CHAPTER SIXTY FOUR

The Physical Basis of The Mild-Slope Wave Equation

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Abstract.

The mild-slope wave equation is derived by demanding minimum in total wave energy.

By demanding conservation of wave energy, two different functionals for the finite element solution of the mild-slope wave equation are constructed. The first functional is based on a finite/infinite element formulation, and the second one is based on a hybrid finite element formulation. Both functionals are constructed in a straight-forward way that leads to a better physical understanding of the functionals and a full understanding of each separate part of them.

1. Introduction.

Waves of arbitrary length propagating in an area of varying depth may be described by the mild-slope wave equation. This equation was first derived by Berkhoff (1972). Later an alternative derivation was given by Smith and Sprinks (1975). A detailed discussion can be found in Jonsson and Brink-Kjær (1973).

Berkhoff (1972) solved the mild-slope wave equation by using the finite element method. Chen and Mei (1974) formulated a hybrid finite element method that, contrary to Berkhoff's formulation, leads to a symmetric stiffness matrix, which is almost a necessity when dealing with larger element grids. Their formulation has later been used in a generalized version by Houston (1981) and Tsay and Liu (1983).

Bettes and Zienkiewicz (1977) solved the mild-slope wave equation by using a simple finite/infinite element method based on a variational formulation different from the one used by Chen and Mei (1974).

In this paper is given a derivation of the mild-slope wave equation based on energy considerations. Then it is demonstrated how — by generalizing the approach used to derive the wave equation — one can construct the functionals used in both the hybrid element and in the finite/infinite element methods when solving the wave equation. General intermediate depth theory will be used, and energy dissipation along an absorbing boundary is included.

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2. Basic Theory.

When the fluid is assumed to be ideal, it is possible to describe the motion of small amplitude surface gravity waves by the complex velocity potential Φ :

$$\Phi(x,y,z,t) = \phi\ e^{-i\omega t}\ \frac{\cosh k(z+h)}{\cosh kh} \tag{1}$$

where x and y are horizontal co-ordinates, z is the vertical co-ordinate, t is the time, h is the local water depth, ω is the angular frequency and k is the wave number. Quantity $\phi=\phi(x,y)$ is a two-dimensional complex potential function.

The complex wave amplitude η is :

$$\eta = \frac{i\omega}{g}\ \phi \tag{2}$$

and the instantaneous complex surface elevation ξ is :

$$\xi = \eta\ e^{-i\omega t} \tag{3}$$

When one deals with a complex potential, a complex wave amplitude and a complex surface elevation, it is only natural also to introduce complex energies.

The complex potential energy of the wave motion per horizontal unit area is defined by :

$$E_p \equiv \int_0^{\xi} \rho g z\ dz \tag{4}$$

and after some calculations one finds :

$$E_p = -\frac{\rho}{g}\ e^{-2i\omega t}\ \tfrac{1}{2}\omega^2\phi^2 \tag{5}$$

The complex kinetic energy per horizontal unit area is defined by :

$$E_k \equiv \int_{-h}^{0} \tfrac{1}{2}\rho(\nabla_3\Phi)^2\ dz \tag{6}$$

∇_3 being the three-dimensional gradient operator. After some calculations this can be written as :

$$E_k = \frac{\rho}{g} e^{-2i\omega t} \tfrac{1}{2}[cc_g(\nabla\phi)^2 + \tfrac{1}{2}(1-G)\omega^2\phi^2] \tag{7}$$

c being the phase velocity, c_g the group velocity and ∇ the horizontal gradient operator.

The total complex wave energy of the wave motion per unit area is the sum of the potential energy (5) and the kinetic energy (7) :

$$E = \frac{\rho}{g} e^{-2i\omega t} \tfrac{1}{2}[cc_g(\nabla\phi)^2 - \frac{c_g\omega^2}{c} \phi^2] \tag{8}$$

The complex energy flux through a section characterized by normal n is per unit length :

$$E_f = \int_{-h}^{0} p^+ \, u \, dz \tag{9}$$

where p^+ is the excess pressure :

$$p^+ = - \rho \frac{\partial\Phi}{\partial t} \tag{10}$$

and u is the horizontal particle velocity in the n-direction :

$$u = \frac{\partial\Phi}{\partial n} \tag{11}$$

Inserting (10) and (11) in (9) and integrating one finds :

$$E_f = \frac{\rho}{g} e^{-2i\omega t} \, i\omega \, cc_g\phi \frac{\partial\phi}{\partial n} \tag{12}$$

It is noted that the energies (5), (7) and (8) and the energy flux (12) are all functions of time t. When dealing with the corresponding quantities in ordinary, real-value wave theory one uses the mean value over one wave period. When one deals with a complex energy and energy flux this has no meaning, since these mean values will all be zero. However, after some simple calculations one easily realizes that the absolut values of the complex energies are equal to twice the value of the corresponding real quantities. The same result is found for the energy flux.

3. The Wave Equation.

An area of calculation, A, is shown in Fig. 1.

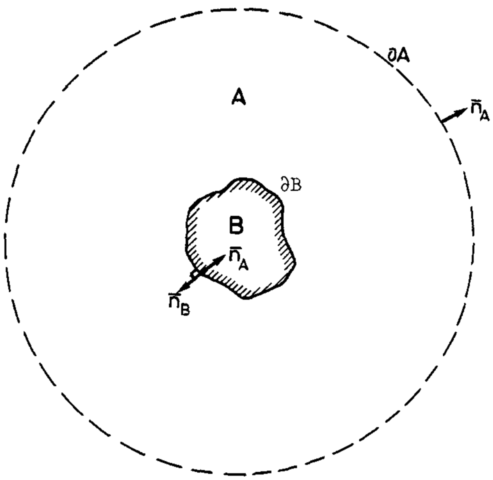

Fig. 1. Area of calculation A in the horizontal plane.

Inside area A some wave scattering structure, area B, is located. The water depth in area A may vary. For simplicity it is assumed in this chapter that the boundary of area B, ∂B, is fully reflecting leading to the boundary condition :

$$\frac{\partial \phi}{\partial n_B} = 0 \quad \text{along } \partial B \tag{13}$$

It is also assumed that the outer boundary of area A, ∂A, is lying at infinity. Far away from the central part of area A, the Sommerfeld radiation condition must be fulfilled :

$$\lim_{r \to +\infty} \sqrt{r}\left(\frac{\partial \cdot}{\partial r} - ik\right)\phi^S = 0 \tag{14}$$

where ϕ^S is the velocity potential of the scattered wave field. ϕ^S may be found from the total potential ϕ by the expression :

$$\phi = \phi^i + \phi^S \tag{15}$$

where ϕ^i is the known velocity potential of the incident wave motion. What (14) says is in fact that the scattered wave system should be outgoing at some distance from the central part of area A.

The total complex wave energy in area A, E_A, can be found from (8):

$$E_A = \frac{\rho}{g} e^{-2i\omega t} \iint_A \tfrac{1}{2}[cc_g(\nabla\phi)^2 - \frac{c_g\omega^2}{c} \phi^2] \, dx \, dy \tag{16}$$

Since the wave motion by definition is stationary E_A must also be stationary. This means that the first variation of E_A must vanish, i.e.

$$\delta E_A = 0 \tag{17}$$

Using Green's theorem one finds from (16) :

$$\delta E_A = \frac{\rho}{g} e^{-2i\omega t} \left\{ -\iint_A \delta\phi \, [\nabla\cdot(cc_g\nabla\phi) + \frac{c_g\omega^2}{c} \phi] \, dx \, dy + \int_{\partial A+\partial B} cc_g \delta\phi \frac{\partial\phi}{\partial n_A} \, ds \right\} \tag{18}$$

The line-integral in (18) along ∂B vanishes by introducing (13). It can be argued that also the integral along ∂A can be ignored. Hereafter by combining (17) and (18) one gets :

$$\iint_A \delta\phi \, [\nabla\cdot(cc_g\nabla\phi) + \frac{c_g\omega^2}{c} \phi] \, dx \, dy = 0 \tag{19}$$

This must hold for any $\delta\phi$. Therefore the term in the square brackets must be zero everywhere in area A :

$$\nabla\cdot(cc_g\nabla\phi) + \frac{c_g\omega^2}{c} \phi = 0 \tag{20}$$

This is the mild-slope wave equation.

So, for the total complex wave energy in the area of calculation to be stationary the mild-slope wave equation is the necessary and sufficient demand.

However, from physical reasoning one can tell that for a known incident wave system there will be no stationary situation except the one that represents the minimum in wave energy. Therefore, the mild-slope wave equation is also a consequence of the total wave energy in the area of calculation being minimized.

4. Two Functionals.

4.1. General Remarks.

For many practical purposes it is convenient not to deal with an area of infinite extension as that shown in Fig.1., but instead to split the horizontal plane in an inner area A, and an outer area R reaching to infinity as shown in Fig.2. Thus, it is possible to treat the inner and the outer area in separate ways as will be done in the following sections.

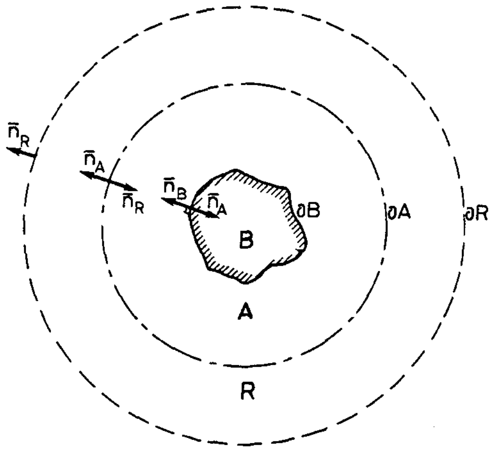

Fig.2. Areas of calculation A and R in the horizontal plane.

Again, inside area A some wave scattering structure B is located. For generality, the boundary of this area, ∂B, may be partially absorbing :

$$\frac{\partial\phi}{\partial n_B} + ik\alpha\phi = 0 \qquad \text{along } \partial B \tag{21}$$

where α is an absorption coefficient. For waves incident at a right angle to the boundary, (21) results in reflected waves with the reflection coefficient $(1-\alpha)/(1+\alpha)$. Thus, $\alpha=1$ corresponds to full absorption of waves incident at a right angle, and $\alpha=0$ corresponds to a fully reflecting boundary.

Over the boundary ∂A which now is not at infinity, the velocity potential has to be smooth which leads to the boundary conditions :

$$\phi_R = \phi_A \qquad \text{along } \partial A \tag{22}$$

$$\frac{\partial\phi_R}{\partial n_A} = \frac{\partial\phi_A}{\partial n_A} \qquad \text{along } A \tag{23}$$

where ϕ_A and ϕ_R are the potentials in area A and R respectively.

In the outer area R the velocity potential can be expressed as :

$$\phi_R = \phi^i + \phi^S \tag{24}$$

where again ϕ^S must satisfy the radiation condition (14) leading to the vanishing of ϕ^S and its derivative at the boundary of area R, ∂R.

4.2. A Simple Element Formulation.

A simple element formulation is here understood as a formulation that only involves element shape functions to describe the solution to the problem. The mild-slope wave equation has been solved by use of such a method by Bettes and Zienkiewicz (1977) where infinite elements were used in the outer area R, and ordinary finite elements were used in the inner area A. Hereby Bettes and Zienkiewicz were able to extend the integration to cover the entire x-y plane.

Using the results from section 2 and 3 will now demonstrate how one in a new and easily understandable way can construct the functional used by Bettes and Zienkiewicz (1977) in their finite/infinite element formulation.

The total complex wave energy in area A is given by (16). In a quite similar way the total energy in area R, E_R, may be found. Energy is lost along the partially absorbing boundary ∂B. This is described by an energy flux through the boundary. Hereby the following equation describing conservation of total complex wave energy arises :

$$\frac{\partial}{\partial t}(E_A+E_R) + \int_{\partial B} E_f^A \, ds + \int_{\partial R} E_f^R \, ds = 0 \tag{25}$$

where E_f^A is positive as a flux out through the boundary ∂B meaning the direction n in (12) should be n_A $(=-n_B)$. E_f^R is positive as a flux out through the distant boundary ∂R, i.e. n should be n_R.

Integrating (25) once with respect to time t yields :

$$E_A + E_R + \int_{\partial B}\left[\int E_f^A \, dt\right] ds + \int_{\partial R}\left[\int E_f^R \, dt\right] ds = \text{Constant} \tag{26}$$

Using (12) and (21) the integral along ∂B in (26) can be calculated :

$$\int_{\partial B}\left[\int E_f^A \, dt\right] ds = -\frac{\rho}{g} e^{-2i\omega t} \int_{\partial B} \tfrac{1}{2} i\alpha\omega c_g \phi^2 \, ds \tag{27}$$

Using (12) and (24) the integral along ∂R in (26) can be written as :

$$\int_{\partial R}\left[\int E_f^R\, dt\right] ds = -\frac{\rho}{g}\, e^{-2i\omega t}\left\{\int_{\partial R} cc_g \tfrac{1}{2}\phi^i \frac{\partial\phi^i}{\partial n_R}\, ds + \int_{\partial R} cc_g \tfrac{1}{2}\phi^S \frac{\partial\phi^S}{\partial n_R}\, ds + \int_{\partial R} cc_g \tfrac{1}{2}\phi^i \frac{\partial\phi^S}{\partial n_R}\, ds + \int_{\partial R} cc_g \tfrac{1}{2}\phi^S \frac{\partial\phi^i}{\partial n_R}\, ds\right\} \tag{28}$$

As in (18) it can be argued that these integrals along ∂R can be ignored.

Introducing (24) in the expression for E_R one gets :

$$E_R = \frac{\rho}{g}\, e^{-2i\omega t}\left\{\iint_R \tfrac{1}{2}\left[cc_g(\nabla\phi^i)^2 - \frac{c_g\omega^2}{c}(\phi^i)^2\right] dx\, dy + \iint_R \tfrac{1}{2}\left[cc_g(\nabla\phi^S)^2 - \frac{c_g\omega^2}{c}(\phi^S)^2\right] dx\, dy + \iint_R \left[cc_g\nabla\phi^i\cdot\nabla\phi^S - \frac{c_g\omega^2}{c}\phi^i\phi^S\right] dx\, dy\right\} \tag{29}$$

The first integral in (29) is stationary, and it can therefore be discarded. The second integral contains the velocity potential ϕ^S of the scattered wave field which is to be modelled by special shape functions in the outer area R. The third integral in (29) may be rewritten as follows :

$$\iint_R \left[cc_g\nabla\phi^i\cdot\nabla\phi^S - \frac{c_g\omega^2}{c}\phi^i\phi^S\right] dx\, dy =$$

$$\int_{\partial R+\partial A} cc_g\phi^S \frac{\partial\phi^i}{\partial n_R}\, ds - \iint_R \phi^S\left[cc_g\nabla^2\phi^i + \frac{c_g\omega^2}{c}\phi^i\right] dx\, dy \tag{30}$$

Again the line integral in (30) along ∂R can be discarded. Furthermore assuming the water depth in the outer area R to be constant, the second integral on the right hand side of (30) vanishes, since ϕ^i naturally must be a solution to the wave equation.

Hereafter the integrated energy equation (26), with (16), (27) and (29) introduced, gives the functional (omitting the unimportant factor $\frac{\rho}{g}\, e^{-2i\omega t}$) :

$$F(\phi) = \iint_A \tfrac{1}{2}[cc_g(\nabla\phi)^2 - \frac{c_g\omega^2}{c}\phi^2]\,dx\,dy$$
$$+ \iint_R \tfrac{1}{2}[cc_g(\nabla\phi^S)^2 - \frac{c_g\omega^2}{c}(\phi^S)^2]\,dx\,dy$$
$$- \int_{\partial A} cc_g\phi^S \frac{\partial\phi^i}{\partial n_A}\,ds$$
$$- \int_{\partial B} \tfrac{1}{2}i\alpha\omega c_g\phi^2 ds \qquad = \text{Constant} \tag{31}$$

By taking the first variation of (31) it is found that :

$$\delta F(\phi) = 0 \tag{32}$$

$F(\phi)$ is exactly the functional used by Bettes and Zienkiewicz (1977). By finite/infinite element discretization (32) can be expressed as a system of linear equations, which easily can be constructed and solved on a computer. When solving (32) one finds the function ϕ — or, more precisely, a close approximation to the true ϕ-function — which with the preconditions introduced fulfils the energy conservation equation (25). So, from a physical point of view, what really happens in this finite/infinite element formulation is that the conservation of total complex wave energy in the entire plane is ensured.

4.3. A Hybrid Element Formulation.

A hybrid element method is a combination of a conventional element method and some other method. The mild-slope wave equation was initially solved by Berkhoff (1972,1975) using a hybrid element method. A sligtly different hybrid element formulation was introduced by Chen and Mei (1974) solving the shallow water wave equation. Later this formulation in its intermediate depth version has been used by Houston (1981) and by Tsay and Liu (1983). The basic principle of the Chen and Mei method is that while the inner area A (see Fig.2.) is divided into finite elements, a semi-analytical solution to the mild-slope wave equation is used in the outer area R to represent the velocity potential ϕ^S. Hereby integration over the outer area R can be avoided.

Using the same new procedure as in the previous section, a functional for the hybrid element formulation as originally given for shallow water by Chen and Mei (1974) will now be constructed. However, as in the previous section, intermediate depth theory will be used, and an absorbing boundary condition will be included.

Conservation of total complex wave energy is again the starting point. However, since the two areas of calculation are going to be treated in two separate ways, it is necessary to describe the conservation of energy in two separate equations. This leads to the following energy conservation equation for the inner area A :

$$\frac{\partial E_A}{\partial t} + \int_{\partial A+\partial B} E_f^A \, ds = 0 \tag{33}$$

and for the outer area R :

$$\frac{\partial E_R}{\partial t} + \int_{\partial A+\partial R} E_f^R \, ds = 0 \tag{34}$$

E_f^A and E_f^R are energy fluxes positive out of areas A and R, respectively. From (33) and (34) one gets by integration :

$$E_A + \int_{\partial A+\partial B} [\int E_f^A \, dt \,] \, ds = \text{Constant} \tag{35}$$

$$E_R + \int_{\partial A+\partial R} [\int E_f^R \, dt \,] \, ds = \text{Constant} \tag{36}$$

The integral in (35) along the partially absorbing boundary ∂B can be calculated according to (27). The integral in (35) along ∂A can be calculated :

$$\int_{\partial A} [\int E_f^A \, dt] \, ds = -\frac{\rho}{g} e^{-2i\omega t} \int_{\partial A} \tfrac{1}{2} c c_g \phi_A \frac{\partial \phi_A}{\partial n_A} \, ds \tag{37}$$

Using Green's theorem one may write the wave energy in the outer area, E_R, as :

$$E_R = \frac{\rho}{g} e^{-2i\omega t} \left\{ -\iint_R \tfrac{1}{2}\phi_R [\nabla\cdot(c c_g \nabla\phi_R) + \frac{c_g \omega^2}{c} \phi_R] \, dx \, dy \right.$$
$$\left. + \int_{\partial A+\partial R} \tfrac{1}{2} c c_g \phi_R \frac{\partial \phi_R}{\partial n_R} \, ds \right\} \tag{38}$$

Now, introducing the precondition that ϕ_R must fulfil the mild-slope wave equation (20), it is seen from (38) that E_R may be written as :

$$E_R = \frac{\rho}{g} e^{-2i\omega t} \left\{ \int_{\partial A} \tfrac{1}{2} c c_g \phi_R \frac{\partial \phi_R}{\partial n_R} \, ds + \int_{\partial R} \tfrac{1}{2} c c_g \phi^i \frac{\partial \phi^i}{\partial n_R} \, ds \right\} \tag{39}$$

where again integrals along ∂R containing ϕ^S or its derivative have been discarded.

When in addition assuming the water depth in the outer area R to be constant, it can be shown that :

$$\int_{\partial A} cc_g \phi^S \frac{\partial \phi^i}{\partial n_R}\, ds = \int_{\partial A} cc_g \phi^i \frac{\partial \phi^S}{\partial n_R}\, ds \tag{40}$$

The integral along $\partial A + \partial R$ in (36) can be calculated :

$$\int_{\partial A+\partial R} [\int E_f^R \, dt] \, ds = - \frac{\rho}{g} e^{-2i\omega t} \left\{ \int_{\partial A} \tfrac{1}{2} cc_g \phi^S \frac{\partial \phi^S}{\partial n_R}\, ds + \int_{\partial A+\partial R} \tfrac{1}{2} cc_g \phi^i \frac{\partial \phi^i}{\partial n_R}\, ds + \int_{\partial A} cc_g \phi^i \frac{\partial \phi^S}{\partial n_R}\, ds \right\} \tag{41}$$

where (40) has been used and some integrals along ∂R again have been discarded.

By use of (27) and (37) to (41), energy equation (35) may now be written as :

$$\frac{\rho}{g} e^{-2i\omega t} \left\{ \iint_A \tfrac{1}{2} [cc_g (\nabla \phi_A)^2 - \frac{c_g \omega^2}{c} \phi_A^2] \, dx\, dy - \int_{\partial A} \tfrac{1}{2} cc_g \phi_A \frac{\partial \phi_A}{\partial n_A}\, ds - \int_{\partial B} \tfrac{1}{2} i \alpha \omega c_g \phi_A^2 \, ds \right\} = \text{Constant} \tag{42}$$

and energy equation (36) may be written as :

$$\frac{\rho}{g} e^{-2i\omega t} \left\{ \int_{\partial A} \tfrac{1}{2} cc_g \phi_R \frac{\partial \phi_R}{\partial n_R}\, ds - \int_{\partial A} \tfrac{1}{2} cc_g \phi^S \frac{\partial \phi^S}{\partial n_R}\, ds - \int_{\partial A} \tfrac{1}{2} cc_g \phi^i \frac{\partial \phi^i}{\partial n_R}\, ds - \int_{\partial A} cc_g \phi^S \frac{\partial \phi^i}{\partial n_R}\, ds \right\} = \text{Constant} \tag{43}$$

Now it is time to include the matching boundary conditions along ∂A, (22) and (23). This is done by using (22) in the first integral in (43) and (23) in the second integral in (42). The reason for this choice is that when it comes to actual calculations, ϕ_A will be represented by finite element shape functions, and therefore the derivative of ϕ_A will in general only be an approximation to the true one and should therefore be avoided. On the other hand, ϕ_R will be represented by an analytical series solution containing some unknown constants. The derivative of ϕ_R will therefore be known exactly.

It is noted that the third integral in (43) only contains ϕ^i and its derivative. Thus, it is stationary and therefore unimportant.

Now, by adding the two expressions (42) and (43), omitting the unimportant factor $(\rho/g)\exp(-2i\omega t)$ and taking advantage of the unimportance of the stationary integral, one gets the 'hybrid functional' :

$$F_H(\phi) = \iint_A \tfrac{1}{2}[cc_g(\nabla\phi_A)^2 - \frac{c_g\omega^2}{c}\phi_A^2]\,dx\,dy$$
$$+ \int_{\partial A} cc_g[\tfrac{1}{2}\phi^S-(\phi_A-\phi^i)]\,\frac{\partial\phi_R}{\partial n_A}\,ds$$
$$- \int_{\partial A} \tfrac{1}{2}cc_g\phi^S\,\frac{\partial\phi^i}{\partial n_A}\,ds$$
$$- \int_{\partial B} \tfrac{1}{2}i\alpha\omega c_g\phi_A^2\,ds \qquad = \text{Constant} \tag{44}$$

Similarly to (32) one finds by taking the first variation of (44) :

$$\delta F_H(\phi) = 0 \tag{45}$$

F_H is the functional first derived in its shallow water version by Chen and Mei (1974). The present formulation is more general since intermediate depth theory has been used, and an absorbing boundary condition has been included. Again it can be concluded that the basic demand expressed in (45) used in finite element calculations simply ensures conservation of total complex wave energy in the entire horizontal plane.

5. Discussion.

Two different functionals, each corresponding to its own element formulation, have been derived. When constructing the functional in section 4.2, constant water depth was assumed in the outer area. When constructing the functional in section 4.3, it was necessary to assume that ϕ^S fulfil the wave equation in the constant water depth outer area. When making the actual calculations, this is ensured by representing ϕ^S by a semi-analytical solution to the Helmholtz equation. This equation is the constant water depth version of the mild-slope wave equation. For practical use the two formulations have the same assumptions behind the functionals. The variational formulation described in section 4.3 is more complicated than the one in section 4.2, but that argument should not be used to downgrade the hybrid element method since this formulation only has to be gone through once. The authors of the present paper believe that when using efficient programming, i.e. mainly sparse-matrix handling techniques, there will be no significant difference in the cost of running the two models.

Finally an example of the results of diffraction calculations that one can get using the finite element method is shown in Fig.3 and Fig.4. Fig.3 shows the relative wave amplitude in and just outside a harbour. The incomming waves are approaching the coastline at a right angle and all boundaries are fully reflecting. Fig.4 shows the instantaneous surface elevetion at a chosen time where the sum of the incident and the reflected waves has zero elevation everywhere outside the harbour. Therefore, what is seen outside the harbour at this moment is alone the surface elevation of the scattered wave field.

More results including absorbing boundary conditions and superposition of waves can be found in Skovgaard et. al. (1984).

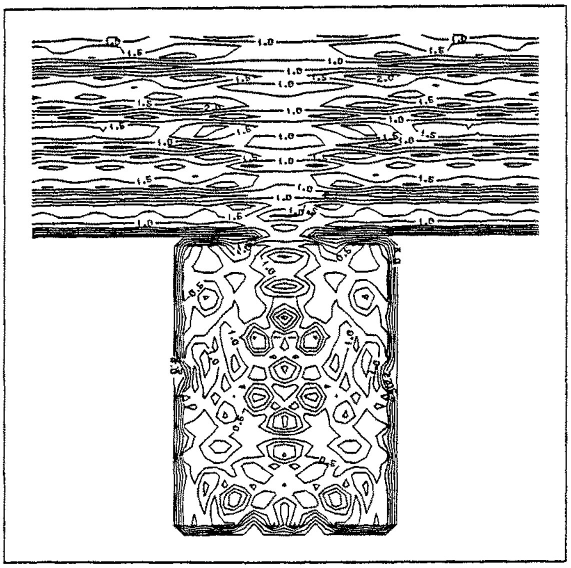

Fig.3. Harbour, relative amplitudes.

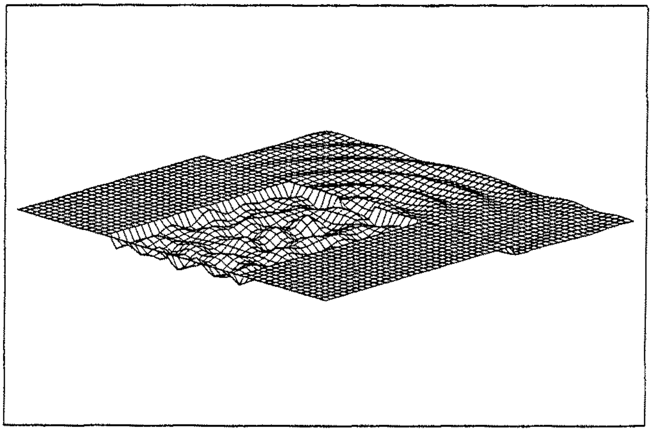

Fig.4. Harbour, relative surface elevation at a chosen time.

6. Conclusion.

The mild-slope wave equation has been derived by demanding minimum of total wave energy in the area of interest. This approach provides a new physical understanding of the mild-slope wave equation.

A functional for a simple element method and a functional for a hybrid element method have been derived from a general energy conservation principle. In both cases the procedure is straight-forward, and there is no need for any trial-and-error methods. Hereby a full understanding of each single part of the functionals is obtained and also a better understanding of the physical basis of the finite element method for the calculation of diffraction of small water waves.

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