PROBABILITY DENSITY FUNCTION OF WAVE HEIGHTS OFF THE WESTERN COAST OF TAIWAN

by

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ABSTRACT

A new probability density function of wave heights off the western coast of Taiwan is submitted in this paper. According to the bathymetry of this area, waves from the central part of Taiwan Strait refract to the point of measurement and minor waves generated by local wind add the energy on the major ones; So an analytical solution is to be worked out by assuming that the wave energies are the linear sum of these two sources and convolution integral is adopted. The new model approaches reality better than Rayleigh's.

1. *INTRODUCTION*

Since 1952, The probability density function of wave heights is supposed to be Rayleigh's distribution. It will be physically sound if the water depth is infinitive, namely in the case of deep water wave. Another requirement is that the frequency band of wave spectrum should be as narrow as possible. In deep water waves, if the wave heights are defined to be the vertical distance

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between the tip of wave crest above mean sea water level and the bottom of wave trough below it. Such a restriction is fulfilled if we use zero-crossing method to read out the wave records. However, the probability density functions of wave heights in deep water are scarcely to be Rayleigh distribution exactly.

The authors measured and analyzed wave data off the western coast of Taiwan, and studied the wave height distributions. The curves can not said to be Rayleigh distributions. These curves have following characteristics:

(1) *The peak is higher than Rayleigh's curve and lower in two sides, namely the distribution is more concentrated.*

(2) *Two points of inflection can be found but there is only one in Rayleigh's curve.*

To explain such phenomena, we performed following two approaches.

(1) *Analyze the wave record directly to work out an empirical formula which can be adopted generally.*

(2) *According to the bathymetry off the western coast of Taiwan. The waves advanced to the point of measurement are refracted from the central part of Taiwan Strait and combined with local wind waves. A new statistical model can be worked out based on this situation. The analytical consideration of the wave height distribution caused by two wave spectra combination is described as follows.*

2. *STATISTICAL DISTRIBUTION OF WAVE HEIGHTS AFTER SPECTRA COMBINATION*

As mentioned above, the statistical features of waves in Taiwan Strait are not coincident with the theory. The reasons of such phenomenon are supposed to be as follows:

(1) *All wave records being analyzed are measured at shallow water area of western coast of Taiwan, where the waves are composed*

of two patterns. The larger one is transmitted from the central part of the strait by refraction, and the smaller one is directly generated by the prevailing NE-NNE wind along the coast as shown in Fig. 1.

(2)*Theoretical distribution of wave features are assuming that the water depth is infinite, i.e., the situation of deep water wave. Waves in Taiwan strait are shallow water waves in essence. According to the wave statistics in Taiwan strait as shown in Fig.* 2, *the Rayleigh distribution shows a lower estimated distribution. A new statistical model sounds necessary.*

Theoretical and practical approaches for explicating this problem are described as below.

Assuming that:

(1)*Wave energy E on a complicated random sea surface is the linear sum of the energies E_i transmitted from different sources with spectral functions $S_i(f)$, i.e.*

$$E = \frac{1}{8}\rho g H^2 = \sum_i E_i = \frac{1}{8}\rho g \sum_i H_i^2 \qquad (1)$$

(2)*Wave height probability density function in each spectrum is Rayleigh distribution.*

$$\psi_{H_i}(H_i) = \frac{H_i}{4\sigma_i^2}\, exp\left(-\frac{H_i^2}{8\sigma_i^2}\right), \quad 0 \le H_i < \infty \qquad (2)$$

where $\sigma_i^2 = \int_0^\infty S_i(f)\,df$

(3)*Wave energies from different spectra are mutually independent random numbers.*

Probability density function of the wave height H of a random sea surface which is composed of two different wave spectra is to be derived as below:

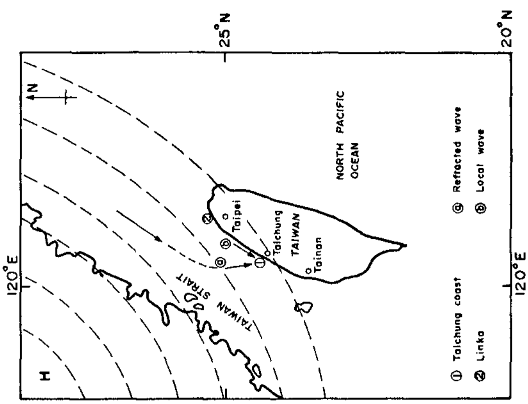
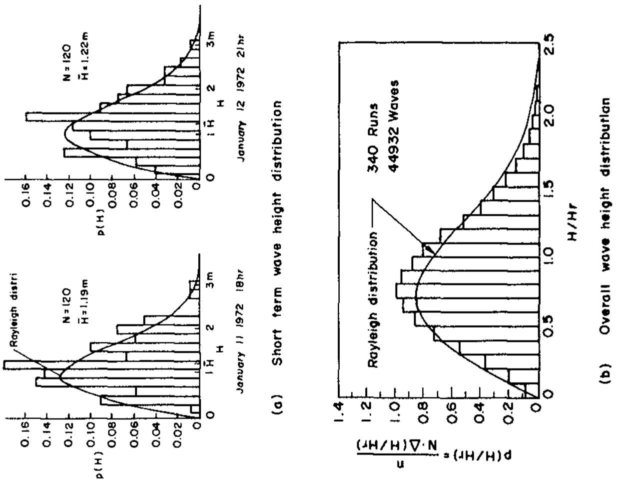


Fig.1 Wind and wave situation in the winter and location of wave measurement



(a) Short term wave height distribution

(b) Overall wave height distribution

Fig.2 Wave statistics in Taiwan Strait

Let $$y_i = \frac{8E_i}{\rho g} = H_i^2 \qquad (i = 1, 2) \tag{3}$$

Density function of y can be found as follows

$$\phi_y(y) = \left|\frac{dH}{dy}\right| \phi_H(H = \sqrt{y}) = \frac{1}{2\sqrt{y}} \phi_H(\sqrt{y})$$

$$\left.\begin{aligned} \phi_y(y_1) &= \frac{1}{2\alpha_1} exp\left(-\frac{y_1}{2\alpha_1}\right) \\ \phi_y(y_2) &= \frac{1}{2\alpha_2} exp\left(-\frac{y_2}{2\alpha_2}\right) \end{aligned}\right\} \tag{4}$$

where $\alpha_i = 4\sigma_i^2$

While two series of waves are mixing, their energies add together.

Let $Z = y_1 + y_2 = H_1^2 + H_2^2 = H^2$

$$\phi_z(Z) = \int_0^z \phi_y(y_1)\, \phi_y(Z - y_1)\, dy_1$$

$$\phi_z(Z) = \frac{1}{2(\alpha_1 - \alpha_2)} \left[exp\left(-\frac{Z}{2\alpha_1}\right) - exp\left(-\frac{Z}{2\alpha_2}\right) \right], \quad 0 \leq Z < \infty \tag{5}$$

$$\phi_H(H) = \frac{H}{\alpha_1 - \alpha_2} \left[exp\left(-\frac{H^2}{2\alpha_1}\right) - exp\left(-\frac{H^2}{2\alpha_2}\right) \right], \quad 0 \leq H < \infty \tag{6}$$

Such a function has two points of inflexion at both sides of the peak whereas there is only one point locating to the right of maximum probability density in Rayleigh distribution.

Wave height distribution functions for the waves transmitted from 3 spectra are able to be calculated by similar way and the *result is as follows.*

$$\phi(H) = \frac{\alpha_1 H}{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)}\left[exp\left(-\frac{H^2}{2\alpha_1}\right) - exp\left(-\frac{H^2}{2\alpha_3}\right)\right]$$

$$- \frac{\alpha_2 H}{(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)}\left[exp\left(-\frac{H^2}{2\alpha_2}\right) - exp\left(-\frac{H^2}{2\alpha_3}\right)\right] \quad (7)$$

While the waves have multiple sources the wave height distribution function can be worked out by repeated convolution integration. A special case $H_1^2 = H_2^2 = \cdots = H_n^2$

let $W = \sum_{i=1}^{n} H_i^2$

$$\phi(W) = \frac{W^{n-1}}{(2\alpha)^n \Gamma(n)} exp\left(-\frac{W}{2\alpha}\right) \quad (8)$$

and
$$\phi(H) = \frac{2H^{2n-1}}{(2\alpha)^n \Gamma(n)} exp\left(-\frac{H^2}{2\alpha}\right) \quad (9)$$

3. *CHARACTERISTICS OF THE NEW DISTRIBUTION*

From eq. (6), *for convenience, put* $\alpha_1 = a$, $\alpha_2 = b$, *the equation becomes*

$$\phi(H) = \frac{H}{a-b}\left[exp\left(-\frac{H^2}{2a}\right) - exp\left(-\frac{H^2}{2b}\right)\right] \quad (10)$$

$$\left.\begin{aligned} a &= 4m_{0,1} = 4\sigma_1^2 \\ b &= 4m_{0,2} = 4\sigma_2^2 \end{aligned}\right\} \quad (11)$$

The moments could be found as follows

$$m_0 = \int_0^\infty \phi(H)\,dH = \int_0^\infty \frac{H}{a-b}\left[exp\left(-\frac{H^2}{2a}\right) - exp\left(-\frac{H^2}{2b}\right)\right]dH$$

$$= 1 \quad (12)$$

$$m_1 = \int_0^\infty H\phi(H)\, dH = \frac{1}{a-b}\left(\sqrt{\frac{\pi a^3}{2}} - \sqrt{\frac{\pi b^3}{2}}\right) \tag{13}$$

$$m_2 = \int_0^\infty H^2\phi(H)\, dH = 2\,(a+b) \tag{14}$$

$$m_3 = \int_0^\infty H^3\phi(H)\, dH = \frac{3}{a-b}\left(\sqrt{\frac{\pi a^5}{2}} - \sqrt{\frac{\pi b^5}{2}}\right) \tag{15}$$

$$m_4 = \int_0^\infty H^4\phi(H)\, dH = 8\,(a^2 + ab + b^2) \tag{16}$$

mean:

$$\mu = m_1 = \frac{\sqrt{\pi/2}}{a-b}\left(a^{3/2} - b^{3/2}\right) \tag{17}$$

variance:

$$\sigma_H^2 = M_2 = m_2 - m_1^2$$

$$= 2\,(a+b) - \frac{\pi}{(a-b)^2}\left(\frac{a^3+b^3}{2} - ab\sqrt{ab}\right) \tag{18}$$

skewness:

$$\sqrt{\beta_1} = \frac{M_3}{(M_2)^{3/2}} = \frac{M_3}{(\sigma_H^2)^{3/2}} \tag{19}$$

$$M_3 = m_3 - 3m_1m_2 + 2m_1^2$$

$$= \frac{3}{a-b}\left\{a^2\sqrt{\frac{\pi a}{2}} - b^2\sqrt{\frac{\pi b}{2}}\right\} - \frac{6\,(a+b)}{a-b}\left\{\sqrt{\frac{\pi a^3}{2}} - \sqrt{\frac{\pi b^3}{2}}\right\}$$

$$+ \frac{2}{(a-b)^3}\left\{\sqrt{\frac{\pi a^3}{2}} - \sqrt{\frac{\pi b^3}{2}}\right\}^3$$

kurtosis:

$$\gamma = \beta_2 - 3 = \frac{M_4}{M_2{}^2} - 3 \tag{20}$$

$$M_4 = m_4 - 4m_1m_3 + 6m_1{}^2m_2 - 3m_1{}^4$$

$$= 8(a^2+ab+b^2) - \frac{12}{(a-b)^2}\left\{\sqrt{\frac{\pi a^3}{2}} - \sqrt{\frac{\pi b^3}{2}}\right\}\left\{\sqrt{\frac{\pi a^5}{2}} - \sqrt{\frac{\pi b^5}{2}}\right\}$$

$$+ \frac{12(a+b)}{(a-b)^2}\left\{\sqrt{\frac{\pi a^3}{2}} - \sqrt{\frac{\pi b^3}{2}}\right\}^2 - \frac{3}{(a-b)^4}\left\{\sqrt{\frac{\pi a^3}{2}} - \sqrt{\frac{\pi b^3}{2}}\right\}^4$$

extreme value occurs at

$$H = \left\{ \frac{2ab}{a-b} \ln \left(\frac{ab - aH^2}{ab - bH^2} \right) \right\}^{1/2} \tag{21}$$

point of inflexion occurs at

$$H = \left\{ \frac{2ab}{a-b} \ln \left[\frac{a^2 (H^3 - 3b)}{b^2 (H^2 - 3a)} \right] \right\}^{1/2} \tag{22}$$

According to the above mentioned equations, we can found that all the characteristic values are influenced by a and b, i.e., influenced by wave energies of refracted and local wave.

4. *PROCEDURE OF DELINEATE PROBABILITY DENSITY CURVES FOR EXISTING RECORDS*

From eq. (17), *we have*

$$\overline{H} = \sqrt{\frac{\pi}{2}} \left(\frac{a^{3/2} - b^{3/2}}{a - b} \right) \tag{23}$$

let $\sqrt{a} = x$ $\qquad \sqrt{b} = y$

then $a = x^2$ $\qquad b = y^2$

$a^{3/2} = x^3$ $\qquad b^{3/2} = y^3$

Eq. (23) *becomes*

$$\frac{\overline{H}}{\sqrt{\frac{\pi}{2}}} = P = \frac{x^3 - y^3}{x^2 - y^2} \tag{24}$$

The local wave energy b can not be equal to that of waves from the central part of Taiwan Strait.

$$a \neq b \quad , \quad i.e. \quad x \neq y$$

then $$x^2 + xy + y^2 - Px - Py = 0 \tag{25}$$

This is the equation of an ellipse as shown in Fig. 3.

The total energy is the sum of local and refracted wave energies

$$a + b = 4 \left(\sigma_1^2 + \sigma_2^2 \right) = 4 \sigma^2 \tag{26}$$

Following relationship exists

$$\overline{H} = k \sigma \tag{27}$$

Eq. (26) *becomes*

$$x^2 + y^2 = \left(\frac{2\overline{H}}{k} \right)^2 = R^2 \tag{28}$$

This is the equation of a circle with radius R.

From equs. (25) *and* (28), *the energies x, y, i.e., a, b can be worked out.*

For standardization

let $$X = \frac{\sqrt{2}}{2} x + \frac{\sqrt{2}}{2} y \qquad Y = -\frac{\sqrt{2}}{2} x + \frac{\sqrt{2}}{2} y$$

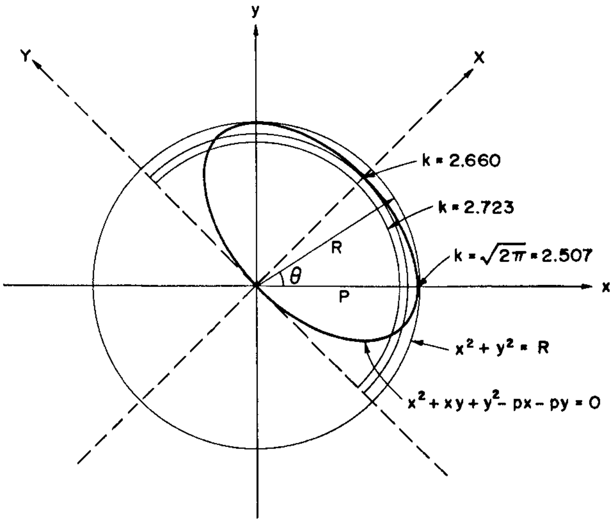


Fig. 3 *Graphical solution of simultaneous equations* *then* $x = \frac{\sqrt{2}}{2}X - \frac{\sqrt{2}}{2}Y$ $\qquad y = \frac{\sqrt{2}}{2}X + \frac{\sqrt{2}}{2}Y$

Eg.(25) *becomes*

$$3X^2 + Y^2 - 2\sqrt{2}\,PX = 0 \tag{29}$$

that is

$$\frac{\left(X - \frac{\sqrt{2}}{3}P\right)^2}{\frac{2}{9}P^2} + \frac{Y^2}{\frac{2}{3}P^2} = 1 \tag{30}$$

The shape of this ellipse was shown as Fig. 3. *Its Center at* $\left(\frac{\sqrt{2}}{3}P, 0\right)$, *long axis* $= \frac{2\sqrt{2}}{\sqrt{3}}P$, *short axis* $= \frac{2\sqrt{2}}{3}P$.

When $Y = 0$, $X = 0$ *and* $\frac{2\sqrt{2}}{3}P$, *the ellipse curve passed at the original point. Besides, when* $y = 0$, $x = P$.

From Fig. 3, *we found that if we want to have solution of equs.*(25) *and*(28), *the condition is* $R \le P$. *But from equs.*(24) *and*(28), *we have*

$$\frac{2\bar{H}}{k} = R \le P = \frac{\bar{H}}{\sqrt{\frac{\pi}{2}}}$$

$$\text{for} \quad \bar{H} \neq 0, \quad \text{so} \quad k \ge \sqrt{2\pi} \tag{31}$$

According to the field data measured at Taichung Harbor (① *in Fig.* 1) *in Taiwan Strait* (*Ou*, 1977), *the value of* $k = 2.723 > \sqrt{2\pi}$. *The new model can be used. But at Linko* (*north coast of Taiwan, as shown in Fig.* 1), $k = 2.461 < \sqrt{2\pi}$, *the new model is not adequate.*

For convenient computation, submit eq.(28) *into eq.*(25) *and* *change it into the polar coordinate*

$$R^2 + R^2 (\sin\theta \cdot \cos\theta) - P \cdot R (\cos\theta + \sin\theta) = 0 \qquad (32)$$

$$\frac{1}{4} (\frac{R}{P})^2 \cdot \sin^2 2\theta + [(\frac{R}{P})^2 - 1] \sin 2\theta + [(\frac{R}{P})^2 - 1] = 0 \qquad (33)$$

let $\frac{R}{P} = A$, $\sin 2\theta = Z$

$$Z^2 + \frac{4 [A^2 - 1]}{A^2} Z + \frac{4 [A^2 - 1]}{A^2} = 0 \qquad (34)$$

again, let $\frac{4 [A^2 - 1]}{A^2} = B$

$$Z^2 + BZ + B = 0$$

The solution is

$$Z = \frac{- B \pm \sqrt{B (B - 4)}}{2} \qquad (35)$$

Use computer for computation, if we have k value (from the field data), then R, P, A, B can solved. Consequently, the values of Z, θ, x, y, a, b can fiud out. Therefore, the new distribution for combined wave spectrum is to be worked out.

5. *COMPARISON WITH TRADITIONAL DISTRIBUTION AND DISCUSSION*

Based on the wave statistical result of 21 *sets of field data in Taiwan Strait, the values of k are to be enumerated. They range from* 2.50 *to* 2.95. *After choicing k =* 2.600, 2.660 *and* 2.723. *The new distribution curves can be dipicted in figure* 4. *Also Rayleigh's curves of the same case are delineated.*

From these figures, it can be realized that the new distribution approaches reality much better than traditional Rayleigh's

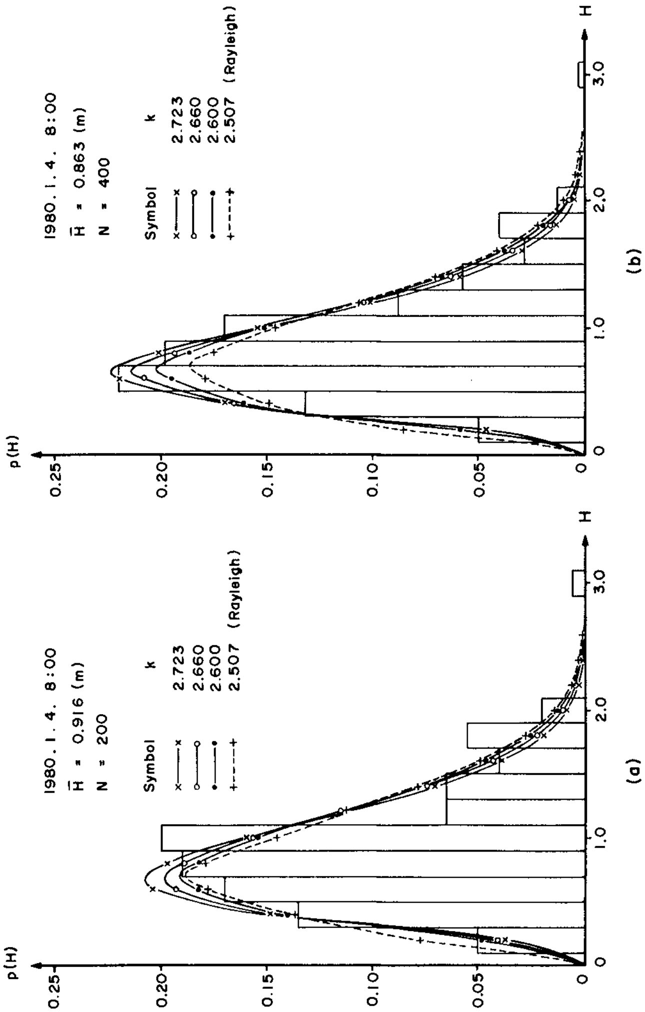


Fig.4 New probability density curve and measured data (1)

density function.

However, in the case of Linko, the winter monsoon attacks the seashore near vertically, the local wind waves are not able to be generated. The assumption of spectral combination is not suitable and the curve of new distribution can not be work out because the values of k is less than $\sqrt{2\pi}$.

In the summer of 1982, *the senior author was invited by Franzius Institute of The University of Hannover to be guest professor. He had the opportunity to analyze the wave records of Sylt. The k value is found to be* 2.46 *for the wind wave is moving depressions are almost approaching frontally to the coast. However, the waves in German Bight are similar to western coast of Taiwan. The wave height distributions over there are to be supposed to fit our new distribution.*

6. *CONCLUSIONS AND SUGGESTION*

1. *The new probability density function of the wave heights fits the reality better than Rayleigh's distribution.*
2. *The new model derived from the idea of the energies of local and refracted waves being combined together, especially conform to situation that the wind direction is parallel to the shoreline.*
3. *At the areas like Linko, local wave can't occur easily, the new model can not be used.*
4. *The forecasting of wave height distribution will be available, provided that the winds can be accurately predicted off the western coast of Taiwan and areas of similar condition.*
5. *Although the new density function approaches reality better than Rayleigh distribution. But the assumption of only two spectra combination will not too sound. Waves in the point of interest are come from every available direction and not able to be divided. However, from the clue used in this paper, it*

can be assumed to be that the wave energy is the linear sum of $S(f)\triangle f_i$, *and* $\triangle f_i$ *is so selected that every* $\triangle E_i = S(f)\triangle f_i$ *are equal, and eq.*(9) *will be a reasonable function to represent the distribution of wave heights if n is suitablely selected.*

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