

## CHAPTER SEVENTY FIVE

### CURRENT DEPTH REFRACTION OF REGULAR WAVES

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#### ABSTRACT

The complete set of equations for the refraction of small surface gravity waves on large-scale currents over a gradually varying sea bed is derived and presented. Wave lengths, direction of propagation and wave heights are all determined along the so-called wave rays as solutions to ordinary, first-order differential equations.

Dissipation due to bed friction in the combined current wave motion is included. The ray tracing method is used in an example. A method for the calculation of current depth refraction of weakly non-linear waves is proposed.

#### 1. INTRODUCTION

When water waves propagate over an area with a variable water depth and current velocity, the ensuing gradients together with the current wave interaction can cause drastical changes in the direction, length and height of the waves, see e.g. Mallory (1974). In this transformation of the waves, diffraction effects are often negligible, and the phenomenon is termed current depth refraction.

This phenomenon is a significant physical process in many coastal and offshore areas, for instance near river mouths and tidal inlets, in the surf zone along a beach, and where wind waves meet major ocean currents. The refraction is of great importance for erosion and deposition in coastal areas, forces on offshore structures, and ship navigation.

Current depth refraction is a complex problem of wave propagation in an inhomogeneous, anisotropic, dispersive, dissipative and moving medium. Attempts to solve the general case have therefore been scarce. Noda et al. (1974) are probably the first to try this. They employed a global finite difference scheme for solving the governing partial differential equations and experienced so many difficulties that they advocated a search for a method which made integration along characteristic curves possible. Later Iwagaki et al. (1977) tried such an approach, but since they mixed different kinds of characteristic curves and furthermore did not introduce wave action conservation, they did

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not really succeed in creating a practical model.

For wave height calculations on a current field the so-called wave action is important. Independently of each other this concept was found by Bretherton and Garrett (1968) for a rather general class of waves and by Hayes (1968) for acoustic waves. Jonsson et al. (1970) introduced the similar concept for water waves on a current, see their eq. (5.6).

Peregrine (1976), Phillips (1977), Jonsson (1978a), Peregrine and Jonsson (1983), and Peregrine et al. (1983) have given detailed reviews of waves on currents.

While today efficient and accurate models exist for the calculation of pure depth refraction, see e.g. Skovgaard et al. (1975), a similar model for the calculation of current and depth refraction does not exist. It is the main purpose of this paper to introduce the complete mathematical framework for such a model, which also takes bed friction and dissipation into account. It will be shown that the existence of a phase function, combined with a dispersion relation and a wave action conservation equation (which will be shown to be just a manipulated energy principle), lead to ordinary (first-order) differential equations along so-called wave rays for all wave quantities.

Small, regular waves are assumed; however, also non-linear waves are briefly dealt with.

Also an example of the practical use of a current depth refraction model will be presented. It is based exclusively on ray theory.

## 2. ASSUMPTIONS AND BASIC CONCEPTS

In most of this study small (and regular) progressive gravity waves are considered. Fluxes of mass, momentum and energy are integrated over depth and averaged over absolute period  $T_a$ , in that order. The problem is therefore formulated in two horizontal dimensions  $x_i, i = 1, 2$ .

A pure wave motion is defined as one with no net drift; thus the average - over-depth current velocity  $U_1$  is a zeroth order quantity. For a further discussion of this subject, see Jonsson (1978b), p. 228, and Jonsson and Wang (1980), pp. 157-158. The current velocity is assumed constant over depth.

Bed shear and accompanying dissipation is included. However, shear stresses in vertical sections are neglected. The bed slope is assumed small, so that the horizontal bed expressions are valid locally, i.e.  $|\nabla h| \ll kh$ , where  $\nabla$  is the horizontal gradient operator ( $\partial/\partial x_1, \partial/\partial x_2$ ), and  $k$  is the magnitude of the wave number vector  $k_i$ , so that  $k = 2\pi/L$ ,  $L$  being wave length. Furthermore, only large-scale currents in space and time are considered, i.e.  $|\nabla U| \ll kU$  and  $|\partial U/\partial t| \ll \omega_a U$ , where  $U$  is current speed,  $t$  is time, and  $\omega_a = 2\pi/T_a$  is the absolute angular frequency.

The study is within the framework of geometrical optics, i.e. wave fronts of small curvature are assumed thus excluding diffractive effects.

Two frames of reference are introduced. One is a coordinate system fixed in space; this is the absolute frame of reference, in which is used subscript 'a'. The other is a Galilean transformation of the first where the transformation velocity is the current velocity. Observations in this moving system are termed 'relative', and subscript 'r' is used here.

Referring to Fig. 1 the following relation exists between absolute and relative phase velocities

$$c_a = c_r + U \cos(\delta - \alpha) \quad (1)$$

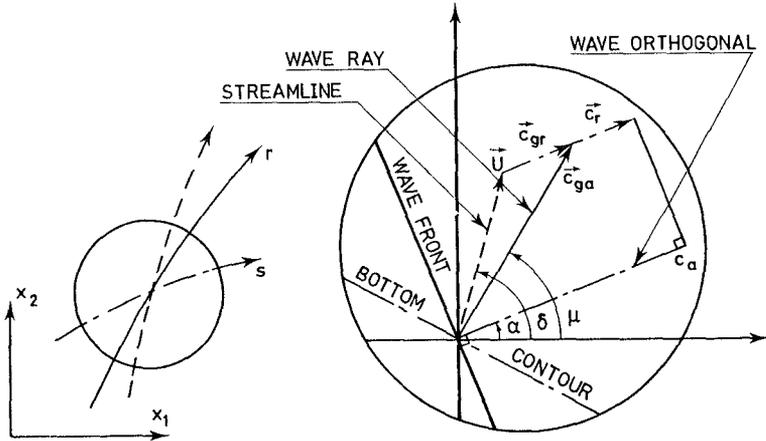


Fig. 1 - Horizontal sketch for characteristic curves, angles and velocities.

where angles  $\delta$  and  $\alpha$  are explained in the figure. (Note that by definition  $c_a$  is at right angles to the wave front, and that the vector  $U_i + (\hat{c}_r)_i$  has no physical meaning). Multiplying (1) by wave number  $k$  yields the Doppler relation

$$\omega_a = \omega_r + k_i U_i \tag{2}$$

in which the tensor summation notation is used, i.e.  $k_i U_i = k_1 U_1 + k_2 U_2 = kU \cos(\delta - \alpha)$ . The relative angular frequency is  $\omega_r = 2\pi/T_r$ ,  $T_r$  being relative period. The dispersion relation is

$$\omega_r^2 = g k \tanh kh \tag{3}$$

with  $g$  being gravity acceleration. Relative phase speed is found from  $c_r = \omega_r/k$ . Note that while  $\omega_a = \omega_a(h, k_i, U_i)$  we have  $\omega_r = \omega_r(h, k)$ .

In Fig. 1 three sets of curves are distinguished, streamlines, wave orthogonals and wave rays. The new concept is the wave ray, which goes in the direction of the absolute group velocity

$$(c_{ga})_i = c_{ga} (\cos\mu, \sin\mu) \tag{4}$$

The wave ray is not a unique concept. In the general (i.e. non-steady) case one must distinguish between *ray paths*, analogous to particle paths in conventional hydrodynamics, and *ray lines*, analogous to streamlines. In steady flow these curves are identical. The term wave ray will in the general case normally be used for the ray path, and in steady flow for ray line. The differential equation for the wave ray is

$$\frac{dx_i}{dt} = (c_{ga})_i \quad i = 1, 2 \quad (5)$$

in which in general  $(c_{ga})_i$  is a function of  $x_i$  and  $t$ .

In Fig. 2 ray paths and ray lines are illustrated together with the concepts of ray tube element and ray tube, which will be introduced in connection with the wave height calculation.

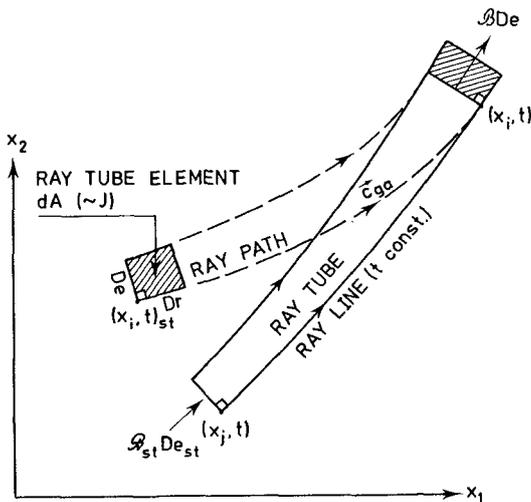


Fig. 2 - Horizontal sketch showing ray path, ray line, ray tube element and ray tube.

From (2) we find the relation between the absolute group velocity and the relative ditto  $(c_{gr})_i$

$$(c_{ga})_i \equiv \frac{\partial \omega_a}{\partial k_i} = \frac{\partial \omega_r}{\partial k_i} + U_i = (c_{gr})_i + U_i \quad (6)$$

which is illustrated in Fig. 1. The relative group velocity goes in the direction of the wave orthogonal and is given by

$$(c_{gr})_i = c_{gr} (\cos \alpha, \sin \alpha) \quad (7)$$

with the relative group speed

$$c_{gr} = \frac{1}{2} c_r (1 + G) \quad (8) \quad G = \frac{2kh}{\sinh 2kh} \quad (9)$$

Also the relative phase velocity goes in the orthogonal direction:

$$(c_r)_i = c_r (\cos \alpha, \sin \alpha) \quad (10)$$

The time derivative 'seen' by an observer moving along a wave ray with speed  $c_{ga}$  is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (c_{ga})_i \frac{\partial}{\partial x_i} = \frac{\partial}{\partial t} + c_{ga} \frac{D}{Dr} \tag{11}$$

where 'r' is the length along the ray. This operator (d/dt) turns out to be extremely important.

3. PROBLEM FORMULATION

In an environment with waves and currents the coastal and ocean engineer is primarily interested in the effect of the fluid flow on structures, the sea bed and vessels. To achieve this goal one must know the wave length (L), direction of propagation ( $\alpha$ ), wave height(H), as well as the mean water level (b) and the current vector ( $U_i$ ), all as functions of space and time ( $x_i, t$ ). This enables a calculation of the total water particle velocities and pressures and thus the forces.

The length and time scales for the waves are most often very much smaller than the corresponding scales for the current. Because of these contrasting scales it is natural to solve the wave field and the current field in two separate steps; this so-called two-level approach was suggested by Skovgaard and Jonsson (1976). Another good reason for this approach is that only at the current level one has to deal with partial differential equations. At the wave level, it turns out that only ordinary differential equations have to be solved. This is done by integration from initial (i.e. one point) boundary conditions along the wave rays. There is naturally, a coupling between the two levels (through bed shear and radiation stress), and an iterative procedure will in principle be necessary.

The known quantities in a water area will be the 'geometrical depth'  $D(x_i)$  (see Fig. 3) describing the bathymetry, and the Nikuradse roughness  $k_{Nj}(x_i)$  of the bed. Also certain lateral boundary conditions must be specified for the current field; for the waves, input data must be specified where the waves 'enter' the calculation area. Details of calculation for the current field will not be presented here; reference is made to Christoffersen (1982).

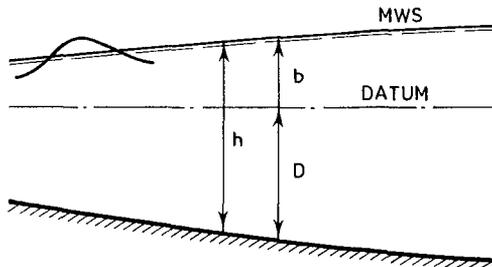


Fig. 3 - Vertical sketch. Definitions of geometrical depth D, mean surface elevation b, and mean water depth h.

In refraction calculations one normally assumes a steady state. In this paper, however, we shall present the general non-steady formulae, since the derivation of these is not more complicated than the steady ones. The steady state is hereafter considered as a special case.

#### 4. THE KINEMATICS

The basic assumption of ray theory (kinematic wave theory) is that changes in wave characteristics are so slow that locally plane waves can be assumed, see Hayes (1970). This means that *locally* we can introduce a phase function

$$\theta = \omega_a t - k_i x_i + \theta_0 \quad (13)$$

By differentiation of (13) we obtain

$$\frac{\partial \theta}{\partial t} = \omega_a \quad (14) \quad \frac{\partial \theta}{\partial x_i} = -k_i \quad (15)$$

These two quantities are then allowed to vary on a larger scale, and by further differentiation we find the so-called consistency relations

$$\frac{\partial k_i}{\partial t} + \frac{\partial \omega_a}{\partial x_i} = 0 \quad (16) \quad \frac{\partial k_i}{\partial x_j} = \frac{\partial k_j}{\partial x_i} \quad (17)$$

Equation (16) expresses conservation of waves, and (17) that the wave number vector is irrotational. On this basis it becomes possible to find the wave number vector, and thus the wave length and propagation direction, since  $L = 2\pi(k_1^2 + k_2^2)^{-1/2}$  and  $\tan \alpha = k_2/k_1$ . (A further consequence of the assumption of locally plane waves is the Doppler relation (3)).

Using the operator (11) on  $k_i$ , and for  $\partial k_i/\partial t$  inserting (16) yields

$$\frac{dk_i}{dt} = -\frac{\partial \omega_a}{\partial x_i} + (c_{ga})_j \frac{\partial k_i}{\partial x_j} \quad (18)$$

Since  $\omega_a = \omega_a(h, k_j, U_j)$  one obtains

$$\frac{\partial \omega_a}{\partial x_i} = \frac{\partial \omega_a}{\partial h} \frac{\partial h}{\partial x_i} + \frac{\partial \omega_a}{\partial k_j} \frac{\partial k_j}{\partial x_i} + \frac{\partial \omega_a}{\partial U_j} \frac{\partial U_j}{\partial x_i} \quad (19)$$

Using (2) to give  $\partial \omega_a/\partial h = \partial \omega_r/\partial h$  and  $\partial \omega_a/\partial U_j = k_j$ , further introducing (17) and (6), (18) and (19) finally yield

$$\frac{dk_i}{dt} = -\frac{\omega_r G}{2h} \frac{\partial h}{\partial x_i} - k_j \frac{\partial U_j}{\partial x_i} \quad (20)$$

Here it is used that  $\partial \omega_r/\partial h = \omega_r G/2h$ , see Christoffersen and Jonsson (1980). Since  $d/dt$  is given by (11), the important result is achieved that through (20) the rate of change of the wave number vector is found along a ray.

In conclusion, the ray path  $x_i$  and the wave number vector  $k_i$  along this path, are determined by the 4 ordinary, first-order differential

equations (5) and (20), with proper initial conditions.

The absolute angular frequency can be determined either from the algebraic Doppler relation (2) or from (Christoffersen, 1982)

$$\frac{d\omega_a}{dt} = \frac{\omega_r G}{2h} \frac{\partial h}{\partial t} + k_j \frac{\partial U_j}{\partial t} \tag{21}$$

It appears that for a steady medium,  $\omega_a$  remains constant for a 'ray tube element', see Fig. 2.

Similarly, an equation for  $d\omega_r/dt$  along a wave ray can be found, see Christoffersen (1982), chapter 4. The relative angular frequency can also be found from the algebraic expression (3).

In the next chapter it will be found that the evolution of the so-called ray tube element (Fig. 2) must be known in order to calculate the wave height along a ray. Since this is a kinematic problem, it will be dealt with in the present chapter.

A ray tube element is a differential concept, to be visualized as an infinitesimally short 'tube' made up of neighbouring rays. Its area being  $dA$ , the 'Jacobian'  $J$  is introduced as

$$J = \frac{dA}{dA_{st}} = \frac{De}{De_{st}} \frac{c_{ga}}{c_{ga,st}} \tag{22}$$

since  $dA = De Dr$  (Fig. 2), and  $Dr = c_{ga} dt$ . In (22) suffix 'st' stands for 'start' or initial value.

Knowledge of how a ray tube element changes as it moves is found in Aris (1962), pp. 83-84. The so-called Euler expansion formula simply states that

$$\frac{1}{J} \frac{dJ}{dt} = \frac{\partial (c_{ga})_i}{\partial x_i} \tag{23}$$

Using (6) and (2) to evaluate the right hand side of (23) (the divergence of  $(c_{ga})_i$ ) it is found that

$$\frac{1}{J} \frac{dJ}{dt} = \frac{\partial^2 \omega_r}{\partial k_i \partial k_j} \frac{\partial k_j}{\partial x_i} + \frac{\partial^2 \omega_r}{\partial k_i \partial h} \frac{\partial h}{\partial x_i} + \frac{\partial U_i}{\partial x_i} \tag{24}$$

The second order derivatives in (24) are

$$\frac{\partial^2 \omega_r}{\partial k_i \partial h} = \frac{\partial^2 \omega_r}{\partial k \partial h} \frac{k_i}{k} \tag{25}$$

$$\frac{\partial^2 \omega_r}{\partial k_i \partial k_j} = \left( \frac{\partial^2 \omega_r}{\partial k^2} - \frac{1}{k} \frac{\partial \omega_r}{\partial k} \right) \frac{k_i k_j}{k^2} + \frac{1}{k} \frac{\partial \omega_r}{\partial k} \delta_{ij} \tag{26}$$

where  $\delta_{ij}$  is the Kronecker delta.

The evaluation of  $\partial k_j / \partial x_i$  in (24) is quite tricky. Playing around with the consistency relations and the Doppler relation finally yields the 'derived ray equation' (Hayes, 1970).

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial k_j}{\partial x_i} \right) = & - \frac{\partial k_k}{\partial x_i} \left[ \frac{\partial^2 \omega_r}{\partial k_k \partial k_\ell} \frac{\partial k_\ell}{\partial x_j} + \frac{\partial^2 \omega_r}{\partial k_k \partial h} \frac{\partial h}{\partial x_j} + \frac{\partial U_k}{\partial x_j} \right] - k_k \frac{\partial^2 U_k}{\partial x_i \partial x_j} \\ & - \frac{\partial k_k}{\partial x_j} \left[ \frac{\partial^2 \omega_r}{\partial k_k \partial h} \frac{\partial h}{\partial x_i} + \frac{\partial U_k}{\partial x_i} \right] - \frac{\partial^2 \omega_r}{\partial h^2} \frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_j} - \frac{\partial \omega_r}{\partial h} \frac{\partial^2 h}{\partial x_i \partial x_j} \end{aligned} \quad (27)$$

in which  $\partial \omega_r / \partial h = \omega_r G / 2h$ . This equation contains three ordinary differential equations, determining the symmetrical tensor  $\partial k_j / \partial k_i$  along rays. For details see Christoffersen (1982), and Jonsson and Christoffersen (1982).

## 5. THE DYNAMICS

The mass conservation equation is simply

$$\frac{\partial}{\partial t} (\rho h) + \frac{\partial}{\partial x_i} (\rho h U_i) = 0 \quad (28)$$

The momentum conservation equations (Phillips, 1977, eq. 3.6.11), with the mean bed shear stress  $(\tau_b)_i$  added, read

$$\frac{\partial}{\partial t} (\rho h U_i) + \frac{\partial}{\partial x_j} (\rho h U_i U_j + S_{ij}) + \rho g h \frac{\partial b}{\partial x_i} + (\tau_b)_i = 0 \quad (29)$$

Standard sign convention is adopted for shear, i.e. the shear stress acting on the *bed* is taken positive in the positive  $x_i$ -direction. Phillips' bed shear stress appears with the opposite sign (his p. 65).

The energy conservation equation (Phillips, 1977, eq. 3.6.18) with the total dissipation per unit area  $E_d$  added reads

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{1}{2} \rho h U_j^2 + E + \frac{1}{2} \rho g (b^2 - D^2) \right) \\ + \frac{\partial}{\partial x_i} \left( \frac{1}{2} \rho h U_i U_j^2 + \rho g h U_i b + E (c_{ga})_i + S_{ij} U_j \right) + E_d = 0 \end{aligned} \quad (30)$$

Note that there is some ambiguity in Phillips' definition of dissipation, see Christoffersen and Jonsson (1980). In the above equations  $S_{ij}$  is the radiation stress tensor given by

$$S_{ij} = \delta_{ij} S_p + \frac{k_i k_j}{k^2} S_m \quad (31)$$

where  $S_p$  and  $S_m$  are the pressure and momentum parts of the radiation stress given by

$$S_p = \frac{1}{2} E G \quad (32) \quad S_m = \frac{1}{2} E (1 + G) \quad (33)$$

Here  $E$  as usual is the wave energy density,  $1/8 \rho g H^2$ .

Christoffersen and Jonsson (1980) derived the wave action equation for a dissipative steady medium. The wave action equation for a general unsteady dissipative medium is here derived in a simpler and more straightforward way.

Eliminating first the mean surface height  $b$  by multiplying (29) by  $U_i$  and subtracting the result from (30) yields

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_i} \left( E (c_{ga})_i \right) + S_{ij} \frac{\partial U_j}{\partial x_i} + E_d - (\tau_b)_i U_i = 0 \quad (34)$$

Dividing by  $\omega_r$  and using also (11) gives

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{E}{\omega_r} \right) + \frac{\partial}{\partial x_i} \left( \frac{E}{\omega_r} (c_{ga})_i \right) + \frac{E_d - (\tau_b)_i U_i}{\omega_r} \\ + \frac{1}{\omega_r} \left( \frac{E}{\omega_r} \frac{d\omega_r}{dt} + S_{ij} \frac{\partial U_j}{\partial x_i} \right) = 0 \end{aligned} \quad (35)$$

Here, the variation of  $\omega_r$  along a ray can be expressed in terms of the radiation stress tensor as (Christoffersen, 1982)

$$\frac{E}{\omega_r} \frac{d\omega_r}{dt} = - S_{ij} \frac{\partial U_j}{\partial x_i} \quad (36)$$

Inserting (36) in (35), the last term vanishes, and the new general wave action equation emerges as

$$\frac{\partial \mathcal{A}}{\partial t} + \frac{\partial}{\partial x_i} \left( \mathcal{A} (c_{ga})_i \right) + E_w = 0 \quad (37)$$

in which wave action  $\mathcal{A}$  and 'wave action dissipation'  $E_w$  are by definition

$$\mathcal{A} = \frac{E}{\omega_r} \quad (38) \quad E_w = \frac{E_d - (\tau_b)_i U_i}{\omega_r} \quad (39)$$

Note that all terms in (37) vanish for a vanishing wave motion. Thus the energy equation (30) for the total flow has been transformed into a neat energy equation for the wave motion. This can be further simplified, though. Introducing (23) in (37) yields

$$\frac{d}{dt} (\mathcal{A} J) + E_w J = 0 \quad (40)$$

This means that wave action  $\mathcal{A}$  - and thus wave height  $H$  - can be calculated along a wave ray by solving the ordinary differential equation (40). By introducing (22), an alternative version of (40) appears as

$$\frac{d}{dt} (\mathcal{B} De) + E_w c_{ga} De = 0 \quad (41)$$

with wave action flux  $\mathcal{B} \equiv \mathcal{A} c_{ga}$ .

### 6. WAVE HEIGHT CALCULATION

The wave height equation is most easily found by looking at the steady case. With  $\partial/\partial t = 0$  we find from (11) and (41),  $D(\mathcal{B}De)/Dr + E_w De = 0$ , where 'r' is distance along the ray, and 'e' is distance at right angles to this (Fig. 2). It thus appears that wave action propagates between neighbouring rays and is dissipated through wave action dissipation. The wave action flux through the ray tube is  $\mathcal{B}_{st} De_{st}$  at one end ('st' stands for start), and  $\mathcal{B} De$  at an arbitrary

point (Fig. 2). Taking the square root of the ratio between these two quantities yields the wave height equation

$$\frac{H}{H_{st}} = \sqrt{\frac{\omega_r}{\omega_{r,st}}} \sqrt{\frac{c_{ga,st}}{c_{ga}}} \sqrt{\frac{De_{st}}{De}} \sqrt{\frac{\mathcal{A}_J}{\mathcal{A}_{st,Jst}}} \quad (42)$$

$$= K_C \times K_S \times K_{ra} \times K_f$$

in which  $K_C$  is the 'Doppler coefficient',  $K_S$  the 'shoaling coefficient',  $K_{ra}$  the 'refraction coefficient', and  $K_f$  the 'friction coefficient'. It can be shown that (42) also is valid in the non-steady case, see e.g. Jonsson (1982). The three first coefficients are determined by the kinematics of the current-wave field, and it is immediately seen from (22) that  $J = (K_S K_{ra})^{-2}$ , i.e.

$$\frac{H}{H_{st}} = K_C J^{-1/2} K_f \quad (43)$$

By dividing (40) with  $\mathcal{A}_{st} J_{st}$  ( $J_{st} \equiv 1$ ) and introducing the expression for  $K_f$  in (42), we find for the friction coefficient

$$\frac{d}{dt} (K_f^2) = - \frac{E_w}{\mathcal{A}_{st}} J \quad (44)$$

i.e. yet another ODE to be solved along the rays.

There remains the calculation of  $E_w$  in (44). From (39) it appears that this calls for the determination of bed shear  $(\tau_b)_i$  and dissipation  $E_d$ . In Christoffersen and Jonsson (1984) a detailed calculation procedure is presented for the calculation of these quantities. The method is based on two-layer models for the current-wave boundary layer, one for 'small roughnesses' and one for 'large roughnesses'. In general the following approximate result was obtained:  $E_d = (\tau_b)_i U_i + \langle (\tau_{wb})_i (u_{wb})_i \rangle$ , whereby from (39) we find  $E_w = \langle (\tau_{wb})_i (u_{wb})_i \rangle / \omega_r$ . Here  $(\tau_{wb})_i$  is the wave part of the bed shear stress and  $(u_{wb})_i$  the wave particle velocity just outside the wave boundary layer at the bed.

## 7. SUMMARY OF RESULTS

The procedure for the calculation of the wave field ( $k_i, H$ ) is hereafter. The wave ray - along which all other quantities can be calculated - and the wave number vector are determined by the four differential equations (5) and (20) with proper initial conditions, assuming water depths and current velocity known. Note that operator  $d/dt$  is given by (11).

The wave height is then determined by (43). Doppler coefficient  $K_C$  is found from its definition in (42), and the Jacobian  $J$  by the four differential equations (24) and (27). Finally  $K_f$  is found from (44).

The above procedure is generally valid. In practise a steady situation is often assumed, and simpler equations emerge, see Christoffersen (1982), and Jonsson and Christoffersen (1982). A discussion of the possible calculation of the current field, and of the interaction between the waves and the currents through radiation stress and bed shear is given in the former reference.

8. EXAMPLE

To illustrate the effect of currents on waves, we look at conditions outside an inlet with a 'jet' ebb flow as shown in Fig. 4.

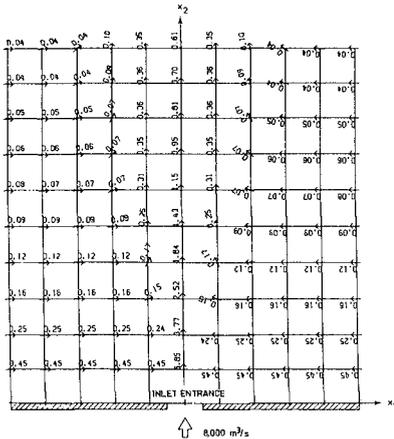


Fig. 4. Horizontal plan. Ebb current velocities (m/s) outside inlet. Arrows indicate current direction and numbers are current speed. The current is symmetrical about the  $x_2$ -axis. Grid spacing is 200 m, and water depth is  $h = 10 \text{ m} + 0.02 x_2$  ( $x_2 > 0$ ), i.e. bed slope 1:50. The inlet entrance is 200 m wide, with water depth  $h_0 = 10 \text{ m}$ . Discharge through inlet is  $Q = 8,000 \text{ m}^3/\text{s}$ , giving an average-over-width current velocity  $U_0 = 4 \text{ m/s}$ .

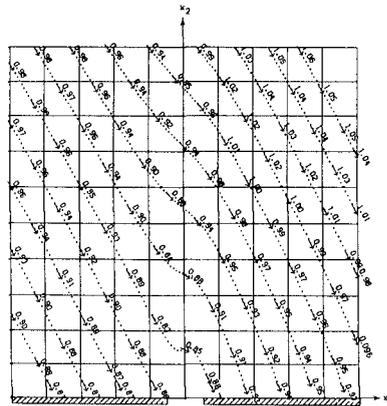


Fig. 5. Horizontal plan. Wave rays (...) on ebb current in Fig. 4. Arrows indicate orthogonal direction, and numbers are wave heights (m) (no dissipation). The thirteen computed rays start at  $x_2 = 5,000 \text{ m}$  ( $h = 110 \text{ m}$ ). The corresponding  $x_1$ -values vary from  $x_1 = -3,600 \text{ m}$  to  $x_1 = -1,200 \text{ m}$ , with 200 m spacing. The angle of incidence is here  $30^\circ$ . Initial wave height is  $H_{st} = 1 \text{ m}$ , and absolute wave period is  $T_a = 8 \text{ s}$ .

With angle of incidence  $30^\circ$  far from the inlet, the computed orthogonal directions and wave heights (no dissipation) are presented along the rays in Fig. 5. The fact that wave rays and orthogonals do not coincide (when currents are present), is most clearly seen near the inlet entrance, where current velocities are large. The most striking - and perhaps unexpected - result of our calculations as presented in Fig. 5, is the marked reduction in wave height in the central part of the ebb current. The reason for this is that the oblique incidence of the waves on the opposing current in ques-

tion, in this area generates a sharp reduction in refraction coefficient. This is a result of the obvious increase in ray spacing here.

9. NON-LINEAR WAVES

In Shen and Keller (1973) and Shen (1975) (considering non-linear wave propagation in shallow water without a current) it was found that the kinematics to a first approximation was governed by linear theory, i.e.  $\omega_r = k \sqrt{gh}$ , which is the shallow water version of (3). This meant that non-linearity only played a role in the calculation of the dynamics, which was governed by a Korteweg-DeVries equation. This equation is known to have cnoidal wave solutions, see Svendsen and Hansen (1978) where the energy flux has been calculated for cnoidal waves.

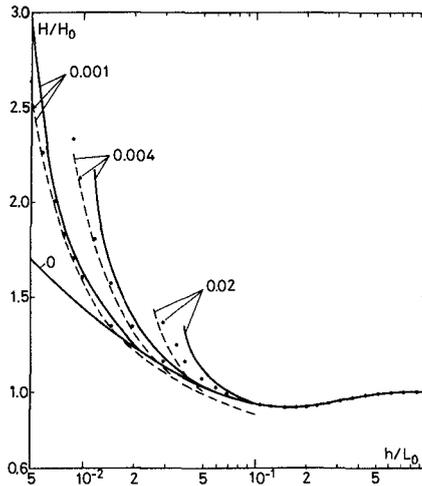


Fig. 6 - Shoaling curves. Full lines: Cokelet's theory. Dashed lines: Svendsen and Brink-Kjær (1972). (From Sakai and Battjes, 1980) Dots: Present study. Values of  $H_0/L_0$  are shown.

Further Ryrie and Peregrine (1982), considering refraction of numerically exact finite-amplitude waves, found that linear wave theory normally proved quite accurate for predicting the wave directions. These observations really indicate that a first simple extension to include weakly non-linear waves, could be to keep the linear-wave kinematics unaltered, and only take non-linearity into account in the wave action equation (40). This means that one just replaces the linear wave energy density in (38) with the non-linear wave energy density  $E = B(m) \rho g H^2$  where

$$B(m) = \frac{1}{3m^2} \left[ \frac{E}{K} (2 + 2m_1 - 3 \frac{E}{K}) - m_1 \right] \tag{45}$$

see e.g. Sarpkaya and Isaacson (1981), p. 185.

Here  $m_1 = 1-m$ , and further  $K(m)$  and  $E(m)$  are the complete elliptic integrals of the first and second kind, respectively. The parameter  $m$  is determined from  $HL^2/h^3 = (16/3) m K^2$ .

Because of  $B(m)$  ( $= 1/8$  for linear waves), a fifth factor  $K_n = \sqrt{B_{st}/B}$  is introduced in (42), to account for non-linearity.

To illustrate the applicability of this new and simple approach, we show in Fig. 6 the results of shoaling without a current.

In the figure, the Sakai and Battjes (1980) results for shoaling of finite amplitude waves (using Cokelet's theory) have been compared with our model, and with the results of Svendsen and Brink-Kjær (1972), showing that our model yields encouraging results for the wave heights, also giving a smooth transition from deep to shallow water. The smaller the steepness, the better the agreement with the Cokelet theory.

The proposed extension does not only give improved results, (as compared with linear theory), but it is also easy to implement in a refraction program capable of treating arbitrary bathymetries and current fields.

## 10. CONCLUSIONS

The assumption of *locally* plane waves, combined with a dispersion relation and wave action conservation, have led to the complete set of equations for current and depth refraction of regular water waves. These are ordinary, first-order differential equations for all wave quantities, when formulated along the so-called wave rays. Bed friction and accompanying dissipation is included. The ray equations form the basis of a computer program, which is applied to a case of refraction on an ebb current outside an inlet. A method to incorporate non-linear waves in the system is proposed.

Dr. Ove Skovgaard, Laboratory of Applied Mathematical Physics, is acknowledged for his unfaltering interest and good advice on the computational problems.

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