CHAPTER THIRTY FOUR

DESCRIPTION OF NATURAL SEA STATES

REQUIREMENTS TO THE REPRODUCTION IN MODELS

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ABSTRACT

In order to attain a satisfactory reproduction in physical and numerical models of Nature's sea states, much development of the description of these sea states has to take place. This paper gives a *discussion* (i) of the problems that can be considered as solved today, and (ii) of the problem areas where much research is still needed.

Firstly, the paper presents the basic definition of the *directional spread* of energy (for a given frequency) as derived from a field record. This definition is necessarily a statistical one and leads directly to the determination in practice of the directional spectrum.

Secondly, the parameterization of spectra is discussed and it is proposed to express one-dimensional storm spectra by means of the *four parameter F.I.-spectrum*, which is more manageable than the JONSWAP-spectrum.

Thirdly, it is concluded that the first order components of any spectrum have *random phases*.

1. INTRODUCTION

In the modelling of a coastal engineering problem it is, of course, desirable to work with the best reproduction of the natural sea states. The degree of sophistication, however, depends on the size of the project and the type of model used.

For economic reasons, the study in *physical models of smaller projects* must be confined to the *1-D f-spectrum*, i.e. the one-dimensional frequency spectrum giving wide-crested waves, i.e. waves with (infinitely) wide crests.

The input wave data will often be given as, say, the

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100 year storm, where H_s and f_p (peak frequency) have been found by *extrapolation* of wave statistics, based on (i) actual wave records, or (ii) meteorologically based wave hindcast models, or (iii) a combination of both, for example, involving the forecast of extrapolated 'historical' storms.

It is quite usual to prescribe a *spectrum* corresponding to the average parameters determined during the JONSWAP-project. Because of the special characteristics of this project (offshore wind), this method is not in general completely satisfactory. It would seem better to parameterize the locally recorded spectra and to extrapolate the parameters to 100 years.

In order to obtain reliable results from models of *larger projects*, it is necessary to reproduce the *2-D (f,θ)-spectrum*, i.e. the directional spectrum giving narrow-crested waves. For physical models it is more expensive to work with the directional spectrum because of the large number of wave generators required and the equipment to control them. On the other hand, in a numerical model, when first established, the extra work involved in the use of the 2-D spectrum is often not essential.

In models of *disturbance inside harbours* the directional spread of wave energy can influence the diffraction considerably, cf. Sand et al., 1983 (Ref. 15). For *rubble mound breakwaters* preliminary tests have shown that unidirectional waves may give results on the conservative side in shallow water (see the reference in Lundgren, 1983 (Ref. 8)). On the other hand, recent failures of major breakwaters indicate that the directional spread may give catastrophic 3-D effects, perhaps as a result of the occurrence of freak waves. For *vertical and composite breakwaters*, which may be exposed also to shock forces, one could imagine that the directional spread of energy would give a higher local force but a smaller total force on a long caisson.

The *determination of directional spectra* is discussed in Sec. 2 in terms of what can be characterized as a complete analysis of the 3-D structure of waves. Unfortunately, rather many papers have dealt with computer simulations of directional spectra, and very few have given actual directional distributions from recorded data. Therefore, it would be premature to propose an empirical formula covering the essential part of the energy in directional spectra for storm waves. On the other hand, such a *parameterized formula* is definitely a prerequisite for the extrapolation to rare events. In addition, there is a great need for studies of the *spectrum development because of shoaling* and *currents*.

For *1-D spectra* the situation is considerably simpler. In this area several parametric formulae have been proposed. The JONSWAP-formula is the most frequently used. It has five parameters and exhibits a discontinuity in curvature at the peak. In Sec. 3 a four-parameter spectrum (without discontinuities) is proposed for use in typical storm situations. It is called the *F.I.-spectrum* because it was conceived in connection with the analysis of thousands of spectra recorded around the *F*aroe *I*slands in the North Atlantic. A complete *procedure* for the determination of the F.I.-parameters is described in the Appendix.

In recent years many papers have been published about the *grouping of waves*. This is discussed in Sec. 4, and it is contended that all information about the grouping of first order components is inherent in the (directional) spectrum. It is therefore proposed that the international discussion of grouping be discontinued. Related to the grouping is the occurrence of *long waves*, a second order effect that is decisive for harbour resonance and for drift forces. In Sec. 4 the imminent necessity of recording long wave directional spectra is pointed out.

In some models it is sufficient to reproduce the first order components of the (directional) spectrum applying random phases. In other models, however, the higher order effects involved in the *wave shapes* must be included, as briefly discussed in Sec. 5.

2. Determination of Directional Spectra

A survey of the methods hitherto used for the determination of directional spectra is given by Lundgren, 1984 (Ref. 9), as Opening Address at the Symposium on Description and Modelling of Directional Seas, organized June, 1984 by Danish Hydraulic Institute & Danish Maritime Institute.

If a fixed support of instruments cannot be provided, a pitch-roll buoy can be used. It records the water surface elevation η together with two slopes, η_x and η_y, of the water surface. If a fixed support, for example, an offshore platform is available, η can be recorded by a wave radar from above or an echo sounder from below, at the same time as two horizontal velocities, u and v, are recorded by a current meter placed below the deepest trough.

In principle, there is little difference between analysing the set (η, η_x, η_y) and the set (η, u, v). The latter will be used as basis for the discussion below.

There are two different methods of analysis:

A. The correlation method, which requires knowledge of all 3 time series, $\eta(t)$, $u(t)$ and $v(t)$, in one vertical. This method can deliver approximate information about only 4 parameters of the directional distribution of energy at each frequency.

B. The complete FFT-method, which - for practical purposes - requires knowledge of only 2 time series, u(t) and v(t). This method will deliver the most complete information about the directional distribution that can be obtained from the data recorded.

The *correlation method* was first developed by Longuet-Higgins et al., 1963 (Ref. 7). It is based on the 6 auto- and cross-spectra, $S_{\eta\eta}$, S_{uu}, S_{vv}, $S_{\eta u}$, $S_{\eta v}$, S_{uv}. From these can be derived the first 5 coefficients, a_0, a_1, b_1, a_2, b_2, in the fourier series development with respect to the direction θ. A special correction had to be introduced in order to eliminate the negative side lobes, which would correspond to propagation of negative energy in directions away from the mean direction.

In recent years the correlation method has been further elaborated in a number of papers, cf. for example, Kuik and van Vledder, 1984 (Ref. 5), who give approximate formulae for 4 directional parameters, viz. the circular mean direction, standard deviation, skewness and curtosis. The basis is still the 6 spectra. The same applies to a recent paper by Isobe et al., 1984 (Ref. 4), who describe an extension of the maximum likelihood method to an arbitrary combination of instruments as wave gauges, current meters etc.

The principle of the *complete FFT-method* was first proposed by the senior author in February 1978 and worked out in details by Dr. S. E. Sand, cf. Sand and Lundgren, 1979 (Ref. 14) and, particularly, Sand, 1979 (Ref. 13). It can be described as follows: The FFT-analysis of the three time series delivers - at a particular frequency $\omega = 2\pi f$ - the fourier coefficients A_η, B_η, A_u, B_u, A_v, B_v, where the three A pertain to cos ωt and the three B to sin ωt. (A_u, A_v) constitute the velocity vector of a wavelet, for which the energy corresponds to $|A_u, A_v|^2 = A_u^2 + A_v^2$. If A_η is positive, the wavelet has a crest for $t = 0$, and the vector indicates the direction of energy propagation. If A_η is negative, the wavelet has a trough for $t = 0$, and the vector has to be rotated 180° in order to give the direction of energy. In this simplified description the numerical value of A_η is an (unused) redundant. It i possible to utilize it in a least square method (Ref. 13).

In 1984 the FFT-method has been further elaborated in connection with the analysis of data from the North Sea where the wave radar (recording η) for practical reasons could not be placed vertically above the current meter. For simplicity, only the latter was used in the analysis. For a strong storm without essential swell the major part of the energy propagates within $\pm 90^\circ$ from the mean direction. Therefore it is easy to distinguish between crests and troughs.

The explanation below will be illustrated with a one hour record from 1983-10-16 with practically constant wave weather with $H_{m0} = 5.8$ m. With a scanning per 0.5 s there is 7200 data available. Traditionally, this can be divided into 14 subseries, each of 512 data. Each subseries gives a frequency resolution of $\Delta f = 1/256$ Hz. For each frequency there are four fourier coefficients, A_u, B_u, A_v, B_v. The total set of fourier coefficients for all frequencies gives a *complete, deterministic description* of the variation of the velocities (u,v).

Hence, for each subseries there is at each frequency *only* two energy vectors

$$|A_u,A_v| \cdot \{A_u,A_v\} \quad \text{and} \quad |B_u,B_v| \cdot \{B_u,B_v\} \tag{2-1}$$

Seen from a deterministic point of view, these vectors represent the energy spread for the subseries considered. Thus, for 14 subseries there are 28 vectors. This ensemble will already give some idea of the statistical spread of energy.

Much more directional information can be obtained, however, by the method of *overlapping subseries*. Overlapping was introduced by Welch, 1967 (Ref. 17) for reduction of the variance by the determination of spectra. Assuming a $\cos^2$-data taper for the subseries and a large number of subseries, giving a variance of one of the spectral value at a certain frequency, the use of overlapping will, according to Welch's formulae, reduce the variance as indicated in Table 1.

Table 1: Reduced variance by overlapping

Overlap of each subseries with the previous one	50%	66.7%	75%	80%	83.3%
5% taper each end	0.730	0.683	0.667	0.660	0.656
10% - - -	0.709	0.660	0.645	0.639	0.636

By 50% overlap the computer work is doubled and the variance reduced to about 70%. For the determination of 1-D spectra it does not pay to go any further.

For 2-D spectra, however, the situation is completely different. Here, *overlapping time series will increase the directional information decisively*. Whereas the one hour record mentioned above gave 28 vectors at each frequency for 14 non-overlapping subseries, an 85% overlap (i.e. a shift of $0.15 \cdot 512 = 77$ data for each subseries) will give 87 subseries, or 174 energy vectors.

This is illustrated by Fig. 1, showing the directional distribution at the peak frequency 102 mHz. The trough vectors have been rotated 180°, and each vector is referred to one of 180 boxes, each 1° wide. For smoothing, a triangular window with a base width of only 5° has been applied.

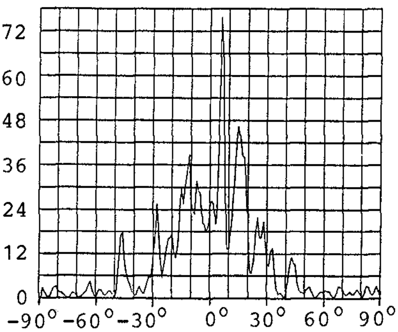

Fig. 1 Directional distribution for 85% overlapping

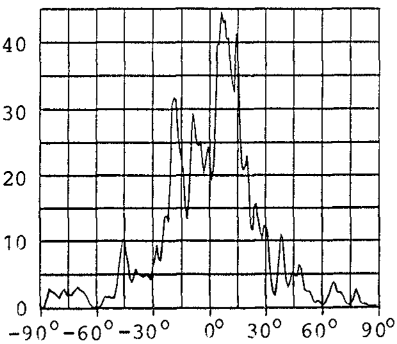

Fig. 2 Directional distribution for 95% overlapping

For comparison, the directional distribution for 95% overlapping is shown in Fig. 2. It is seen to be much smoother. In both cases a 10% $\cos^2$-taper has been applied at the ends of a subseries.

It is a fundamental fact that the 1-D spectrum of a field record can be defined only statistically, by a sufficiently long record or by a spectral window. Analogously, *the overlapping method leads to the very statistical definition of the directional distribution of energy in a field record.*

If the length of the velocity vector (A_u, A_v) is squared, the vector $(A_u, A_v) \cdot (A_u^2 + A_v^2)^{1/2}$ is obtained. It is proportional to the energy. Thus, each of these vectors defines an "energy point" in the plane where $\theta = 0$ is the u-axis and $\theta = 90°$ the v-axis. Half the points correspond to crests of wavelets, the other half to troughs, which – on the whole – are in the opposite directions.

The *first principal axis*, with direction θ_0, is the *mean energy direction.* It is found from the equation

$$\tan 2\theta_0 = \frac{\sum A_u A_v (A_u^2 + A_v^2) + \sum B_u B_v (B_u^2 + B_v^2)}{\sum (A_u^2 - A_v^2)(A_u^2 + A_v^2) + \sum (B_u^2 - B_v^2)(B_u^2 + B_v^2)} \quad (2\text{-}2)$$

Before Figs. 1 – 2 were plotted, the coordinate system was rotated the angle θ_0, so that the mean energy is in direction 0°. In addition, all energy vectors with directions between 90° and 270° were assumed to correspond to troughs and were rotated 180°. This method implies that the small amount of energy that propagates in directions beyond + 90° or – 90° is mirrored in these directions. At frequencies far from the peak frequency, or if essential swell is involved, it is necessary to include information from the fourier series of $\eta(t)$ in order to determine the full distribution over 360°.

It should be noted that with the method developed in 1978 a good resolution of the spectrum with respect to frequencies was accompanied by a poor resolution with respect to directions and vice versa. With the overlapping method a *good resolution* can be obtained simultaneously in frequencies and directions.

With this method it should also be possible to study the spectral properties that give rise to *freak waves*.

Excellent *deterministic reproduction* of the 3-D structure of waves may be obtained by means of 60 wave generators, as shown by Aage and Sand, 1984 (Ref. 1).

3. F.I.-SPECTRUM

Four waveriders placed around the *F*aroe *I*slands have delivered a comprehensive material, for the analysis of which the socalled F.I.-spectrum has been chosen

$$S(f) = A_p \; (f/f_p)^{-\varphi_1} \exp\left[- B_p \; (f/f_p)^{-\varphi_2}\right] \tag{3-1}$$

This formula is a generalization of the P-M-spectrum, for which $\varphi_1 = 5$ and $\varphi_2 = 4$. The Ochi-spectrum is the sum of two similar terms, both with $\varphi_2 = 4$. At the 19th ICCE in Houston, 1984, Liu (Ref. 6) has, independent of the present authors, reported the use of a formula that differs from (3-1) only in the symbols.

The F.I.-formula can be fitted with good accuracy to more than 95% of all spectra of larger waves. It has only four parameters, because the powers of f_p may be included in A_p and B_p.

A procedure for the determination of A_p, B_p, φ_1, φ_2 is described in the Appendix. It is based on the moments of (3-1), which can be expressed by means of the Γ-function, see Eq. (A-4). This circumstance has previously been utilized by Ochi and Hubble, 1977 (Ref. 11) and by Houmb, 1984 (Ref. 3). The advantage of using the moments is that they have a much smaller variance than the individual spectral values S(f).

An example of an F.I.-fit is shown in Fig. 3 for a 20 minute record with 9 non-overlapping subseries, each of 256 data. (With a 10% data taper it is recommended to use 55% overlapping, cf. Table 1, Sec. 2.) The abscissa is the frequency number $i = f/\Delta f$, where $\Delta f = 1/128$ Hz. The moment M(n) in (A-2) and its derivatives in (A-12/14) have been taken at $n = -2$, with summations extending from $i_{min} = 5$ to $i_{max} = 30$.

The spectrum of the wave record is shown with dots. For this spectrum, with $H_{m0} = 11$ m, the values $\varphi_1 = 3.5$ and $\varphi_2 = 10.6$ have been found.

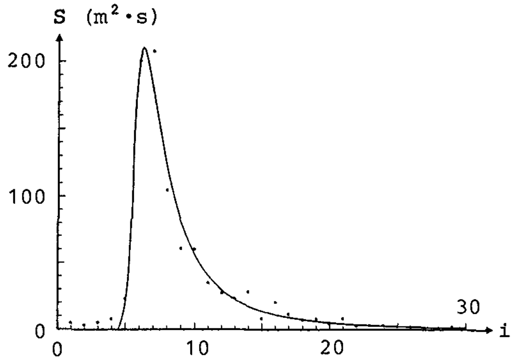

Fig. 3 F.I.-spectrum

4. WAVE GROUPING AND LONG WAVES

It is well known that there is a steadily ongoing *multi-exchange of energy* (or rather wave action) between infinitely many combinations of four wave number vectors. Because of this exchange, *the phases of the infinite number of first order wave components are random.* It is excluded to define a "phase spectrum".

Hence, the *grouping of waves* in a stochastic ensemble of shorter and longer groups is determined uniquely by the spectrum. On the average, a narrow spectrum gives long groups and a wide spectrum short ones. A very long wave record is required in order to obtain a complete statistical picture of the grouping.

For a *steady wave state* the grouping is associated with a pattern of *second order long waves* (SOLW) in such a manner that the long waves have troughs where the short waves are high, and crests where the short waves are small. Formulae for the SOLW for an arbitrary directional sea in arbitrary depth were given by Sharma and Dean, 1979 (Ref. 16). Independently, Ottesen Hansen et al., 1981 (Ref. 12) established the corresponding formulae for a unidirectional sea. The SOLW have frequencies and wave number vectors equal to the differences between frequencies and wave number vectors of pairs of first order wave components.

In practice, however, *all wave states are unsteady.* Because of the movement and the change of the wind field, the change in water depth and current as the waves propagate, wave-wave interaction and energy decay, there is a continuous change of the (directional) spectrum. Since the characteristic frequencies of the long waves are the differences between frequencies of the short ones, the long waves can only gradually adjust themselves to the changing short wave spectrum. If, for instance, the wind field changes direction, the short waves may turn relatively fast, whereas the long waves will turn much more slowly. Hence, it is possible to speak of a *partial separation* of the long waves from the short ones.

The long waves are responsible for harbour resonance and will often be the dominant cause of drift forces, cf. Lundgren et al., 1982 (Ref. 10). Hence, there is an *urgent need for good records of directional spectra of long waves.* Since long waves have a much larger directional spread than the short ones, it is necessary to determine their distribution over the full circumference of 360^o. This cannot be done with a pitch-roll buoy because of its poor response for low frequency heave motions. Therefore, it is necessary to record three time series (η,u,v) with a fixed installation and analyse the data with the overlapping method.

5. Wave Shape Effects

The spectrum represents the superposition of a large number of small sinusoisdal components. In contrast to sine curves, high waves have *elevation skewness*, i.e. the crest height is larger than the trough depth, and they may have *slope skewness*, i.e. the wave front is steeper than its back.

In some model tests, for example with vertical and composite breakwaters, it is necessary to also reproduce the natural wave shapes. The deviations from the "sinusoidal" shapes are *second order short waves* (SOSW), generated by the first order components through the nonlinear surface conditions or the bed slope. The SOSW have frequencies and wave number vectors equal to the sums of frequencies and wave number vectors of pairs of first order wave components, cf. Sharma and Dean, 1979 (Ref. 16).

Because of the high frequencies of the SOSW, they have a good possibility of adjusting themselves to the local first order wave conditions, provided that the bed slope is not too large. By abrupt changes, however, a *partial separation* may take place. This was shown by Biésel, 1963 (Ref. 2) in connection with the diffraction of regular high waves into a harbour model and the refraction across a shoal. In both cases the height of the first order waves changes over a rather short distance and, if the first order wave is reduced to 90%, the corresponding SOSW is reduced to 81%. Therefore, some SOSW energy is released and diffracted/refracted in a different manner.

A particularly strong nonlinear effect is present in *freak waves*, where a very high 3-D crest is surrounded by troughs of normal depth, cf. Fig. 4.

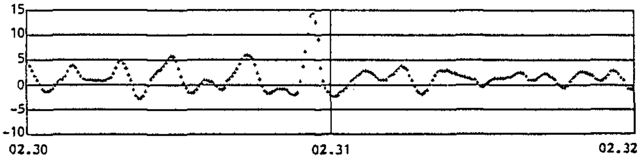

Fig. 4 Record with freak wave from the North Sea, 1981-11-25

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APPENDIX: DETERMINATION OF F.I.-PARAMETERS

From the FFT-analysis the empirical spectrum S(f) is determined at the equidistant values $f = i \cdot \Delta f$, with i = 1, 2, 3, ... For simplicity, the spectrum is considered a function of the *frequency number* i, i.e.

$$S(i) = A\, i^{-\varphi_1} \exp\left[-B\, i^{-\varphi_2}\right] \tag{A-1}$$

Expressed in frequency numbers, the moment of order n is

$$\begin{aligned} M(n) &= \sum_{i=1}^{\infty} i^n\, S(i) \approx \int_0^{\infty} i^n\, S(i)\, di \\ &= A \int_0^{\infty} i^{n-\varphi_1} \exp\left[-B\, i^{-\varphi_2}\right] di \end{aligned} \tag{A-2}$$

After introduction of the parameter

$$z = z(n) = (\varphi_1 - n - 1)/\varphi_2 \tag{A-3}$$

and the substitution $t = B\, i^{-\varphi_2}$, (A-2) may be written

$$M(n) = A\, \varphi_2^{-1}\, B^{-z} \int_0^{\infty} t^{z-1}\, e^{-t}\, dt = A\, \varphi_2^{-1}\, B^{-z}\, \Gamma(z) \tag{A-4}$$

the infinite integral being the definition of the gamma function.

Hence, the problem is, from a knowledge of the moments M(n), to determine the four parameters A, B, φ_1, φ_2. Since φ_1 and φ_2 through the parameter z are 'well hidden' under the Γ-function, this is a seemingly difficult problem. It may be tackled in the following manner: The *logmoment* P(n) is introduced by taking the logaritm of (A-4)

$$P(n) = \ln M(n) = \ln A - \ln \varphi_2 - z \ln B + \ln \Gamma(z) \tag{A-5}$$

In the following, M(n) and P(n) are considered functions of the *continuous variable* n and may be differentiated with respect to n.

Since $z'(n) = -\varphi_2^{-1}$, the first derivative of (A-5) is

$$P' = P'(n) = M'/M = \varphi_2^{-1} \ln B - \varphi_2^{-1}\, \psi(z) \tag{A-6}$$

where

$$\psi(z) = \Gamma'(z)/\Gamma(z) \tag{A-7}$$

is the digamma function, also called the psi function.

The derivative of (A-6) is

$$P'' = M''/M - (M'/M)^2 = \varphi_2^2\, \psi'(z) \tag{A-8}$$

where $\psi'(z)$ is the trigamma function. Finally, the derivative of (A-8) is

$$\begin{aligned} P''' &= M'''/M - 3\,(M''/M)\,(M'/M) + 2\,(M'/M)^3 \\ &= -\varphi_2^3\, \psi''(z) \end{aligned} \tag{A-9}$$

where $\psi''(z)$ is the tetragamma function.

In (A-8) and (A-9) both φ_2 and z are unknown. Therefore, φ_2 is eliminated by introducing the *ratio function*

$$R(z) = (P''')^2/(P'')^3 = (\psi'')^2/(\psi')^3 \qquad \text{(A-10)}$$

where P is differentiated with respect to n and ψ with respect to z. In principle, R(z) may be tabulated from published tables of the ψ'' and ψ' functions.

Assuming that P''' and P'' have been calculated from the given spectrum for a chosen value of n, the *procedure* is now: Calculate the value of R from the first expression in (A-10), and find the corresponding value of z as the *inverse function*

$$z = z(R) \qquad \text{(A-11)}$$

from the table mentioned. In practice, no table is needed because an approximate formula for z(R) is given below.

According to (A-6,8,9), the derivatives P', P'', P''' of the logmoment P can be found from the moment M and its derivatives M', M'', M'''. For these derivatives the following formulae are easily derived from (A-2)

$$M'(n) = \sum_{i=1}^{\infty} \ln i \cdot i^n\, S(i) \qquad \text{(A-12)}$$

$$M''(n) = \sum_{i=1}^{\infty} \ln^2 i \cdot i^n\, S(i) \qquad \text{(A-13)}$$

$$M'''(n) = \sum_{i=1}^{\infty} \ln^3 i \cdot i^n\, S(i) \qquad \text{(A-14)}$$

Based on analyses of a number of Faroe Island spectra, the present authors recommend the following *procedure*:

(a) In the sums (A-2,12,13,14) choose $n = -2$. ($n = -1$ has been found not to reproduce a sharp peak in the spectrum sufficiently well. $n = -3$ has been found to give practically the same spectral shape as $n = -2$.) In order to compare the F.I.-parameters for different storms, it is advantageous to use the same n for all spectra.

(b) The summations in (A-2,12,13,14) must *not* start from $i = 1$. The reason is that the left hand side of the F.I. spectral shape is very steep down to extremely small values. Therefore, if the empirical spectrum has points above this steep slope - for example, because of the presence of some swell - such points would influence the F.I.-parameters in a *most disturbing* manner if they were included in the summations. In most cases the following rules may be applied:
(i) To the left of the maximum, S_{max}, of the given spectral values all points are included for which $S(i) > 0.25\ S_{max}$.
(ii) Below $0.25\ S_{max}$ at least one more point is included but

otherwise the summations stop at the value i_{min} for which

$$S\left(i_{min}\right) - S\left(i_{min}-1\right) < 0.8\left[S\left(i_{min}+1\right) - S\left(i_{min}\right)\right] \quad \text{(A-15)}$$

this inequality indicating that the steep slope does not continue further to the left.

(c) The summations (A-2,12,13,14) need only extend to a value, i_{max}, for which the spectral values are so small that they do not influence the sums essentially. In this respect the value $n = -2$ helps in reducing the influence of large i-values.

(d) M, M', M" and M''' are calculated.

(e) P', P" and P''' are calculated.

(f) R is calculated from (A-10).

(g) q is calculated from

$$q = \left(1 - \frac{R}{4}\right)^{\frac{1}{2}} \quad \text{(A-16)}$$

(h) z is calculated from the approximation

$$R\,z = 1.8024\,q + 0.7906\,q^2 - 1.5920\,q^{4.640} - 2.252\,q^{3.379}\,(1-q)^{1.589} \quad \text{(A-17)}$$

The error on z according to this formula is numerically less than 0.17% for z < 1.5 and less than 0.29% for all z.

(i) Calculate $\Gamma(z)$ from the formula

$$1/\Gamma(z) = \sum_{k=1}^{\infty} c_k\,z^k \quad \text{(A-18)}$$

where

k	c_k	k	c_k
1	1.00000 00000 000000	11	0.00012 80502 823882
2	0.57721 56649 015329	12	—0.00002 01348 547807
3	—0.65587 80715 202538	13	—0.00000 12504 934821
4	—0.04200 26350 340952	14	0.00000 11330 272320
5	0.16653 86113 822915	15	—0.00000 02056 338417
6	—0.04219 77345 555443	16	0.00000 00061 160950
7	—0.00962 19715 278770	17	0.00000 00050 020075
8	0.00721 89432 466630	18	—0.00000 00011 812746
9	—0.00116 51675 918591	19	0.00000 00001 043427
10	—0.00021 52416 741149	20	0.00000 00000 077823

(j) Calculate $\psi(z)$ from the formula

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{m=1}^{10} \frac{z}{m\,(z+m)} + z\,\Sigma_2 - z^2\,\Sigma_3 + z^3\,\Sigma_4 - \ldots \quad \text{(A-19)}$$

where

$$\gamma = 0.57721\ 56649 \quad \text{and} \quad \Sigma_k = \sum_{m=11}^{\infty} m^{-k} \quad \text{(A-20)}$$

[The formula for $\psi(z)$ has been derived by differentiation of the logaritm of Euler's infinite product for $1/\Gamma(z)$. Since $\psi(z)$ has simple poles at the points $0, -1, -2, \ldots,$ the convergence of the series (A-19) has been improved by separating the first 11 poles with their respective residues.] The numerical values of Σ_k are *)

$\Sigma_2 = 9.516\ 633\ 568 \cdot 10^{-2}$; $\Sigma_3 = 4.524\ 917\ 486 \cdot 10^{-3}$;

$\Sigma_4 = 2.866\ 502\ 175 \cdot 10^{-4}$; $\Sigma_5 = 2.041\ 379\ 870 \cdot 10^{-5}$;

$\Sigma_6 = 1.549\ 542\ 998 \cdot 10^{-6}$; $\Sigma_7 = 1.224\ 317\ 311 \cdot 10^{-7}$;

$\Sigma_8 = 9.942\ 690\ 896 \cdot 10^{-9}$; $\Sigma_9 = 8.236\ 974\ 0 \cdot 10^{-10}$;

$\Sigma_{10} = 6.927\ 213 \cdot 10^{-11}$; $\Sigma_{11} = 5.89777 \cdot 10^{-12}$;

$\Sigma_{12} = 5.0683 \cdot 10^{-13}$; $\Sigma_{13} = 4.388 \cdot 10^{-14}$;

$\Sigma_{14} = 3.80 \cdot 10^{-15}$; $\Sigma_{15} = 3.3 \cdot 10^{-16}$.

(k) Calculate $\psi'(z)$ from the formula

$$\psi'(z) = \sum_{m=0}^{10} \frac{1}{(z+m)^2} + \Sigma_2 - 2z\,\Sigma_3 + 3z^2\,\Sigma_4 - 4z^3\,\Sigma_5 + \ldots \tag{A-21}$$

(l) Find φ_2 from (A-8) as

$$\varphi_2 = \left[\psi'(z)/P''(n)\right]^{\frac{1}{2}} \tag{A-22}$$

(m) Find φ_1 from (A-3) as

$$\varphi_1 = z\,\varphi_2 + n + 1 \tag{A-23}$$

(n) Find B from (A-6) as

$$\ln B = P'(n)\,\varphi_2 + \psi(z) \tag{A-24}$$

(o) Find A from (A-4) as

$$A = M(n)\,\varphi_2\,B^z/\Gamma(z) \tag{A-25}$$

(p) Plot the graph of S(i) according to the F.I.-formula (A-1) and compare with the given spectral values. In special cases it may be necessary to choose another value of i_{min} than given by the rules (i) and (ii) under item (b) above. In some cases another value than $n = -2$ may have to be chosen; see (a) above.

(q) Calculate the peak frequency from

$$i_p = \left[B\,\varphi_2/\varphi_1\right]^{1/\varphi_2} \tag{A-26}$$

and the peak spectral value from

$$S_p = A\,i_p^{-\varphi_1}\,\exp\left(-\varphi_1/\varphi_2\right) \tag{A-27}$$

*) The last few decimals in these values are not reliable.