

CHAPTER SEVENTY

THE EXACT SOLUTION OF THE HIGHEST WAVE DERIVED FROM A UNIVERSAL WAVE MODEL

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ABSTRACT

The solitary wave is first established in this paper by extending the series solution of periodic gravity wave as the wavelength approaches to infinite. Then, the highest gravity wave of permanent type in finite depth of water is immediately analyzed. The maximum ratio of wave height to water depth is obtained as 0.854654..., and the angle at the crest for the considered highest wave is estimated to be 90° .

INTRODUCTION

Since Stokes (1847) developed the theory of oscillatory waves, the analysis for the periodic gravity waves of permanent form in the water has comprehensively been expanded, particularly, a very interested and important problem is the highest waves. However, up to now, all wave theories which have presented are not continuously available to describe the flow field of a wave motion in which both amplitude and wavelength vary from small to possible maximum in any fixed depth of water. Therefore, to solve this highest wave problem, several special wave theories have been employed, for example, as the cnoidal wave by Laitone (1960), the solitary wave by McCowan (1894), Yamada (1957), Lenau (1966) and Longuet-Higgins (1974), and an integral equation of wave motion by Byatt-Smith (1970) etc. The purpose of this paper is to establish an universal model which can adequately be used to analyze the waves motion in arbitrary uniform depth, from deep to shallow water, up to a solitary wave especially, and its wave height from small to just breaking; i.e. this model is adjustable to all the region for the periodic gravity waves of permanent type can be existed.

In §I, we recapitulate to formulate the series solution for the periodic gravity waves of permanent type in water of arbitrary uniform depth, which has been submitted in 19th International Conf. on Coastal Eng. In §II, the solitary wave similar to McCowan's type is established, in §III, §IV and §V, the structure of the highest waves is discussed.

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1. THE NEW MODEL OF WATER WAVES

The solution of Laplace equation in water wave motion are to be set as follows:

$$\psi(x, y, t) = \sum_{n=1}^{\infty} C_n e^{-nkd} [e^{nk(d+y)} - e^{-nk(d+y)}] \cos nk(x-ct) \dots (1.1)$$

$$\phi(x, y, t) = \sum_{n=1}^{\infty} C_n e^{-nkd} [e^{nk(d+y)} + e^{-nk(d+y)}] \sin nk(x-ct) \dots (1.2)$$

$\psi(\cdot)$: Stream function

$\phi(\cdot)$: Potential

C_n : constants to be worked out

n : 1, 2, 3,

e : exponential

k : wave number, $k = 2\pi/L$

L : wavelength

d : water depth under x - axis

c : wave celerity

The co-ordinate system is illustrated in Fig 1.

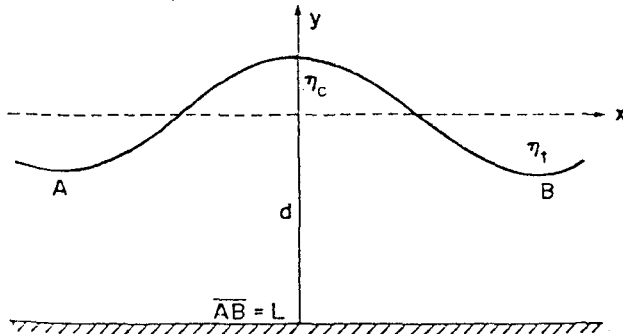


Figure 1 The coordinate system of wave motion

The motion is altered to be steady flow by adding an opposite velocity c . $(x-ct)$ in above equations will be replaced by x , then

$$\psi(x, y) = -cy + \sum_{n=1}^{\infty} C_n e^{-nkd} [e^{nk(d+y)} - e^{-nk(d+y)}] \cos nkx \dots (1.3)$$

$$\phi(x, y) = -cx + \sum_{n=1}^{\infty} C_n e^{-nkd} [e^{nk(d+y)} + e^{-nk(d+y)}] \sin nkx \dots (1.4)$$

The equation of surface elevation is to be obtained by $\psi(x, \eta) = 0$ as follows

$$\eta(x) = \sum_{n=1}^{\infty} \frac{C_n}{c} e^{-nkd} [e^{nk(d+\eta)} - e^{-nk(d+\eta)}] \cos nkx \dots\dots\dots (1.5)$$

η : elevation of wave surface with reference to x axis

The constants are worked out by boundary conditions as shown in the previous paper published in Captain conference (19th proc. ICCE page 508-520).

The solution is as follows:

In deep water, $d \rightarrow \infty$, when $n=1$, $\dots\dots\dots (1.6)$

$$\psi(x, y) = -cy + \frac{c\theta}{ke^\theta} e^{ky} \cos kx \dots\dots\dots (1.7)$$

$$\eta(x) = \frac{\theta}{ke^\theta} e^{k\eta} \cos kx \dots\dots\dots (1.8)$$

$$\theta = \frac{2\pi\delta}{1 + e^{-2\pi\delta}}, \quad \delta = \frac{H}{L}, \quad \text{the wave steepness.}$$

In water area of finite depth, when $n=1$,

$$\psi(x, y) = -cy + c\eta_c \frac{\sinh k(d+y)}{\sinh k(d+\eta_c)} \cos kx \dots\dots\dots (1.9)$$

$$\eta(x) = \eta_c \frac{\sinh k(d+y)}{\sinh k(d+\eta_c)} \cos kx \dots\dots\dots (1.10)$$

and η_c is to be computed through following equation by newton-Raphson's method

$$\delta = \frac{H}{L} = \frac{\eta_c}{L} \{ [1 + \cosh(2\pi\delta)] - \coth k(d+\eta_c) \sinh(2\pi\delta) \} \dots\dots\dots (1.11)$$

η_c : the elevation of wave crest above x - axis

wave celerity c , period T and the difference between still water level and the water depth of $\psi(x, \eta) = 0$, namely the depth under x-axis, $\zeta = d_s - d$ have been worked out (d_s the depth of still water).

After elaborate experiments, the new model represents the reality very well. (19th Proc. ICCE page 510-520)

II. EXTENSION OF THE MODEL

For extending the new model to the region of solitary wave, the co-ordinate system is set as Fig-2.

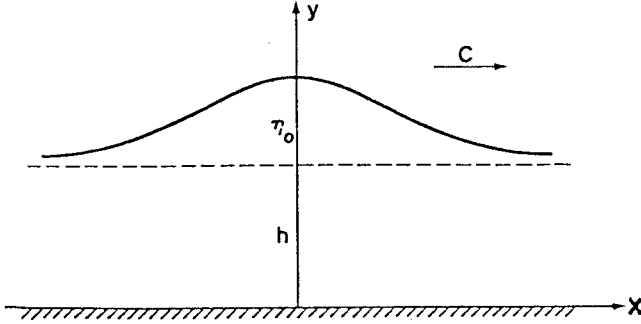


Figure 2 The coordinate system of solitary wave

The equations expressing ψ, ϕ, η are changed to be

$$\psi(x, y) = -c(y-h) + \sum_{n=1}^{\infty} C_n^1 (e^{nky} - e^{-nky}) \cos(nkx) \quad \dots\dots (11.1)$$

$$\phi(x, y) = -cx + \sum_{n=1}^{\infty} C_n^1 (e^{nky} + e^{-nky}) \sin(nkx) \quad \dots\dots\dots (11.2)$$

$$\eta(x, y) = \sum_{n=1}^{\infty} \frac{C_n^1}{c} [e^{nk(h+\eta)} - e^{-nk(h+\eta)}] \cos(nkx) \quad \dots\dots\dots (11.3)$$

$$C_n^1 = C_n e^{-nkd}$$

let

$$K_n = \frac{2C_n^1}{c} \sinh[nk(h+\eta_0+\epsilon_0)] \quad \dots\dots\dots (11.4)$$

η_0 is the height of wave crest on water level and ϵ_0 is an constant, $\epsilon_0 > 0$

Above equations become

$$\psi(x, y) = -c(y-h) + c \sum_{n=1}^{\infty} K_n \frac{\sinh(nky)}{\sinh[nk(h+\eta_0+\epsilon_0)]} \cos(nkx) \quad \dots (11.5)$$

$$\phi(x, y) = -cx + c \sum_{n=1}^{\infty} K_n \frac{\cosh(nky)}{\sinh[nk(h+\eta_0+\epsilon_0)]} \sin(nkx) \quad \dots\dots (11.6)$$

$$\eta(x) = \sum_{n=1}^{\infty} K_n \frac{\sinh[nk(h+\eta_0+\epsilon_0)]}{\sinh[nk(h+\eta_0+\epsilon_0)]} \cos(nkx) \quad \dots\dots\dots (11.7)$$

While $L \rightarrow \infty$, $k = \frac{2\pi}{L} \rightarrow 0$, let $k = d\tau$, $nk = \tau$, $K_n = K(\tau)$

$$\lim_{L \rightarrow \infty} \sum_{n=1}^{\infty} \frac{K_n \sinh\left(\frac{2n\pi}{L}y\right)}{\sinh\left[\frac{2n\pi}{L}(h+\eta_0+\epsilon_0)\right]} \cos\left(\frac{2n\pi}{L}x\right) \quad \dots\dots\dots (11.8)$$

$$= \frac{1}{2\pi} \int_0^{\infty} LK(\tau) \frac{\sinh(y\tau)}{\sinh[(h+\eta_0+\epsilon_0)\tau]} \cos(\tau x) d\tau \dots\dots (11.8)$$

The integration should be a finite quantity even though the wave-length becomes infinitive, let

$$\lim_{L \rightarrow \infty} \frac{L \cdot K(\tau)}{2\pi} = \beta' \dots\dots\dots (11.9)$$

β' will be a constant of finite value, consequently

$$\begin{aligned} & \lim_{L \rightarrow \infty} \sum_{n=1}^{\infty} \frac{K_n \sinh(nky)}{\sinh[(h+\eta_0+\epsilon_0)\tau]} \cos(nkx) \\ &= \beta' \int_0^{\infty} \frac{\sinh(y\tau)}{\sinh[(h+\eta_0+\epsilon_0)\tau]} \cos(\tau x) d\tau \\ &= \frac{\beta'}{2} \int_{-\infty}^{\infty} \frac{e^{i\tau x} \sinh(y\tau)}{\sinh[(h+\eta_0+\epsilon_0)\tau]} d\tau \dots\dots\dots (11.10) \\ & \quad i = \sqrt{-1} \end{aligned}$$

let

$$\tau(h+\eta_0+\epsilon_0) = \pi\xi \dots\dots\dots (11.11)$$

$$\begin{aligned} & \lim_{L \rightarrow \infty} \sum_{n=1}^{\infty} \frac{K_n \sinh(nky)}{\sinh[nk(h+\eta_0+\epsilon_0)]} \cos(nkx) \\ &= \frac{\beta'}{2} \frac{\pi}{(h+\eta_0+\epsilon_0)} \int_{-\infty}^{\infty} \exp\left(-\frac{i\pi x\xi}{h+\eta_0+\epsilon_0}\right) \frac{\sinh\left(\frac{\pi y\xi}{h+\eta_0+\epsilon_0}\right)}{\sinh(\pi\xi)} d\xi \dots\dots (11.12) \end{aligned}$$

A complex integral as follows is invoked here, and which is integrated through the route in Fig. 3.

$$I = \int_{-\infty}^{\infty} e^{i\lambda_0 u} \frac{\sinh(a_0 u)}{\sinh(\pi u)} du \quad -\pi < a_0 < \pi \dots\dots\dots (11.13)$$

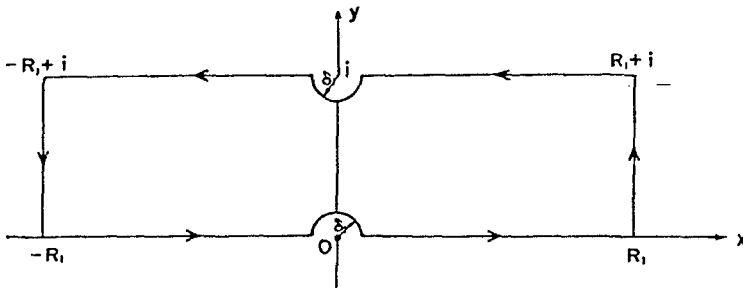


Figure 3 The contour of complex integration $I, R_1 \rightarrow \infty, \delta_1 \rightarrow 0$

From mathematical operation, the integral is evaluated to be

$$I = \frac{\sin a_0}{\cosh \lambda_0 + \cos a_0} \quad -\pi < a_0 < \pi \dots\dots\dots (11-14)$$

As a result, it is proved that

$$\begin{aligned} & \lim_{L \rightarrow \infty} \sum_{n=1}^{\infty} \frac{K_n \sinh(nky)}{\sinh[nk(h + \eta_0 + \epsilon_0)]} \cos(nkx) \\ &= \frac{\pi \beta'}{2(h + \eta_0 + \epsilon_0)} \frac{\sin\left(\frac{\pi y}{h + \eta_0 + \epsilon_0}\right)}{\cosh\left(\frac{\pi x}{h + \eta_0 + \epsilon_0}\right) + \cos\left(\frac{\pi y}{h + \eta_0 + \epsilon_0}\right)} \dots\dots (11-15) \end{aligned}$$

put

$$\frac{\pi \beta'}{2(h + \eta_0 + \epsilon_0)} = \beta \dots\dots\dots (11-16)$$

The stream function and potential of the wave motion in the case of wave length approaching infinitive are to be obtained as follows:

$$\psi(x, y) = -c(y-h) + c\beta \frac{\sin(my)}{\cosh(mx) + \cos(my)} \dots\dots\dots (11-17)$$

$$\phi(x, y) = -cx + c\beta \frac{\sinh(mx)}{\cosh(mx) + \cos(my)} \dots\dots\dots (11-18)$$

Consequently, the wave profile can be expressed to be

$$\eta(x) = \frac{\beta \sin[m(h + \eta)]}{\cosh(mx) + \cos[m(h + \eta)]} \dots\dots\dots (11-19)$$

$$\text{where } m = \frac{\pi}{(h + \eta_0 + \epsilon_0)} \dots\dots\dots (11-20)$$

These equations identify with McCowen's theory. Therefore, the mathematical model submitted by the authors is proved to be able to represent the waves of symmetrical profile progressing in area of any water depth with any wavelength. So that this model can be nominated to be "universal".

III. EXACT SOLUTION OF FLOW FIELD IN SOLITARY WAVE

From the equations (11.17), (11.18) and (11.19), the partial velocity q can be calculated as follows:

$$q^2 = c^2 \left\{ 1 + \frac{m^2 \beta^2 - 2m\beta(1 + \cosh mx \cos my)}{(\cosh mx + \cos my)^2} \right\} \dots\dots\dots (111.1)$$

Expand and rearrange this equation, then substitute (11.17) to it, following expression is to be obtained.

$$q^2 = c^2 + m^2 [\psi(x, y) + c(y-h)]^2 \cdot \left[\frac{1}{\sin^2(my)} - \frac{2}{m\beta} \right] - 2mc [\psi(x, y) + c(y-h)] \cot(my) \dots\dots\dots (111.2)$$

The partial velocity on the surface can be evaluated by substituting $\psi(x, h+\eta) = 0$ to (111.2) and then have

$$(q^2)_{y=h+\eta} = c^2 \left\{ 1 - 2m\eta \cot[m(h+\eta)] + m^2 \eta^2 \cdot \left[\frac{1}{\sin^2[m(h+\eta)]} - \frac{2}{m\beta} \right] \right\} \dots\dots\dots (111.3)$$

The water elevation in very distant location will have no disturbance, so

$$x \longrightarrow \pm \infty, \quad \eta = 0, \quad (q^2)_{x \rightarrow \pm \infty} = c^2 \dots\dots\dots (111.4)$$

Also from the dynamical boundary condition on the surface, the Bernoulli's constant Q can be evaluated as

$$Q = g(h+\eta) + \frac{1}{2} \left[\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right]_{y=h+\eta} = \frac{1}{2} c^2 + gh \dots\dots (111.5)$$

so that

$$\left[\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right]_{y=h+\eta} = (q^2)_{y=h+\eta} = c^2 - 2g\eta \dots\dots\dots (111.6)$$

From equations (111.3) and (111.6), another equation of wave profile can be worked out as follows

$$\eta(x) = \frac{2m \cot[m(h+\eta)] - 2g/c^2}{m^2 \left\{ \frac{1}{\sin^2[m(h+\eta)]} - \frac{2}{m\beta} \right\}} \dots\dots\dots (111.7)$$

Use the condition of $X \rightarrow \pm \infty, \eta = 0$ again, c is to be calculated as follows

$$c^2 = \frac{g}{m} \tan(mh) \dots\dots\dots (111.8)$$

This is just the same as McCowan's theory (1891).

At $x=0$, $\eta = \eta_0 = H$, H is the wave height of solitary wave, from (III.19)

$$H = \beta \tan \left[\frac{1}{2} m (h+H) \right] \dots\dots\dots (III.9)$$

and

$$m\beta = mH \cot \left[\frac{1}{2} m (h+H) \right] \dots\dots\dots (III.10)$$

Substitute (III.8) and (III.10) to (III.7) in the case of $\eta = H$ at $x=0$. Then we obtain

$$H = \frac{2m \cot [m(h+H)] - 2m \cot (mh)}{m^2 \left\{ \frac{1}{\sin^2 [m(h+H)]} - \frac{2 \tan \left[\frac{1}{2} m (h+H) \right]}{mH} \right\}} \dots\dots\dots (III.11)$$

Rearrange this equation

$$\begin{aligned} & mH \{ 1 + \cos [m(h+H)] \} \\ &= 2 \{ \cot [m(h+H)] - \cot (mh) \} \cdot \sin^2 [m(h+H)] \cdot [1 + \cos m(h+H)] \\ & \quad + 2 \sin^3 [m(h+H)] \dots\dots\dots (III.12) \end{aligned}$$

It is soon seen that, when the wave height H and the depth h of water are given, m can be worked out by solving this equation, then c , β are also to be calculated through (III.8) and (III.9). The profile of solitary wave is to be delineated.

In this section, not any approximate calculation is used, so it can be called the exact solution.

IV. ON THE HEIGHT OF CRITICAL WAVE

When the crest particle velocity is equal to the wave celerity, the wave begins to break. The critical wave height is the height of wave just before breaking. In a coordinate system of steady motion the critical condition is that the particle velocity at the wave crest is zero. From (III.6)

$$c^2 = 2g\eta_0 = 2gH \dots\dots\dots (IV.1)$$

From (III.8)

$$c^2 = 2gH = \frac{g}{m} \tan (mh), \quad mH = \frac{1}{2} \tan (mh) \dots\dots\dots (IV.2)$$

Substitute this equation to (III.12) and rearrange, following equation is obtained

$$\begin{aligned}
& \frac{1}{2} \tan(mh) \left\{ 1 + \cos \left[mh + \frac{1}{2} \tan(mh) \right] \right\} \\
& = 2 \left\{ \cot \left[mh + \frac{1}{2} \tan(mh) \right] - \cot(mh) \right\} \\
& \quad \times \left\{ \sin^2 \left[mh + \frac{1}{2} \tan(mh) \right] \right\} \left\{ 1 + \cos \left[mh + \frac{1}{2} \tan(mh) \right] \right\} \\
& \quad + 2 \left\{ \sin^3 \left[mh + \frac{1}{2} \tan(mh) \right] \right\} \dots \dots \dots (IV. 3)
\end{aligned}$$

Solve this equation

$$mh = 1.070733 \dots \dots \dots (IV. 4)$$

Substitute to equation (IV.2)

$$mH = 0.915106 \dots \dots \dots (IV. 5)$$

Here the ratio of critical wave height to the water depth is worked out as follows

$$\frac{H}{h} = \frac{H}{d} = 0.854654 \dots \dots \dots (IV. 5)$$

This value is very agreement with those results obtained by Yamada (1957), Lenou (1966), Byatt-smith (1970), and Longuet-Higgins (1974). Also not any approximate computation is adopted in this section, this number is reliable and correct.

V. ON THE SHAPE OF CRITICAL WAVE

Substitute (II.19) (III.9) and (III.10) to equation (III.12), after complicated algebraical operation, following relationship in critical wave situation can be worked out.

$$mH = \sin[m(h+H)] \dots \dots \dots (V.1)$$

Now, to find the shape of the crest of critical wave, differentiate the wave profile with respect to x

$$\begin{aligned}
\frac{d\eta}{dx} &= m\beta \left\{ \frac{\cos[m(h+\eta)]}{\cosh(mx) + \cos[m(h+\eta)]} + \frac{\sin^2[m(h+\eta)]}{\{\cosh(mx) + \cos[m(h+\eta)]\}^2} \right\} \\
&+ \frac{d\eta}{dx} - \frac{m\beta \sinh(mx) \sin[m(h+\eta)]}{\{\cosh(mx) + \cos[m(h+\eta)]\}^2} \dots \dots \dots (V.2)
\end{aligned}$$

Substituting equation (II.19) to this equation and simplifying

$$\frac{d\eta}{dx} = \frac{\frac{m\eta^2}{\beta \sin[m(h+\eta)]} \cdot \sinh(mx)}{\{m\eta \cot[m(h+\eta)] + \frac{m}{\beta} \eta^2 - 1\}} \dots \dots \dots (V.3)$$

In critical case, $x=0, \eta=H$, using the relationship of (V.1)

$$\lim_{x \rightarrow 0} \left(\frac{d\eta}{dx} \right)_{\eta=H_{\text{critical}}} = \frac{0}{0} \dots\dots\dots (V.4)$$

The wave crest of critical wave is a singular point, using L'Hospital's rule and through some complicated operations, then, we obtain

$$\lim_{x \rightarrow 0} \left(\frac{d\eta}{dx} \right)^2 = \frac{mH}{m\beta \left[\frac{\cos m(h+H)-1}{\sin m(h+H)} \right] + 2mH} = \frac{mH}{mH}$$

Immediately, we have

$$\lim_{x \rightarrow 0} \left(\frac{d\eta}{dx} \right) = \pm 1 \dots\dots\dots (V.5)$$

This means that the critical wave crest is a branch point, the slopes of left and right side are

$$\left(\frac{d\eta}{dx} \right)_{x=0^-} = -1 \quad \left(\frac{d\eta}{dx} \right)_{x=0^+} = +1 \dots\dots\dots (V.6)$$

Consequently, the crest is a summit where two symmetrical curves intersect, the slope angle of both side is 45°, as a result, the angle of the critical wave crest is 90° in stead of 120° stated by Stokes (1880) and identifys with Rankine's theory (1865).

The shope of the wave is illustrated in Fig. 4

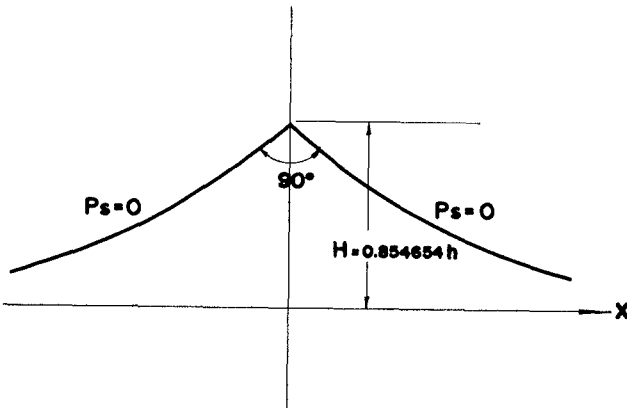


Figure 4 The Structure of highest wave

CONCLUSION

The new model submitted in this paper is proved to be applicable to any wavelength and any water depth so it can be nominated to be "universal model".

The ratio of critical wave height to water depth derived from this model is

$$\frac{H}{d} = 0.8546....$$

and the angle of the critical wave crest is 90° .

No approximate calculation is adopted in this theory, the solution is to be called "exact".

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