CHAPTER THIRTY

A Dynamical Expression of Waves in Shallow Water

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Making the assumptions that solitons are one of the most elementary excitation in random nonlinear waves in shallow water and that the waves have a coherent dynamic structure of solitons, we attempt to describe the swell-like waves theoretically by deriving the asymptotic multi-soliton solution for the KdV equation. Formulations of the random wave profiles and internal properties are also made. We conclude from the comparisons between observed and theoretical results of the propagation characteristics of the swell-like random waves and their water particle velocities, that the waves in shallow water have a coherent dynamic structure of solitons and that the theoretical expression for the waves has practically sufficient accuracy in estimating their propagation.

1. INTRODUCTION

One of the outstanding features of swell-like waves propagating on shallow water is the coupling between their nonlinearity and randomness. How to express them theoretically is clearly of basic importance for coastal engineering because of the need to accurately calculate forces on coastal structures and to rationally estimate design waves and so on. Various attempts have been made to formulate them theoretically, for example, by using the nonlinear spectral analysis based on a perturbative approach from a linear mode, which was developed by Tick(1959), Phillips (1961) and Hasselmann(1962). However, the application of the nonlinear spectral analysis requires much labor because the nonlinear effects of the waves are remarkably stronger than those of ocean waves. As the result, the design of coastal structures is generally carried out by using the individual wave method which is a mere statistic approach and has no basis of the dynamics.

The coupling between nonlinearity and randomness of the waves makes their dynamics seriously difficult, because it provides active exchanges of energy among different modes and brings the above perturbative approach to the chaotic situation. From the new viewpoint of the dynamics of nonlinear waves, therefore, another approach may be needed in establishing the dynamics of the waves, instead of the usual perturbative approach mentioned above.

In this study, we attempt to propose a dynamical expression of the swell-like waves based on soliton modes, including simultaneously both effects of their nonlinearity and randomness. Further, applicability of the expression to field waves is examined by comparing it with observed data of wave propagation in the field.

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2. WAVE EQUATION OF SWELL-LIKE WAVES IN SHALLOW WATER

Consider swell-like waves in shallow water of uniform depth and define the two-dimensional coordinate system as shown in Fig. 1. By assuming that viscosity is ignorable in the swell-like waves, we derive the equations governing them as

$$\left.\begin{aligned} &\nabla^2 \Phi = 0 \\ &\Phi_t + (\Phi_x^2 + \Phi_z^2)/2 + gz' \big|_{z=h+z'} = 0 \\ &\eta_t + \Phi_x \eta_x - \Phi_z \big|_{z=h+z'} = 0 \\ &\Phi_z \big|_{z=0} = 0 \end{aligned}\right\} \quad (1)$$

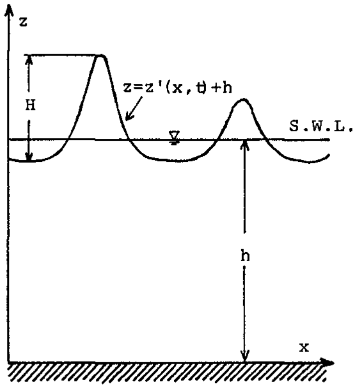


Fig. 1 Coordinate system and symbols used

where ϕ is the velocity potential, z' the water surface displacement from the mean water level, x and z the coordinate system, t the time, h the water depth, and g the acceleration of gravity.

Furthermore, assuming that the effects of nonlinearity and frequency dispersion of waves are of the same order and considering progressive waves of temporal evolution, we introduce the so-called Gardner-Morikawa transform

$$\xi = \varepsilon^{1/2}(x^* - t^*), \quad \tau = \varepsilon^{3/2} t^*, \quad x^* = x/h, \quad t^* = t\sqrt{g/h}, \quad z^* = z/h, \quad \varepsilon = (h/L)^2 \qquad (2)$$

and the perturved solution

$$\left.\begin{aligned} &z'/h \; (= \varepsilon\eta) = \varepsilon\eta_1 + \varepsilon^2\eta_2 + \cdots \\ &\Phi/h\sqrt{gh} = \varepsilon^{1/2}\Phi_1 + \varepsilon^{3/2}\Phi_2 + \cdots \end{aligned}\right\} \qquad (3)$$

where L is the representative wave-length.

By substitution of Eqs. (2) and (3) into Eq.(1), we can rewrite Eq. (1) as

$$\eta_\tau + 3\eta\eta_\xi/2 + \eta_{\xi\xi\xi}/6 = \varepsilon F(\eta, \eta_\xi, \Omega_\tau, \cdots) + O(\varepsilon^2) \qquad (4)$$

$$\Omega_\xi - \eta = \varepsilon(\eta_{\xi\xi}/2 + \eta^2/2) + O(\varepsilon^2) \qquad (5)$$

The relation between the new velocity potential Ω and the original one ϕ is given as

$$\Phi/h\sqrt{gh} = \varepsilon^{1/2}(\Omega - \varepsilon z^{*2}\Omega_{\xi\xi}/2 + \cdots) \qquad (6)$$

Taking ε = 0 in Eq.(4), we obtain the well-known Korteweg de Vries(KdV) equation as the lowest order equation of the swell-like waves as

$$\eta_\tau + 3\eta\eta_\xi/2 + \eta_{\xi\xi\xi}/6 = 0 \qquad (7)$$

Fig. 2 which was calculated by the authors(1982) indicates the comparison of wave transformation in evolution process between the numerical solutions of Eqs.(4) and (7) and the experimental results obtained by giving sinusoidal waves a initial condition of waves. From this figure, it can be considered that the second order correction when compared with that of the first order is relatively small for the wave transformation and that the soliton development can sufficiently be explained by the KdV equation. Hence, from the viewpoint that the swell- like waves should be described as simple as possible, we may use the KdV equation as the governing equation.

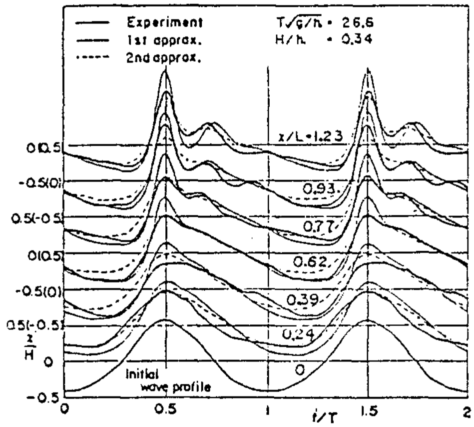


Fig. 2 Coparison between numerical and experimental wave profiles

3. WAVE SOLUTION OF THE KDV EQUATION BASED ON SOLITON MODES

3.1 Elementary Excitation in the Wave Field Governed by the KdV Equation

We can derive the solution of the KdV equation perturbed from a sinusoidal wave solution by using Stokes expansion as

$$\left.\begin{aligned} &\eta = A\cos\theta + (3A^2/4\gamma)\cos 2\theta + (27A^3/64\gamma^2)\cos 3\theta + \cdots, \\ &\gamma = (2\pi h/L)^2, \qquad \theta = 2\pi x/L - \omega t \end{aligned}\right\} \qquad (8)$$

where A is the Ursell number and ω the angular frequency. The above expression describes that a sinusoidal wave becomes an elementary excitation of the wave field to be governed by the KdV equation. It is, however, noticed that the reduction of higher order terms requires much labor and that the number of sinusoidal modes is undetermined. We may conclude that sinusoidal waves are inadequate as elementary excitation in the wave field to be governed by the KdV equation.

The exact stationary solution of the KdV equation is known as cnoidal waves which is given as

$$\eta = A\left[\mathrm{cn}^2\{(\sqrt{3A}/2k)\vartheta\} - (E/K + k^2 - 1)/k^2\right] \qquad (9)$$

where $\vartheta = (x/h - ct/h)$, c is the wave-celerity, K and E the complete elliptic integrals of the first and the second kind, respectively, of which the modulus is denoted by k and cn the Jacobian cn-function. Eq.(9) describes the stationary waves which is simpler than Eq.(8), although it contains an elliptic function.

It is well-known that Eq.(9) expresses a solitary wave(soliton), when the wave-period of cnoidal waves is taken infinite, as

$$\eta = A\,\mathrm{sech}^2\{(\sqrt{3A}/2)\,\vartheta\} \tag{10}$$

This means that the cnoidal waves contain solitons as their limiting and are rather elementary than solitons. However, using the relation between Jacobian dn-elliptic and hyperbolic functions obtained by Toda(1970)

$$\mathrm{dn}^2\chi = (\pi/2K')^2 \sum_{l=-\infty}^{\infty} \mathrm{sech}^2\{(\pi K/K')(\chi/2K - l)\} - k/2KK' + E/K \tag{11}$$

we can transform Eq.(9) into

$$\eta = \sum_{l=-\infty}^{\infty} a\,\mathrm{sech}^2\{(\sqrt{3a}/2)\vartheta - l\pi K/K'\} - (4aK'^2/3\pi^2)\{3(E/K) + 2k^2 - 2\} \tag{12}$$

where

$$\left.\begin{aligned} &\vartheta = \varepsilon^{-1/2}(\xi - c^*\tau) = (x/h - ct/h),\quad c = 1 - (2aK'^2/\pi^2)\{3(E/K) + k^2 - 2\}. \\ &a = A(\pi/2kK')^2,\quad \varepsilon = (h/L)^2 \end{aligned}\right\} \tag{13}$$

Eq.(12) shows that a cnoidal wave train with the amplitude A consists of a periodic sequence of pulse-like waves, that is, solitons with the amplitude a, and that the cnoidal wave train can be expressed by making solitons its elementary excitation. As the modulus k becomes large, the number of solitons expressing a single wave crest of the cnoidal wave decreases. When the value exceeds about 0.98, a is approximated by A so that the influence of the adjacent waves become negligible and the cnoidal waves can be expressed as the regular train of solitons with the same amplitude. This means that periodic wave motion can be expressed by using a particle-like waves such as solitons, and that solitons are superior to cnoidal waves as elementary excitation in the wave field governed by the KdV equation. We, therefore, may describe theoretically nonlinear random waves in shallow water by making solitons elementary excitation.

3.2 Expression of the Swell-Like Waves Based on Soliton Modes

It is possible to express approximately the swell-like waves by the perturbed approach from sinusoidal modes as made by Freilich and Guza (1984). However, the calculation becomes very complicated, so that the approach is not applicable to explain nonlinear random waves completely. Some attempts[Nakamura and Matsuno(1980) and Hirota and Ito(1981)] were carried out of the theoretical expression of nonlinear random waves by making cnoidal waves elementary excitation , but they merely succeed in the case of two-periodic waves in shallow water. Consequently, the approach making solitons elementary excitation may give the sole and possible expression of nonlinear random waves in shallow water. Hence, considering that the swell-like waves in shallow water are the typical nonlinear random waves, we assume that the swell-like waves have a coherent dynamic structure making solitons elementary excitations, that is, the soliton structure, as far as concerned with the dominant portions of them. In order to describe theoretically the swell-like waves under the above assumption and the condition of nondegeneration, we derive the exact multi-soliton solution of the KdV equation satisfying the continuity condition for wave profiles around the mean water level(1984) as

$$\eta = (4/3)\,(\log F)_{\xi\xi} - \eta_0 \tag{14}$$

where

$$F = \lim_{N\to\infty} \det \left| \delta_{ij} + \frac{2B_i}{B_i + B_j} f_i \right|,\quad 1 \leq i,\ j \leq N \tag{15}$$

$$\left.\begin{aligned} & f_i = \exp\{B_i(\xi - c_i\tau - \tilde{\delta}_i)\},\quad B_i = \sqrt{3A_i}, \\ & c_i = A_i/2 - 3\eta_0/2 \end{aligned}\right\} \tag{16}$$

in which both of A_i and $\tilde{\delta}_i$ are integral constants, A_i denotes the amplitude of a soliton expressing its energy level, $\tilde{\delta}_i$ the phase constant determining the position of the soliton with the amplitude A_i, c_i the wave-celerity of the soliton, and δ_{ij} Kronecker's δ. And, η_0 is a statistical quantity defined as an ensemble mean of solitons satisfying the continuity condition and given as

$$\eta_0 = \lim_{\zeta\to\infty} (2/3\zeta)\,(\log F)_\xi \Big|_{-\zeta}^{\zeta} \tag{17}$$

under the assumption that disturbances exist at infinity and the period of observation($\xi: -\zeta \sim \zeta$) is sufficiently long to describe the wave phenomena.

In analyzing the swell-like waves with the soliton structure, we may assume the following relation for spacings between the crests of solitons, which states that interactions of solitons are negligibly small.

$$|\xi_i - \xi_j| > \beta,\quad e^{-\beta} \ll 1,\quad -\infty < \cdots < \xi_i < \xi_j < \cdots < +\infty \tag{18}$$

where ξ_i is the coordinate of the crest position of the soliton with the amplitude A_i at $\tau = \tau_0$. Under the above assumption, Eq.(15) can asymptotically be rewritten as

$$F = \prod_{i=1}^{\infty} \{1 + f_i \exp(-B_i\Delta_i)\} \tag{19}$$

where

$$\Delta_i = -(1/B_i)\log\left\{\prod_{l=1}^{i-1} (B_l - B_i)^2/(B_l + B_i)^2\right\} \tag{20}$$

Substituting Eq.(19) into Eq.(14), we can obtain the asymptotic expression of the swell-like waves based on soliton modes as

$$\eta = \sum_{i=1}^{\infty} A_i \operatorname{sech}^2 \vartheta_i - \eta_0,\quad \vartheta_i = (\sqrt{3A_i}/2)\,(\xi - c_i\tau - \delta_i),\quad \delta_i = \tilde{\delta}_i + \Delta_i \tag{21}$$

This expression asymptotically agrees with the exact solution of the KdV equation even when the value of τ is finite not but $\tau \to \infty$, only if the relation of Eq.(18) is satisfied. It is emphasized that this expression admitts arbitrary degeneration of the amplitudes of solitons and is independent of evolution type. Moreover, we can easily determine the integral constants A_i and δ_i from the observed data of swell-like waves by using the asymptotic solution, although it is very difficult to determine them and transform the exact solution with unknown constants into the particular solution expressing actually the swell-like waves as far as the exact solution is used.

Taking into account these facts in addition to that the usual swell-like waves can be expressed by the asymptotic multi-soliton solution as the authors recently pointed out(1984), any wave motion to be governed by the KdV equation may asymptotically be expressed as a train of solitons with various amplitudes under the assumption of Eq.(18). Macro-scopic properties of the waves then may uniquely be described by using the ensembles of A_i and δ_i.

3.3 A Complete Orthonormal System

If the right-hand side of Eq.(21) is transformed into a linear combination of orthonormal system, $\{\phi_i\}$, by using Schmidt's method, it becomes a complete orthonormal system because Eq.(21) is a solution of the swell-like waves having a soliton structure. Eq.(21) is then rewritten by the linear combination of complete orthonormal systems

$$\eta = \lim_{N\to\infty} \sum_{i=1}^{N} A_i \psi_i(\vartheta_i) - \eta_0 = \lim_{N\to\infty} \sum_{i=1}^{N} A_i \sum_{l=1}^{i} c_{il}\, \phi_l(\vartheta_l) - \eta_0 \tag{22}$$

where

$$\left.\begin{aligned}
&\psi_i = \mathrm{sech}^2 \vartheta_i \\
&\phi_1 = \psi_1 / \|\psi_1\|, \\
&\phi_l = |F| \cdot G(\psi_1, \cdots, \psi_{l-1})^{-1/2} \cdot G(\psi_1, \cdots, \psi_l)^{-1/2}, \quad (l \geqq 2) \\
&[F] = \begin{bmatrix} (\psi_1, \psi_1) & & (\psi_1, \psi_l) \\ \vdots & & \vdots \\ (\psi_{l-1}, \psi_1) & \cdots & (\psi_{l-1}, \psi_l) \\ \psi_1 & \cdots\cdots & \psi_l \end{bmatrix} \\
&G(\psi_1, \cdots, \psi_l) = \begin{vmatrix} (\psi_1, \psi_1) & \cdots & (\psi_1, \psi_l) \\ \vdots & & \vdots \\ (\psi_l, \psi_1) & \cdots & (\psi_l, \psi_l) \end{vmatrix} \\
&c_{il} = D_{li}\,(d_{11} \cdot d_{22} \cdots d_{NN})
\end{aligned}\right\} \tag{23}$$

in which (ψ_l, ϕ_j) is the dot product defined by

$$(\psi_l, \phi_j) = \int_0^\infty \psi_l\, \phi_j\, d\xi \tag{24}$$

and D_{li} a cofactor of d_{li} element of the matrix [D] defined by

$$[D]=\begin{bmatrix} d_{11} & & 0 \\ \vdots & \ddots & \\ d_{N1} & \cdots & d_{NN} \end{bmatrix} \tag{25}$$

where

$$\left.\begin{aligned} d_{li} &= \sum_{j=i}^{l-1} d_{ji}(\varphi_l, \varphi_j) \Big/ \Big\| \varphi_l - \sum_{j=1}^{l-1} \phi_j(\varphi_l, \phi_j) \Big\|, \qquad (l > i \geqq 1) \\ d_{ll} &= 1 \Big/ \Big\| \varphi_l - \sum_{j=1}^{l-1} \phi_j(\varphi_j, \phi_l) \Big\| \end{aligned}\right\} \tag{26}$$

4. EXPRESSION OF THE INTERNAL PROPERTIES OF SWELL-LIKE WAVES BASED ON SOLITON MODES

4.1 Reduction of the Velocity Potential

Not only the number of solitons but both of the amplitude A_i and the integral constant η_0 are invariant, so that the wave-celerity c_i also become invariant and each soliton propagates with a constant speed. We then introduce the following transform of variables into each soliton.

$$\sigma_i = \xi - c_i \tau \tag{27}$$

Assuming that both η_i expressing the wave profile of the soliton with the amplitude A_i and Ω_i being the velocity potential are functions of σ_i, we obtain the relations

$$\left.\begin{aligned} &\partial\eta_i/\partial\tau = -c_i\,\partial\eta_i/\partial\sigma_i\,, \quad \partial\eta_i/\partial\xi = \partial\eta_i/\partial\sigma_i \\ &\partial\Omega_i/\partial\tau = -c_i\,\partial\Omega_i/\partial\sigma_i\,, \quad \partial\Omega_i/\partial\xi = \partial\Omega_i/\partial\sigma_i \end{aligned}\right\} \tag{28}$$

Substituting Eqs.(27) and (28) into the relation between Ω and η derived from Eqs.(5) and (6),

$$(\partial\Phi/\partial x)/\sqrt{gh} = \varepsilon\eta + \varepsilon^2\{\eta^2/2 + (1/2)(1 - z^{*2})\,\partial^2\eta/\partial\xi^2 + \partial\Omega/\partial\tau\} + \cdots \tag{29}$$

which may be transformed to

$$\begin{aligned} (\partial\Phi/\partial x)/\sqrt{gh} = {} & \sum_i^\infty \eta_i - \eta_0 + \Big(\sum_i^\infty \eta_i - \eta_0\Big)^2\Big/2 + (1 - z^{*2})/2 \sum_i^\infty \partial^2\eta_i/\partial\sigma_i^2 \\ & - \sum_i^\infty c_i(\eta_i - \eta_0) + \cdots \end{aligned} \tag{30}$$

It is found that the water particle velocities of the waves having the soliton structure have the particle-like property to be governed directly by the amplitude of each soliton and the periodic property dependent on the ensemble of solitons through the continuity condition for wave profiles.

4.2 Expression of the Water Particle Velocities

Substituting Eq.(21) into Eq.(30) and evaluating the velocity potential of the waves to the second-order approximation, the horizontal and vertical velocities in the Eulerian coordinates u and w are obtained respectively as

$$\frac{u}{\sqrt{gh}}=\sum_i^\infty A_i\,\mathrm{sech}^2\vartheta_i-\eta_0+\sum_i^\infty A_i\,\mathrm{sech}^2\vartheta_i\Big[\frac{\eta_0}{2}+\frac{A_i}{2}\{2-3(\frac{z}{h})^2\}-\frac{9}{4}A_i\{1-(\frac{z}{h})^2\}\mathrm{sech}^2\vartheta_i\Big]$$
$$+\frac{1}{2}(\sum_i^\infty A_i\,\mathrm{sech}^2\vartheta_i)^2-\eta_0{}^2+\frac{1}{2N}\eta_0\sum_i^\infty A_i \tag{31}$$

$$\frac{w}{\sqrt{gh}}=\frac{1}{2}(\frac{z}{h})\sum_i^\infty\sqrt{A_i^3}\,\mathrm{sech}^2\vartheta_i\tanh\vartheta_i\Big[2+\{(\frac{z}{h})^2-1\}A_i(3\,\mathrm{sech}^2\vartheta_i-1)$$
$$+2\sum_j^\infty A_j\,\mathrm{sech}^2\vartheta_j+2\eta_0-A_i\Big] \tag{32}$$

Further, those in the Lagrangian coordinates U and W are expressed as

$$\frac{U}{\sqrt{gh}}=\frac{u}{\sqrt{gh}}+2\sum_i^\infty\sum_j^\infty\frac{\sqrt{A_iA_j}}{c_i}\tanh\vartheta_i\Big|_0^\infty\tanh\vartheta_j\,\mathrm{sech}^2\vartheta_j+\eta_0 t^*\sum_i^\infty\sqrt{3A_i^3}\tanh\vartheta_i\,\mathrm{sech}^2\vartheta_i \tag{33}$$

$$\frac{W}{\sqrt{gh}}=\frac{w}{\sqrt{gh}}-(\frac{z}{h})\Big[\sum_i^\infty\sum_j^\infty\frac{A_j^2\sqrt{3A_i}}{c_i}\tanh\vartheta_i\Big|_0^\infty\mathrm{sech}^2\vartheta_i\,(3\,\mathrm{sech}^2\vartheta_j-2)$$
$$-\frac{3}{2}\eta_0+\sum_j^\infty A_j^2\,\mathrm{sech}^2\vartheta_j(3\,\mathrm{sech}^2\vartheta_j-2)-\sum_i^\infty\sum_j^\infty\frac{A_i\sqrt{3A_j^3}}{c_i}\mathrm{sech}^2\vartheta_i\,\mathrm{sech}^2\vartheta_j\tanh\vartheta_j\Big] \tag{34}$$

4.3 Expression of the Mass Transport Velocities

In this study, the expression of the velocity potential has been derived under the assumption of Eq.(28) as shown already, so that the water particle velocity is independent of the restriction caused by the so-called Stokes' definition of wave-celerity. The mass transport velocities can then be defined both in the Eulerian and Lagrangian coordinates. Denoting the observed period as T^*, the mass transport velocity in the Eulerian coordinate $\bar{u}$ is defined as

$$\frac{\bar{u}}{\sqrt{gh}}=\frac{1}{T^*}\int_0^{T^*}\frac{u}{\sqrt{gh}}dt^* \tag{35}$$

Rewriting Eq.(31) into the expression of N solitons which are included in observed data within the period of T^* and introducing this into Eq.(35), we obtain

$$\frac{\bar{u}}{\sqrt{gh}}=\frac{1}{2T^*}\int_0^{T^*}(\sum_i^N A_i\,\mathrm{sech}^2\vartheta_i)^2dt^*-\frac{1}{2}\eta_0^2-\frac{1}{2T^*}\{1-(\frac{z}{h})^2\}\sum_i^N\frac{\sqrt{3A_i^3}}{c_i}$$

$$\times \left\{ 2 \tanh \vartheta_i \Big|_0^{T^*} - \tanh \vartheta_i (\operatorname{sech}^2 \vartheta_i + 2) \Big|_0^{T^*} \right\}$$

$$- \sum_i^N c_i \left\{ \frac{\eta_0}{N} + \sqrt{\frac{A_i}{3}} \frac{2 \tanh \vartheta_i}{c_i T^*} \Big|_0^{T^*} \right\} \tag{36}$$

By the same means, the mass transport velocity $\tilde{U}$ in the Lagrangian coordinates is derived by the horizontal velocity for the observed period as

$$\frac{\tilde{U}}{\sqrt{gh}} = \frac{\bar{u}}{\sqrt{gh}} + \frac{2\eta_0}{T^*} \sum_i^N \frac{1}{c_i^2} \sqrt{\frac{A_i}{3}} \left\{ \frac{\sqrt{3A_i}}{2} c_i t^* \operatorname{sech}^2 \vartheta_i + \tanh \vartheta_i \right\} \Big|_0^{T^*}$$

$$- \frac{2}{T^*} \sum_i^N \sum_j^N \frac{A_j}{c_i c_j} \sqrt{\frac{A_i}{3}} \tanh \vartheta_i \big|_0 \operatorname{sech}^2 \vartheta_j \big|_0^{T^*} + \frac{2}{T^*} \sum_i^N \sum_j^N \frac{\sqrt{A_i A_j^3}}{c_i}$$

$$\times \int_0^{T^*} \tanh \vartheta_i \tanh \vartheta_j \operatorname{sech}^2 \vartheta_j dt^* \tag{37}$$

5. APPLICATION TO FIELD WAVES

5.1 Soliton Analysis and Synthesis of Observed Data

For expressing theoretically observed waves on soliton modes, we must decompose them into a number of solitons to determine their amplitude A_i and phase constant δ_i. Then, employing these values of A_i and δ_i into Eq.(21) may yield a theoretical expression of the waves with a soliton structure.

Under the assumption that the right-hand side of Eq.(23) is a complete orthonormal system, moreover, the ensemble of the amplitude $\{A\}$ by making the inner product between the theoretical wave profile η given by Eq.(23) and the observed wave profile Y can be calculated by

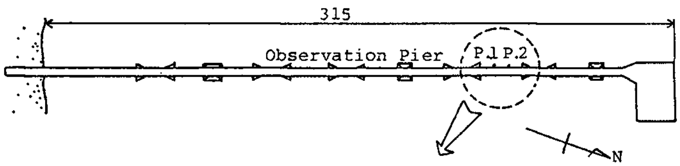
$$\begin{Bmatrix} A_1 \\ \vdots \\ A_N \end{Bmatrix} = \begin{bmatrix} d_{11} & \cdots & d_{N1} \\ & \ddots & \vdots \\ 0 & & d_{NN} \end{bmatrix} \begin{Bmatrix} (Y + \eta_o , \ \phi_1) \\ \vdots \\ (Y + \eta_o , \ \phi_N) \end{Bmatrix} \tag{38}$$

5.2 Accuracy of the Expression of the Swell-Like Waves

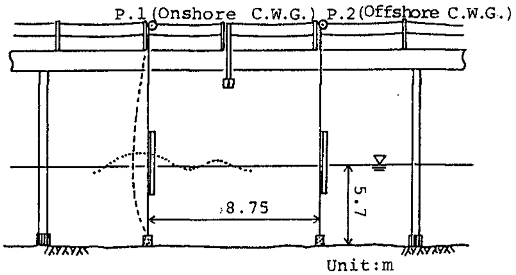
An observation of the swell-lke waves was carried out on the 7th of March, 1981, by using the wave observation system, at the Ogata Wave Observatory, Disaster Prevention Research Inatitute, Kyoto University.

Fig. 3 describes the position of the wave gauges of capacitance type installed along the observation pier 315m long in the offshore direction. The water depths at which the wave gauges are installed are shown in the figure. Wind speed and direction during the observation were about 3m/s and S, respectively. Incident waves were long crested and approaching parallel with the pier as swell-like waves suffering from no influence of wave breaking.

A comparison of power spectra computed by the FFT method between the waves observed at P. 1 and P. 2 is shown in Fig. 4 with their values of skewness and kurtosis of the water surface displacement. And, the Ursell



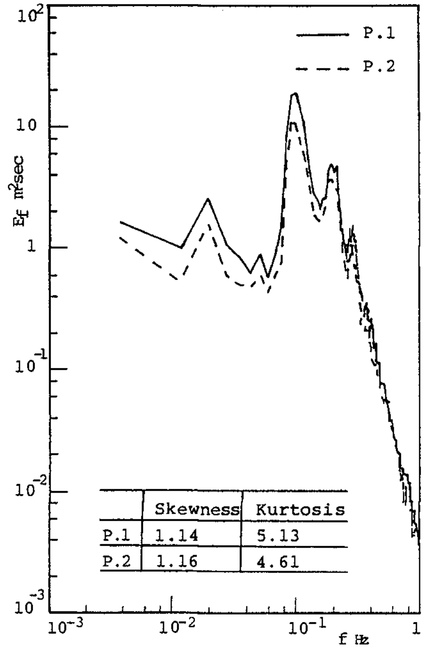
(a) Plane view of observation pier



(b) Side view of observation pier and wave gauges installed

Fig. 3 Positions and number of installed wave gauges

number(U_r) at P. 1 calculated by the zero-up crossing method is 45.9. The power spectra exhibit pronounced secondary peaks at frequencies about twice the main peak frequencies due to nonlinear interaction among sinusoidal modes. It is considered from this figure that the waves are swell-like with fully developed nonlinearity. Low frequency components of the waves have considerable energy which is probably due to water surface displacements with long periods, such as surf-beat, independent of the soliton modes. Mechanism of their excitation is different from that of the waves. The appraoch based on soliton modes can therefore be applied to the observed data, if the low frequency components under 0.04Hz are excluded with a numerical low-cut filter of critical frequency of f_c = 0.04Hz.



	Skewness	Kurtosis
P.1	1.14	5.13
P.2	1.16	4.61

Fig. 4 Spatial variation of power spectra

Fig. 5 shows comparisons of wave profiles between the data observed at P. 1 and P. 2 mentioned above and the theoretical results obtained by the expression of Eq.(21). Clearly, Eq.(21) has sufficient accuracy in expressing the wave profiles under the swell-like sea condition.

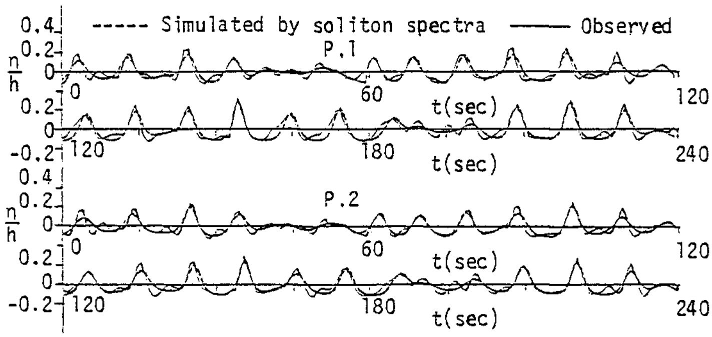


Fig. 5 Comparisons of wave profiles between the observed data and the theoretical results

One of the comparisons between the observed data and the theoretical profiles of the waves propagated from P. 1 to P. 2 is shown in Fig. 6. This figure states that this approach can express the propagation of the waves, as well as the wave profiles with satisfactory accuracy.

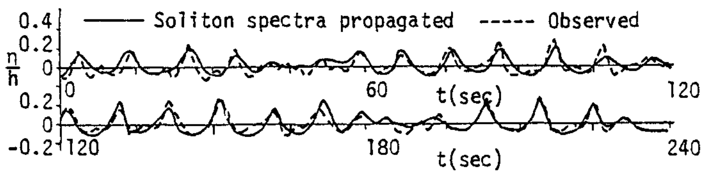


Fig. 6 Comparison of wave propagation between the observed and theoretical data

Figs. 7 and 8 show another comparison on swell-like waves with the Ursell number 22.4 at P. 1. The observed data in these figures were obtained by the same observation system under the similar sea condition and analyzed as well. The expression has sufficient accuracy, so that the dominant part of these waves have the soliton structure for which the formulation can be made by the soliton modes.

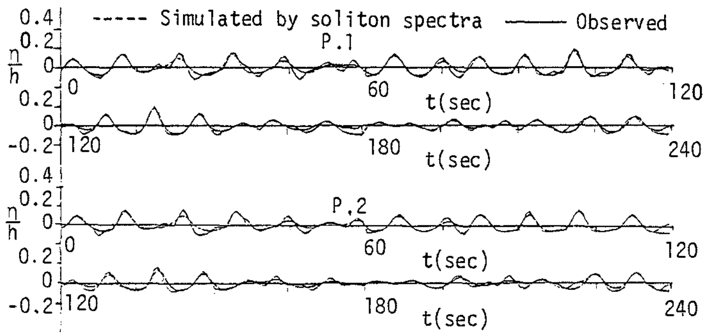


Fig. 7 Comparisons of wave profiles between the observed data and the theoretical results

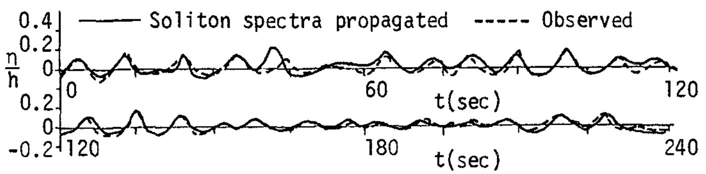


Fig. 8 Comparison of wave propagation between the observed and theoretical data

5.3 Extensive Application to Waves Under the Various Sea Conditions

Observations of waves were carried out by use of both the array consisting of nine capacitance-type wave gauges installed along the observation pier mentioned above and the line array of four ultra - sonic-type wave gauges installed at the offshore end of the pier to obtain directional spectra. Fig. 9 shows the positions of the wave gauges at which the water depths are also shown. Fig. 10 indicates the changes in wind speed and direction within a period of the observation and the relation between them together with the data numbers of which abbreviation is expressed by DNO.

Fig. 11 shows some comparisons of wave profiles at the points of U4, CW5, CW7 and CW9 between the observed data of DNOs 27, 31 and 34 and the theoretical results by soliton modes. The Ursell numbers at U4 of DNOs 27, 31 and 34 are 4.7, 18.0 and 11.1, respectively. We may conclude that the expression by soliton modes has sufficient accuracy in expressing the wave profiles under various sea conditions, as far as their dominant portions are concerned, although the influence of wave breaking is not negligible in these comparisons where about a few ten's percent or more of the waves are broken.

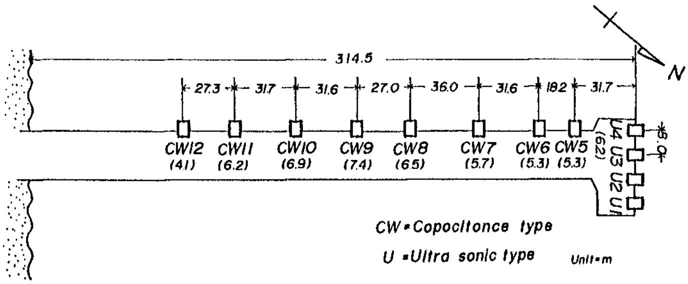


Fig. 9 Positions and their numbers of the installed wave gauges (figures in parenthese indicate the water depths)

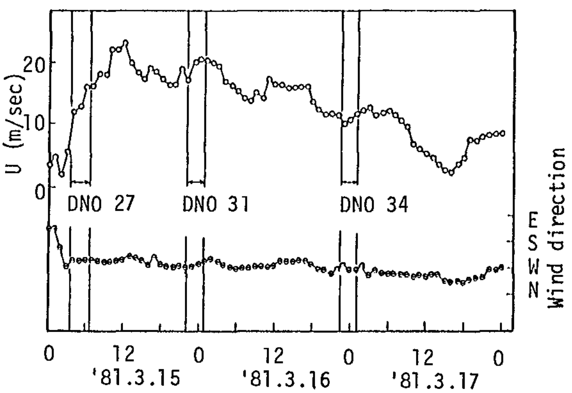
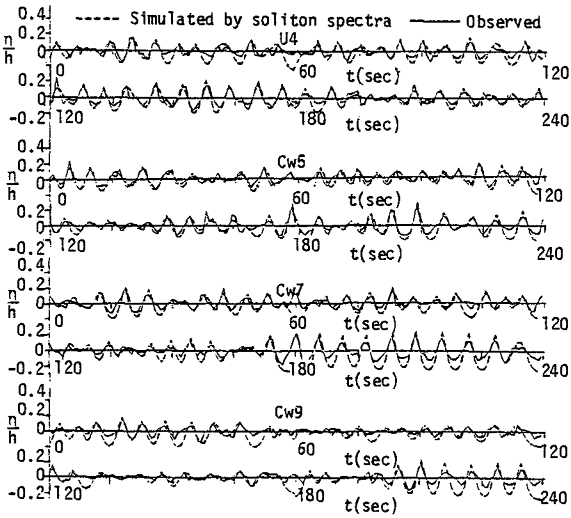
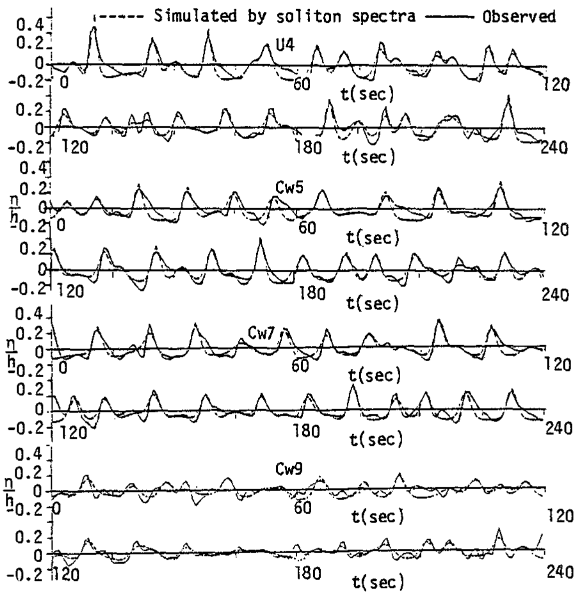


Fig. 10 Wind speed and direction during the wave observations

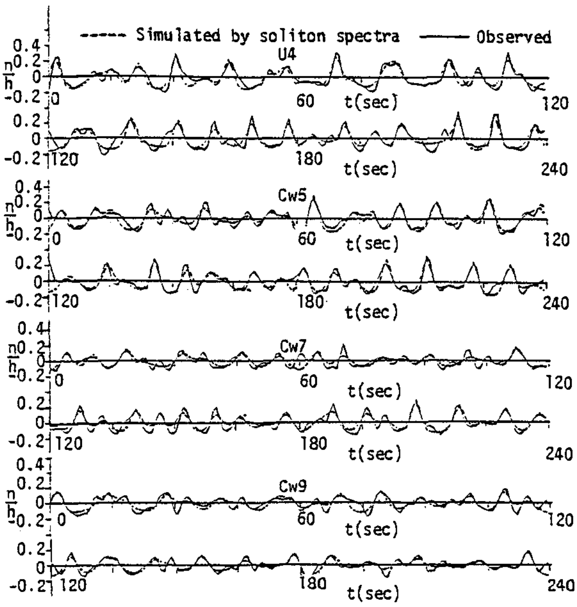


(a) DNO 27



(b) DNO 31

Fig. 11 Comparisons of wave profiles in a period of growing state to damping one between the observed data and the theoretical results



(c) DNO 34

Fig. 11 Comparisons of wave profiles in a period of growing state to damping one between the observed data and the theoretical results

5.4 Accuracy of the Expression of Internal Properties

An observation of horizontal water particle velocities of the swell-like waves with the Ursell number 19.6 was carried at Ajigaura coast facing the Pacific ocean by Horikawa et al. The velocities were measured at the water depth of 2.28m by using an electro-magnetic current meter which is installed at the height of 0.62m above the sea bottom.

Fig. 12 shows part of the comparison of the velocities between their observed data and the theoretical result calculated by Eq.(31). Fig. 13 also shows similar comparison of wave profiles. It is found from these

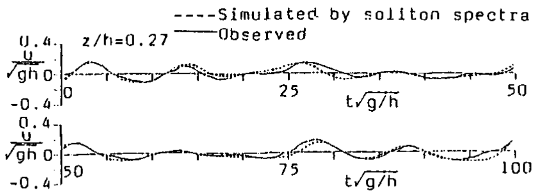


Fig. 12 Comparison of horizontal water particle velocity between the observed data and the theoretical results

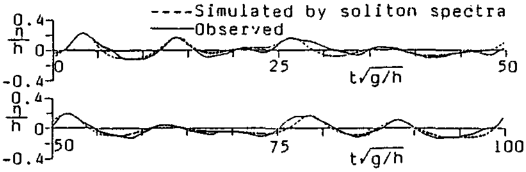


Fig. 13 Comparison of wave profiles between the observed data and the theoretical results

results that the expression of the water particle velocity based on soliton modes has accuracy similar to that for wave profiles as far as the expression of wave profiles has sufficient accuracy.

6. CONCLUSIONS

From the viewpoint that the soliton is one of the most elementary excitation in the nonlinear random waves, we have attempted to describe swell-like waves in shallow water theoretically by deriving the asymptotic multi-soliton solution under the assumption that the waves have a coherent dynamic structure composed of solitons alone. The expressions of wave profiles and internal properties such as water particle and mass transport velocities were derived by using the asymptotic solution, independently of both the evolution type and the degeneration of the amplitudes.

Further, applicability of the expressions by soliton modes was examined by comparisons between observed and theoretical wave profiles and internal properties. We conclude from these comparisons that the expressions by soliton modes practically have sufficient accuracy in expressing various properties of the swell-like waves.

7. ACKNOWLEDGEMENT

Figs. 12 and 13 were drawn from the wave data observed by Professor Horikawa et al. of Tokyo University and their permission for publication is greatfully acknowledged.

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