_1+k_2|\tanh\{|k_1|h\}\tanh\{|k_1+k_2|h\}} \\ &dB(K_1)dB(K_2)dB(K-K_1-K_2) \end{aligned} \tag{5}$$

2.1 water particle velocity

The increment of water particle velocity is also defined by

$$\vec{Q}=\int_K dq e^{i\chi} \tag{6}$$

However, the particle velocity vector $\vec{Q}$ can be obtained by taking the gradient of velocity potential Φ. Therefore, from the equations (2) and (6), we have

$$dq^H(K,Z)=\frac{i|k|\cosh\{|k|(h+z)\}}{\cosh|k|h} dA(K) \tag{7}$$

where dq^H is the increment of horizontal water particle velocity. Substituting equation (5) into equation (7), we obtain

$$\begin{aligned} dq^H(K,z)=&f_1(K,z)dB(K)+\int_{K_1} f_2(K,K_1,z)dB(K_1)dB(K-K_1) \\ &+\int_{K_1}\int_{K_2} f_3(K,K_1,K_2,z)dB(K_1)dB(K_2)dB(K,K_1,K_2) \end{aligned} \tag{8}$$

where
$$f_1(K,z)=\frac{\omega k\cosh\{|k|(h+z)\}}{|k|\sinh|k|h}$$

$$f_2(K,K_1,z)=-\frac{1}{2}\left\{\frac{\omega_1 k<k,k_1>}{\tanh|k_1|h}+\frac{(\omega-\omega_1)k<k,k-k_1>}{\tanh\{|k-k_1|h\}}\right\}\frac{\cosh\{|k|(h+z)\}}{\sinh|k|h}$$

$$f_3(K,K_1,K_2,z)=\frac{\omega_1 k\cosh\{|k|(h+z)\}}{\sinh|k|h}\left\{\frac{1}{2}|k_1|^2+|k_1||k_2|<k_1,k_2>\right.$$
$$\left.-\frac{|k||k_1+k_2|<k_1,k_1+k_2><k,k_1+k_2>}{\tanh|k_1|h\,\tanh\{|k_1+k_2|h\}}\right\} \qquad (9)$$

Here $<k,k'>\equiv k.k'/|k||k'|$ denotes the cosine of the angle formed by this vectors k and k'. For further development, let the increment dB(K) is represented as the sum of primary component $dB_1(K)$, second component $dB_2(K)$ and so on. However, following a previous study (Masuda et al 1979), $dB_2(K)$ and $dB_3(K)$ can also be expressed in terms of $dB_1(K)$. Taking the mean square value of the increment dq^H and using the definition of power spectrum

$$\overline{\frac{dq^H(K)\,dq^H(K')}{dK\,dK'}}=\phi_H^q(K)\,\delta(K+K') \qquad (10)$$

we finally obtained the horizontal water particle velocity spectrum ϕ_H^q as

$$\phi_H^q(K,z)=f_1(K,z)f_1(-K,z)\phi_1(K)$$
$$+2\int_{K_1}\frac{f_1(K,z)f_1(-k,z)}{W(K)W(-K)}\{fv_2(K,K_1)\}^2\phi_1(K_1)\phi_1(K-K_1)dK_1$$
$$+4\int_{K_1}\frac{f_1(-k,z)}{W(k)}f_2(k,k_1,z)\{fv_2(k,k_1)\}\,\phi_1(k_1)\phi_1(k-k_1)dK_1$$
$$+2\int_{K_1}f_2(k,k_1,z)f_2(-k_1,-k_1,z)\phi_1(k_1)\phi_1(k-k_1)dK_1$$
$$+4\int_{K_1}\frac{f_1(-k,z)}{W(k-k_1)}f_2(k,k_1,z)\{fv_2(k-k_1,-k_1)$$
$$+fv_2(k-k_1,k)\}\phi_1(k)\phi_1(k_1)dK_1$$
$$+2\int_{K_1}f_1(-k,z)\{f_3(k,k_1,-k_1,z)+f_3(k,k_1k,z)$$
$$+f_3(k,k,k_1,z)\}\phi_1(k)\phi_1(k_1)dK_1$$
$$+2\int_{K_1}\frac{f_1(k,z)f_1(-k,z)}{W(k)}\left\{2\frac{fv_2(k,k_1)}{W(k-k_1)}+fv_3(k,k_1,-k_1)\right.$$
$$\left.+fv_3(k,k_1,k)+fv_3(k,k,k_1)\right\}\phi_1(k)\phi_1(k_1)dK_1 \qquad (11)$$

where

$$W(K)=g-\frac{\omega^2}{|k|\tanh|k|h} \tag{12}$$

$$fv_2(K,K_1)=\frac{1}{2}\{\omega^2-\omega\omega_1+\omega_1^2-\omega\omega\frac{<k,k_1>}{\tanh\{|k|h\}\tanh\{|k_1|h\}}$$
$$-\omega(\omega-\omega_1)\frac{<k,k-k_1>}{\tanh|k|h\ \tanh\{|k-k_1|h\}}$$
$$-\omega_1(\omega-\omega_1)\frac{<k_1,k-k_1>}{\tanh|k_1|h\ \tanh\{|k-k_1|h\}} \tag{13}$$

$$fv_3(K,K_1,K_2)=\frac{1}{4}\{\frac{\omega_1^2|k_1|}{\tanh|k_1|h}+\frac{\omega_2^2|k_2|}{\tanh|k_2|h}\}$$
$$+\frac{1}{2}\omega_1\omega_2\{\frac{|k_1|}{\tanh|k_1|h}+\frac{|k_2|}{\tanh|k_2|h}\}$$
$$+\frac{1}{4}\frac{\omega}{|k|\tanh|k|h}\{\omega_1|k_1|^2+\omega_2|k_2|^2\}$$
$$-\frac{1}{2}(\omega_1+\omega_2)|k_1+k_2|\{\frac{\omega_1<k_1,k_1+k_2>}{\tanh|k_1|h}+\frac{\omega_2<k_2,k_1+k_2>}{\tnah|k_2|h}\}$$
$$-\frac{1}{2}\omega(\omega_1+\omega_2)\frac{|k_1||k_2|<k_1,k_2>}{|k|\quad\tanh|k|h}$$
$$+\frac{\omega}{2}|k_1+k_2|\frac{<k,k_1+k_2>}{\tanh|k_1|h\ \tanh\{|k_1+k_2|h\}}$$
$$\{\frac{\omega_1<k_1,k_1+k_2>}{\tanh|k_1|h}-\frac{\omega_2<k_2,k_1+k_2>}{\tanh|k_2|h}\}$$
$$-\frac{1}{2}\omega_1\omega_2|k|\{\frac{<k,k_1>}{\tanh|k_1|h}+\frac{<k,k_2>}{\tanh|k_2|h}\}$$
$$+\frac{1}{2}\frac{\omega_1\omega_2}{\tanh|k_1|h\ \tanh|k_2|h}\{\frac{|k-k_1|}{\tanh\{|k-k_1|h\}}<k_1,k-k_1><k_2,k-k_1>$$
$$+\frac{|k-k_2|}{\tanh\{|k+k_2|h\}}<k_1,k-k_2><k_2,k-k_2>\} \tag{14}$$

In equation (11), the first term on right is the linear component and the others are nonlinear components. $\phi_1(K)$ is the first order spectrum. Although usually difficult to obtain, the first order spectrum can often be replaced by the total power spectrum to compute the nonlinear components approximately (Masuda at al 1979). It is clear that, based on equation (11), the nonlinear spectral component of horizontal particle velocity can be calculated directly from the directional spectrum of water surface displacement.

On the other hand, from the equation (2) and (6) for vertical velocity component, we have

$$dq^V(k,z)=\frac{i|k|\sinh\{|k|(h+z)\}}{\cosh|k|h}dA(K) \tag{15}$$

where dq^V is the increment of vertical water particle velocity. Similarily, using the same procedure described above, we can obtain a formula to calculate the nonlinear spectral component of vertical water particle velocity. This formula is just as same as the equation (11) except that the functions f_1, f_2 and f_3 are completely different from equation (9). The functions for vertical velocity spectrum are shown as follows

$$f_1(K,z)=\frac{i\omega\ \sinh\{|k|(h+z)\}}{\sinh|k|h}$$

$$f_2(K,K_1,z)=-\frac{i}{2}\{\frac{\omega_1|k|<k,k_1>}{\tanh|k_1|h}$$

$$+\frac{(\omega-\omega_1)|k|<k,k-k_1>}{\tanh\{|k-k_1|h\}}\}\frac{\sinh\{|k|(h+z)\}}{\sinh|k|h}$$

$$f_3(K,K_1,K_2,z)=-i\frac{\omega\ \sinh\{|k|(h+z)\}}{\sinh|k|h}$$

$$\{\frac{1}{2}|k_1|^2+|k_1||k_2|<k_1,k_2>$$

$$-\frac{|k||k_1+k_2|<k_1,k_1+k_2><k,k_1+k_2>}{\tanh|k_1|h\ \tanh\{|k_1+k_2|h\}} \tag{16}$$

2.2 wave pressure

Neglecting the Bernoulli's constant, we have the Bernoulli's equation as

$$\frac{p}{\rho}+\frac{1}{2}(\nabla\Phi)^2+\frac{\partial\Phi}{\partial t}=0 \tag{17}$$

Here the increment of wave pressure dC(K) is defined by

$$\frac{p}{\rho}=\int_K dC(k,z)e^{i\chi} \tag{18}$$

(where p is the dynamic wave pressure, and ρ is the water density). Substituting the equation (2) into equation (17), we can obtain

$$dC(k,z)=f_1^p(k,z)dB(k)+\int_{K_1}f_2^p(k,k_1,z)dB(k)dB(k-k_1)$$

$$+\int_{K_1}\int_{K_2}f_3^p(k,k_1,k_2,z)dB(k_1)dB(k_2)dB(k-k_1-k_2) \tag{19}$$

where

$$f_1^p(K,z)=\frac{\omega^2}{|k|\sinh|k|h}\cosh\{|k|(h+z)\}$$

$$f_2^p(K,K_1,z)=-\frac{1}{2}\{\frac{\omega\omega_1<k,k_1>}{\tanh|k_1|h}+\frac{\omega(\omega-\omega_1)<k,k-k_1>}{\tanh\{|k-k_1|h}\}\frac{\cosh\{|k|(h+z)\}}{\sinh|k|h}$$

$$+\frac{1}{2}\omega_1(\omega-\omega_1)(1-\frac{<k_1,k-k_1>}{\tanh|k_1|h\ \tanh\{|k-k_1|h\}})$$

$$\frac{\sinh\{|k-k_1|(h+z)\}}{\sinh\{|k-k_1|h\}}\ \frac{\sinh\{|k_1|(h+z)\}}{\sinh|k_1|h}$$

$$f_3^p(K,K_1,K_2,z)=-\frac{\omega\omega_1}{|k|\sinh\{|k|h\}}\cosh\{|k|(h+z)\}\{\frac{1}{2}|k_1|^2+|k_1||k_2|<k_2,k_2>$$

$$-\frac{|k||k_1+k_2|<k,k_1+k_2><k_1,k_1+k_2>}{\tanh|k_1|h\ \tanh\{|k_1+k_2|h\}}-\frac{\omega_1\omega_2}{|k_2|}$$

$$(1-\frac{<k,k-k_1>}{\tanh\{|k_1|(h+z)\}\tanh\{|k-k_1|(h+z)\}})\frac{|k-k_1|k_2}{\tanh|k_2|h}$$

$$\frac{\sinh\{|k-k_1|(h+z)\}}{\sinh\{|k-k_1|h\}}\ \frac{\sinh\{|k_1|(h+z)\}}{\sinh|k_1|h} \qquad (20)$$

In the same way, by using of perturbation expasion of dB(K) and the definition of power spectrum, the formula for calculating the nonlinear spectral component of wave pressure can be obtained. This formula is also as same as equation (11) except that the functions f_1, f_2 and f_3 must be replaced by functions f_1^p, f_2^p, and f_3^p (equation (20)).

3. COMPARISON WITH EXPERIMENTAL RESULTS

An experimental result concerning water particle velocities is used to verify the above nonlinear theory. The measurement was made by Anastasiou (1981) in laboratory random waves by using Laser Doppler anemometry. The random waves were simulated by using the spectral form of Pierson-Moskowitz type

$$S(\omega)=\frac{0.081}{\omega^2}\exp(\frac{-0.74\omega^4}{\omega^4}) \qquad (21)$$

and generated by an irregular wave maker.

Figure 1 shows the comparison of partical velocity spectrum between the theoretically computed results and the experimental results with a water depth of 0.7m, wave steepness of 0.0478 and measurement elevation of 0.1m under water surface, where "I" denotes the measured spectrum including 95% confidence limits. From the comparison, we can find that the measured data are better consistent with the nonlinear theory than the linear theory. But in this case, the nonlinear phenomenum is not obvious because of the experimental wave conditions. In future, we will try to

make the comparison with more suitable data.

4. DISCUSSION

After normalizing the formula (11) with total energy and the peak frequency of power spectrum, we use some standard form of power spectrum in place of linear spectrum $\phi_i(k)$ in equation (11) to investigate some nonlinear characteristics of water particle velocity and wave pressure.

First, based on Pierson-Moskowitz spectrum, the influence of wave steepness on the nonlinear effect is investigated. The computational results with relative water depth (h/Lm) of 0.15, and relative measured elevation (Z/Lm) of -0.08 (measured from water surface) are shown in Figure 2. In the figures, the significant wave steepness δ is defined by significant wave height ($H_{1/3}$) and wave length (Lm), while the latter is correoponding to the peak frequency fm. The results calculated by linear theory are also shown in these figures. From the fact that the nonlinear component appear in higher and lower frequency ranges, we can learn from figure 2 that the nonlinear effect increases as the wave steepness increases.

Figure 3 shows the influence of surface displacement spectral form on the nonlinear effect for horizontal water particle velocity. Figure 3(a) and 3(b) show the results computed by using Pierson-Moskowitz spectrum and JONSWAP spectrum respectively. From the comparison between these two figures, it is found that the sharp form like JONSWAP spectrum produces clearer nonlinear effect than the milder form like Pierson-Moskowitz spectrum.

Figure 4 shows the influence of water depth on the nonlinear effect. In the figures, A_N/A_L denotes the relative magnitude of nonlinear total energy to linear total energy. The results are computed by using Pierson-Moskowitz spectrum. The breaking limit drawing with dash line in these figures is calculated from Hamada's wave breaking formula (Hamada 1951). It is shown that the horizontal velocity, vertical velocity and wave pressure all exhibite the same properties. From the figures, it is found that the nonlinear effect increases as relative water depth h/Lm decreases when the h/Lm in the range between 0.1 and 0.3. However when h/Lm is larger than 0.3, the nonlinear quantity is independent of the relative water depth. On the other hand, when relative water depth approaches 0.1, the nonlinear components approach to infinitive. In fact, in this water depth or shallower area, this formula can no longer be used because the formula is derived based on the weakly nonlinearity assumption.

Figure 5 shows the influence of measured elevation on the nonlinear effect. Z denotes the measured elevation from mean water surface. The results are computed by using Pierson-Moskowitz spectrum and with relative water depth of 0.15. For particle velocities, horizontal component and vertical component exhibite the same properties. From the figure (a), it is found that the measured elevation tends not to bear influence on the nonlinear effect. But as to the wave pressure (Figure(b)), it is found that the nonlinear effect increase as the elevation decreases.

In the above description, all simulate computational results only concern one-dimensional spectrum. In figure 6, the influence of direction-

al dispersion on the nonlinear effect is investigated with relative depth of 0.15, relative measured elevation of -0.04 and wave steepness of 0.06. The nonlinear spectral components calculated by the directional wave spectrum is compared with that of calculated by uni-directional wave spectrum. For convenience, the form of $\cos^2\theta$ was used as the wave directional distribution to calculate the nonlinear spectral components. In the computations, we found that it has to take a long computing time to integrate the directional distribution to obtain the ordinary power spectra of particle velocity or wave pressure. In Figure 6, from the comparison, it is found that the wave directional dispersion diminishes the nonlinear effect.

5. CONCLUSION

This paper is composed of two parts. First, a weakly nonlinear random wave theory in finite water depth was developed. The nonlinear spectral components for water particle velocity and wave pressure can be calculated directly from the directional spectrum of water surface displacement. The theoretical computed results were verified by experiment. Next, a simulate computation was performed. In accordance with the different characteristics of wave properties, such as wave steepness, water depth and so on, the nonlinear effects on wave kinematic and pressure properties were extensively investigated by using some standard power spectrum. From the investigation, we found some important conclusions as follows. First, the important characteristics of nonlinear effect for water particle velocity and wave pressure are similar to that for water surface displacement spectrum. Second, it takes much more computing time for directional spectrum than that for one-dimensional spectrum. Third, this theory can not be used in shallow water area when relative water depth is below 0.1.

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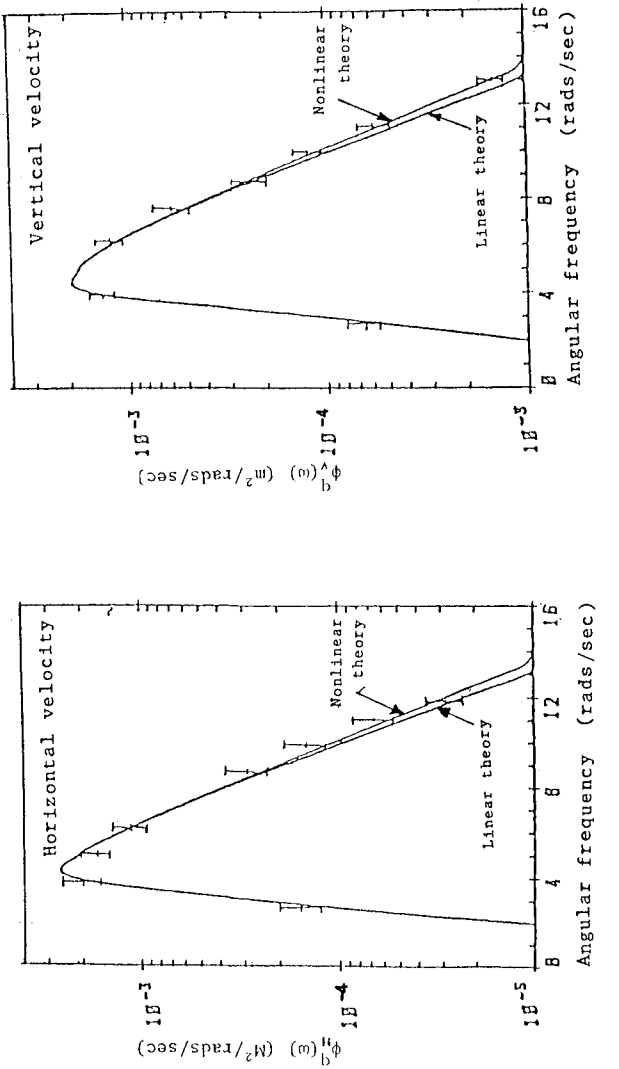
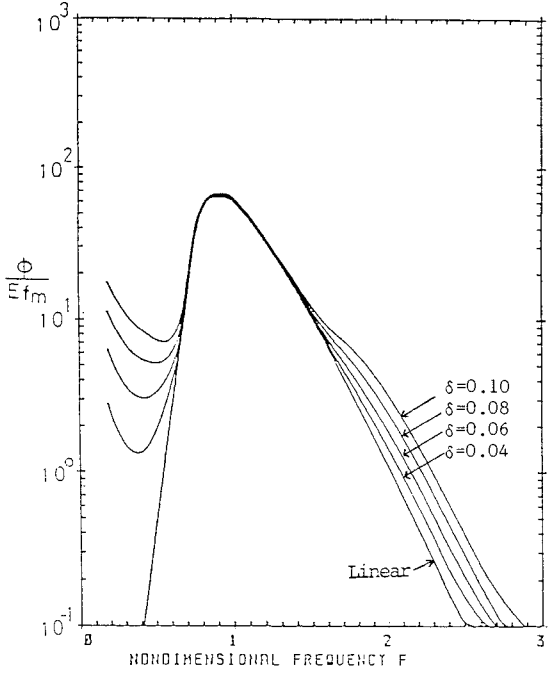
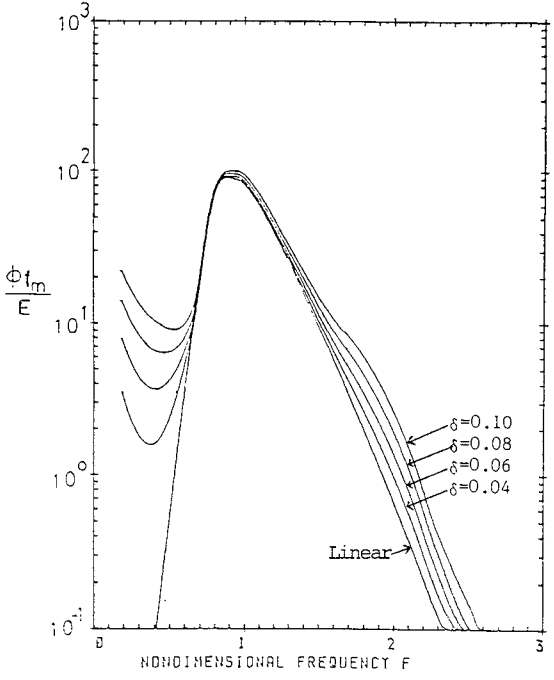


Figure 1 Comparison of the linear and nonlinear theory with the measured data

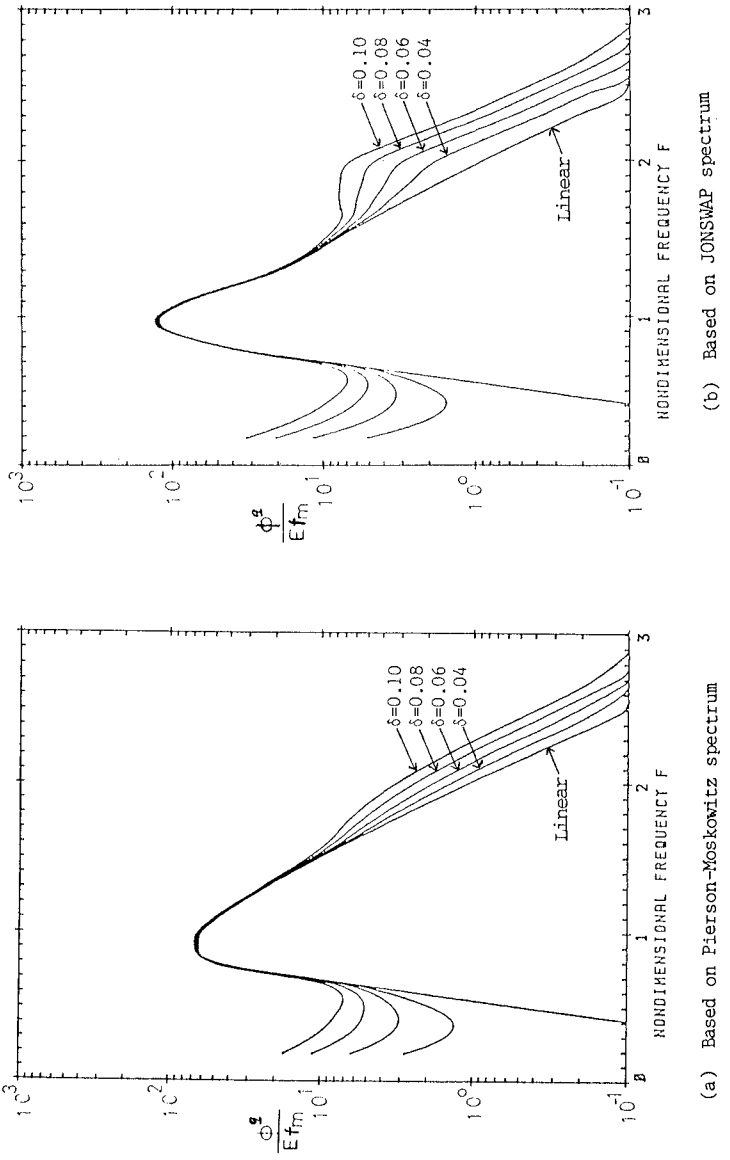


(a) Normalized horizontal particle velocity spectrum



(b) Normalized wave pressure spectrum

Figure 2 The influence of wave steepness on the nonlinear effect
(Based on Pierson-Moskowitz spectrum)



(a) Based on Pierson-Moskowitz spectrum

(b) Based on JONSWAP spectrum

Figure 3 The influence of spectral form on the nonlinear effect for horizontal water particle velocity

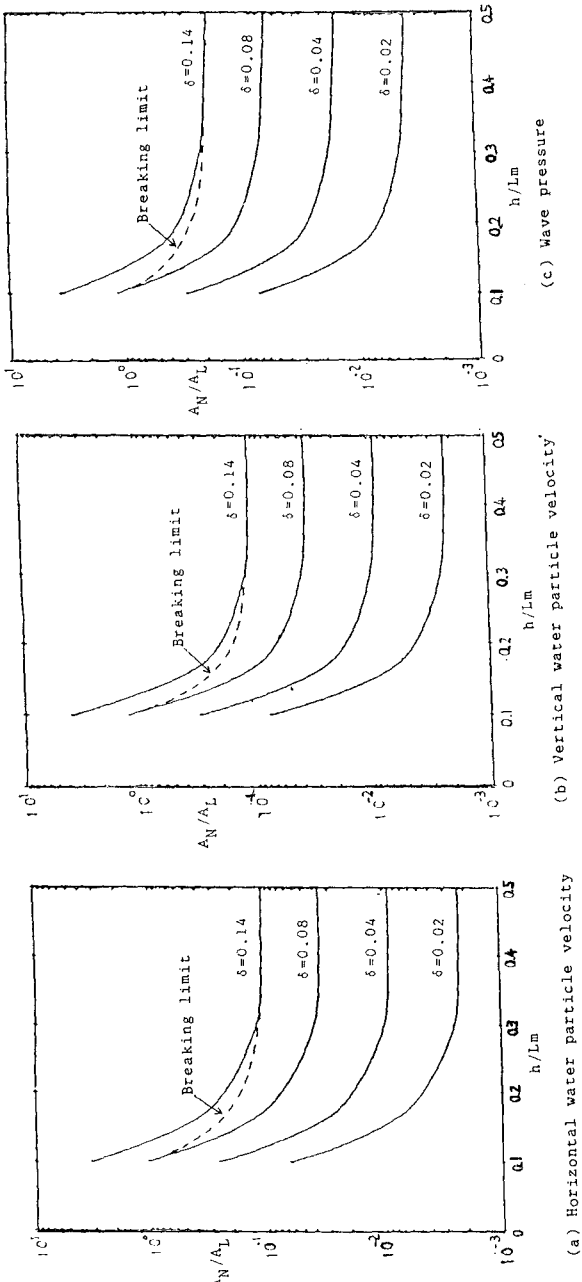


Figure 4 The nonlinear characteristics in terms of relative water depth and significant wave steepness.

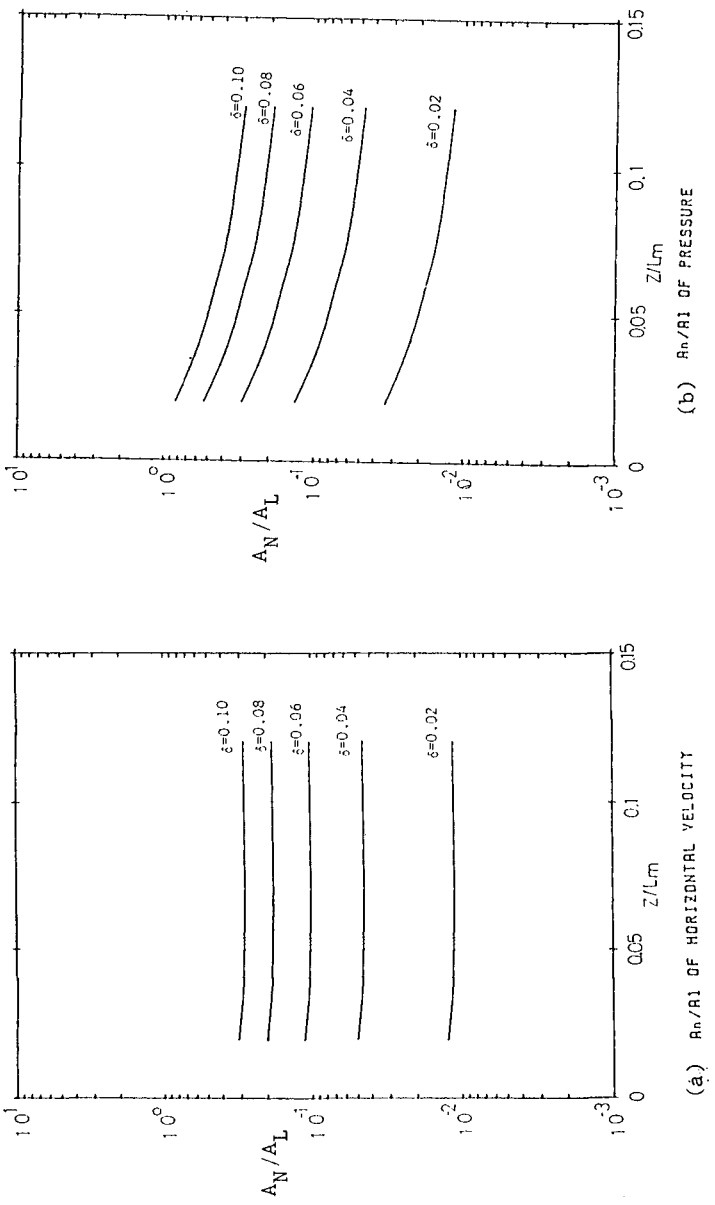


Figure 5 The influence of measured elevation on the nonlinear effect

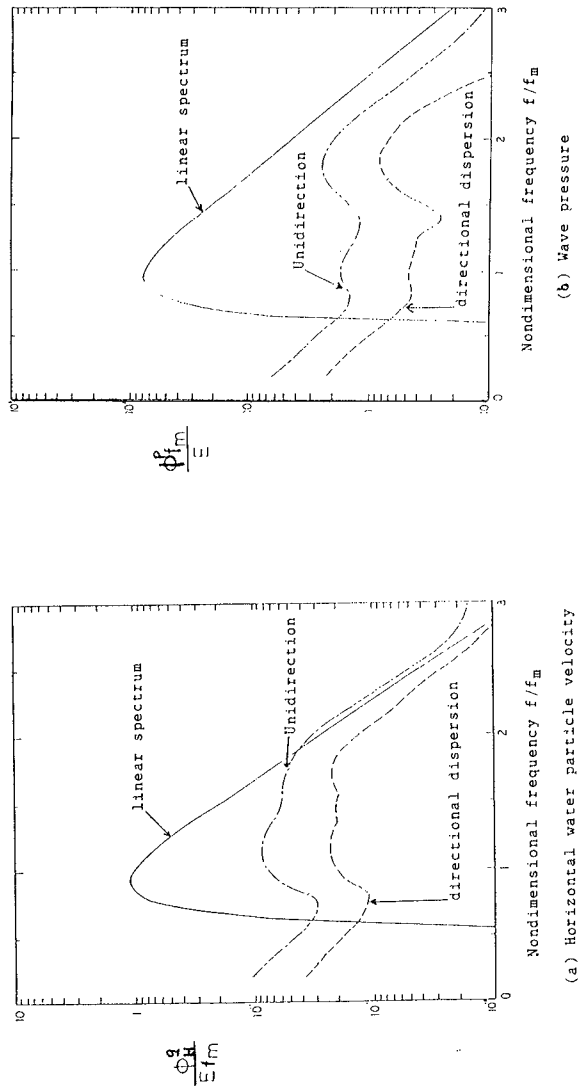


Figure 6 The influence of directional dispersion on the nonlinear effect for water particle velocity and water pressure.