CHAPTER TWENTY FIVE

# On A Design Wave Spectrum*

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We propose the use of a generalized representation for acquiring a design wave spectrum. The generalized form, free from any predetermined coefficients and exponents, requires only significant wave height and average wave period as input for practical applications. The usefulness of this representation has been demonstrated with over 2000 measured deep-water wave spectra recorded from NOMAD buoys in the Great Lakes during 1981.

## Introduction

Since Neumann (1953) first introduced his representation for wind-wave spectra to facilitate wave prediction, coastal engineers have endeavored for over three decades to develop a realistic representation of wave spectra. The two-parameter spectrum of Bretschneider (1959), the fully developed spectrum of Pierson and Moskowitz (1964), and the fetch-limited spectrum of JONSWAP (Hasselmann et al., 1973) are notable examples. Among these, the JONSWAP spectrum is perhaps the one most widely used in wave modeling and engineering designs during the last decade.

All of these proposed spectral forms consist of a number of empirical coefficients and exponents that differ among various authors, while the overall forms are basically similar. The applicability and universality of these coefficients and exponents has frequently been the subject of question or dispute. For instance, the rear face of the spectrum in the high frequency range was first intuitively set by Neumann to be proportional to the -6th power of frequency. Later Phillips (1958) deduced from dimensional considerations that the exponent should be -5. The -5th power dependence on the high frequency side of the spectrum seems to have been substantiated by many laboratory and field measurements. Most of the spectral forms use the -5 formulation. More recently, however, Kitaigorodskii (1983) found theoretical as well as experimental evidence that the exponent should be -4 instead. The situation remains unsettled. Moreover, the widely used average JONSWAP spectrum consists of four numerical parameters, three of which are averaged from largely scattered empirical data points. When applying the formula with these empirical numbers in

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practice, it is seldom certain how close the representation is as compared to the actual modeling or design conditions.

In this paper, we propose an alternative approach by using a generalized form (Liu, 1983) that avoids predetermining any coefficient and exponents in the spectrum representation. They can all be obtained from known spectral parameters. Hence, if design wave conditions are given, relevant spectral parameters can be estimated and a design wave spectrum can be readily determined.

## The Spectral Form

The generalized spectral form given by Liu (1983) in representing a one-dimensional, single peak frequency wave spectrum is

$$S(f) = C_1(E/f_m)(f/f_m)^{-C_2}\exp[-C_3(f/f_m)^{-C_2/C_3}] \qquad (1)$$

where $E = \int S(f)df$ is the variance of the surface displacement, $f_m$ is the frequency of the spectral peak, and $C_i$'s are dimensionless coefficients and exponents that can be determined from given spectral parameters.

To show how the $C_i$'s are determined, we note first in eq. (1) that for $f = f_m$

$$C_1 = \exp(C_3)S(f_m)f_m/E \qquad (2)$$

Furthermore from the definition of E and using the gamma function $\Gamma$ with eqs. (1) and (2), we find

$$E/(S(f_m)f_m) = \exp[C_3 + (1 - C_3 + C_3/C_2)\ln C_3]\Gamma(C_3 - C_3/C_2)/C_2 \qquad (3)$$

which is an equation for $C_2$ and $C_3$. Now if we use the average frequency $f_a$ defined by Rice (1944)

$$f_a^2 = [\int f^2S(f)df]/[\int S(f)df] \qquad (4)$$

and substituting S(f) from eq.(1) and using the gamma function again, we have another equation for $C_2$ and $C_3$

$$(f_a/f_m)^2 = \exp(2C_3\ln C_3/C_2)\Gamma(C_3 - 3C_3/C_2)/\Gamma(C_3 - C_3/C_2) \qquad (5)$$

Thus we have two equations [(3) and (5)] that can be solved for $C_2$ and $C_3$ if the parameters E, $f_a$, $f_m$, and $S(f_m)$ are given. Then $C_1$ can be found from eq. (2) and the spectrum is fully determined.

The representation eq. (1) has been applied to over 2000 sets of deep-water wave spectra measured from NOMAD buoys moored in the Great Lakes with water depth ranging from 15 m to 250 m. Sample comparisons are presented in Liu (1983). It was found that the spectral form fits the measured spectra very well under a variety of wave conditions. The close fit was especially evident for large storm waves. Perhaps the spectra that eq. (1) does not fit well were those multimodal ones that occur mainly during the early stage of wave growth.

The process for solving eqs. (3) and (5) can best be achieved iteratively. Liu (1983) presented a workable empirical trial and error procedure. A more effective approach is to apply the iteration procedure using the Newton method to solve eq. (3) (Liu, 1984a; Fullerton, 1984) for $C_3$ with an initial $C_2$. These values are then substituted into the right-hand side of the following equation obtained from eqs. (3) and (5)

$$C_2 = \exp[C_3 + (1 - C_3 + 3C_3/C_2)\ln C_3]\Gamma(C_3 - 3C_3/C_2)/D \tag{6}$$

where $D = (f_a/f_m)^2E/(S(f_m)f_m)$. If the $C_2$ calculated from eq. (6) is not nearly equal to the previous one, then the new $C_2$ will be used for input and the procedure is repeated again until $C_2$ converges.

## The Applications

The practical applications of the generalized spectrum form eq. (1) require the parameters E, $f_a$, $f_m$, and $S(f_m)$ to be known. With these four parameters, all the useful spectral properties can be readily deduced. For example, the nth moment of the wave spectrum is given by

$$\begin{aligned} M_n &= (2\pi)\int f^n S(f)df \\ &= E(2\pi f_m)^n(C_1/C_2)\exp[(1-Z)\ln C_3]\Gamma(Z) \end{aligned} \tag{7}$$

where $Z = C_3 - (n + 1)C_3/C_2$. From eq. (7), the spectrum-width parameter is then

$$\begin{aligned} \varepsilon^2 &= 1 - M_2^2/(M_0M_4) \\ &= 1 - \Gamma^2(C_3 - 3C_3/C_2)/[\Gamma(C_3 - C_3/C_2)\Gamma(C_3 - 5C_3/C_2)] \end{aligned} \tag{8}$$

Therefore, relevant spectral characteristics can generally be estimated if we know the four essential parameters.

Among the four parameters, E and $f_a$ are obtainable from design wave information (e.g., E = $(H_{1/3}/4)$, where $H_{1/3}$ is the significant wave height). The parameters $S(f_m)$ and $f_m$ are in general, however, not readily obtainable without actual measurements. For practical applications, Liu (1983) found that for $f_a$ and $f_m$ in Hz, $S(f_m)$ in $m^2$/Hz, and E in $m^2$ the following empirical correlations exist for deep-water wave spectra

$$f_a = 0.82(f_m)^{0.74} \tag{9}$$

and

$$S(f_m) = 17.0(E)^{1.13} \tag{10}$$

which effectively reduced the necessary parameters to only two. Applying eqs. (9) and (10) in practice, we can always acquire a reasonably accurate deep-water design wave spectrum with only customarily available design wave conditions, e.g., a significant wave height and an average wave period. The applicability of eqs. (9) and (10) has been further corroborated with over 2000 measured wave spectra as shown in Fig. 1. These measured data, recorded from the eastern Lake

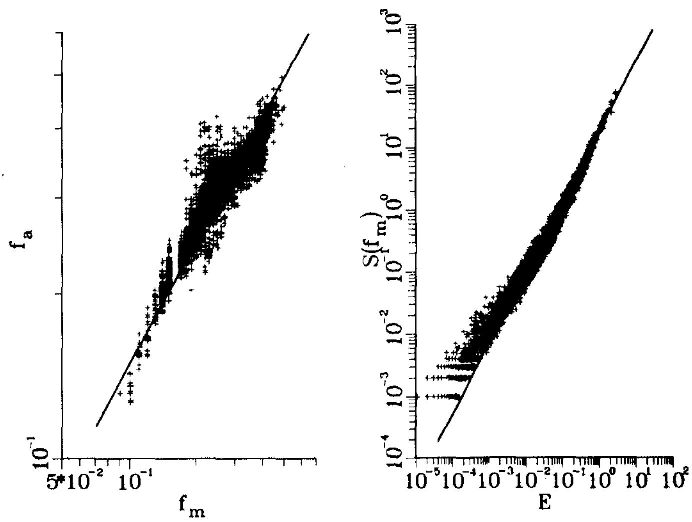


Figure 1. The correlations between $f_a$ and $f_m$ and between $S(f_m)$ and E. The straight lines correspond to eqs. (9) and (10), respectively.

Superior buoy during 1981, consist of waves ranging from 0.2 to over 8 m in significant wave height. Although dimensionally inhomogeneous, the obvious existence of the empirical correlations eqs. (9) and (10) has provided a useful modification that simplifies and enhances the applicability of the generalized wave spectrum representation eq. (1).

## Assessment of the Representations

To demonstrate the actual applicability of the generalized spectrum form, we use the following deviation index, D.I., defined in Liu (1983) as a measure of the accuracy of the representations:

$$D.I. = \sum_i \left[\frac{S(f_i) - S_R(f_i)}{S(f_i)} \times 100\right] \left[\frac{S(f_i)\Delta f}{E}\right] \tag{11}$$

where $S(f_i)$ and $S_R(f_i)$ are, respectively, measured spectral density at frequency $f_i$ and that calculated from eq. (1); $\Delta f$ is the frequency interval used in calculating the spectrum. Simply stated, this deviation index is the sum of percentage deviations of calculated from measured and weighted by the relative magnitude of the measured spectral density. A perfect representation will yield a zero D.I. Hence, a smaller D.I. implies a better fit.

Based on over 2000 measured wave spectra recorded from the eastern Lake Superior NOMAD buoy during 1981, our representation with all four required parameters had an average D.I. of 30.875 ± 8.756. The modified application with only two parameters leads to an average D.I. of 41.896 ± 13.425.

For comparison, we also applied the JONSWAP representation to the same data set with the coefficients determined through fitting the measured spectrum. The resulted average D.I. is 44.056 ± 27.203. Furthermore, when the average JONSWAP representation with predetermined parameters is used, the average D.I. becomes 101.31 ± 82.022.

Since the data used in the analysis are from both large and small waves, it is of interest to eliminate the smaller waves for design purposes. For over 900 spectra with significant wave heights greater than 1 m, the average D.I.'s are 28.510 ± 6.720 and 38.136 ± 11.011, respectively, for the original and modified applications of representation (1). Similarly, average D.I.'s of 27.802 ± 8.428 and 52.549 ± 23.032 are found for the JONSWAP and average JONSWAP representations, respectively.

From the above discussion, it appears that the JONSWAP representation gives a better fit for waves having a significant wave height greater than 1 m. However, JONSWAP only has the advantage when the spectra are given and the parameters can be obtained through fitting the known spectrum. For practical applications where a wave spectrum is not expected to be available, the proper JONSWAP parameters cannot be readily obtained; then the generalized representation is certainly more advantageous. A comparison of the various average D.I.'s we have calculated makes it evident that the modified approach of eq. (1), which renders a reasonably accurate representation with only $H_{1/3}$ and $f_a$ as required inputs, is the most useful and feasible approach for acquiring a deep-water design wave spectrum.

## The Equilibrium Range Exponent

The generalized wave spectrum representation eq. (1) is characterized by the nonpredetermined coefficients and exponents $C_i$'s. What is the significance of these $C_i$'s? Basically, they are simply scale factors for the spectrum. Liu (1983) found evidence that the $C_i$'s are correlated with the wave growth process; i.e., the $C_i$'s are large during early growth and approach some asymptotic value as waves become well developed. This is generally the case. We are particularly interested in $C_2$, which is the equilibrium range exponent corresponding to the high frequency side of the spectrum. Its exact value has been the subject of some controversy.

Fig. 2 presents a correlation between $C_2$ and the significant wave height $H_{1/3}$ based on the 2000 and more data sets used in this study. For smaller wave heights, $C_2$ is generally larger and quite scattered. For larger wave heights, the scatter reduces significantly and $C_2$ appears to cluster around the value 5, which is consistent with many of the previous studies. There is no indication, however, that the exponent $C_2$ should approach a value of 4.

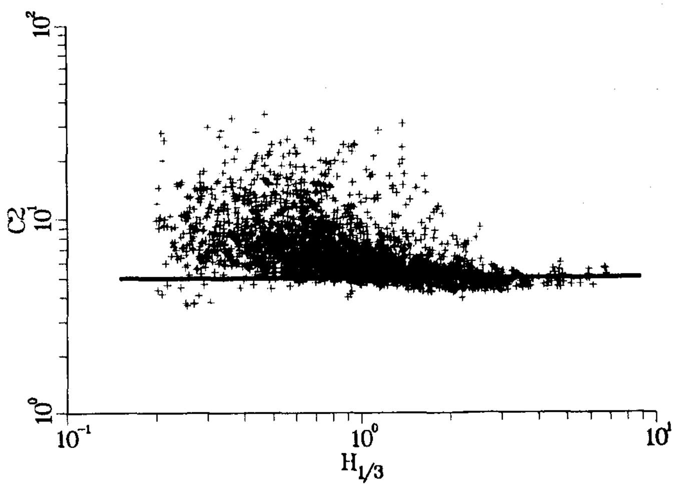


Figure 2. The correllation between $C_2$ and $H_{1/3}$. The straight line represents $C_2 = 5$.

It should be noted that in most of the previous studies the equilibrium range exponent was generally obtained by fitting only the high frequency side of the spectrum, which basically ignores the total spectrum. The $C_2$'s deduced in this study relate to the entire spectrum and are therefore more representative of the true equilibrium range exponent.

## Concluding Remarks

We have shown that the generalized form eq. (1) provides a reasonably accurate representation for the deep-water wave spectrum. The spectrum is fully determined from four internal parameters: the variance of the surface displacement E, the average frequency $f_a$, the frequency of the spectral peak $f_m$, and the energy density at the spectral peak $S(f_m)$. Because some of these parameters are not readily accessible, we have also found empirical correlations between $S(f_m)$ and E and between $f_a$ and $f_m$ that can be used to reduce the essential parameters to two. This is useful in many coastal and oceanic engineering applications when spectral properties are needed and only design wave height and wave period are available.

The analysis presented here has concentrated on deep-water waves. One of the immediate interests related to coastal engineering problems is the finite depth effect. We have found from a separate analysis (Liu, 1984b) that, with four parameters given, the representation also applies satisfactorily to shallow-water waves. The depth clearly affects the spectral parameters but not the generalized form of representation. There is at present not enough shallow-water wave measurements to allow a more detailed analysis.

The contention that spectral characteristics can be obtained from given wave height and wave period can also lead to a simplified wave prediction process. That is, instead of using complicated spectral model predictions, a simpler model can be developed to predict wave height and wave period only. One such model has been developed for the Great Lakes (Liu et al., 1984; Schwab et al., 1984) with satisfactory results. Thus, by combining the output from this model with the generalized spectrum representation, we can obtain results similar to those produced from a general spectral model, but using a simpler approach and with greater computational economy.

## Appendix A.--References

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