CHAPTER 64

A NEW APPROACH FOR TIDAL COMPUTATIONS

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During the twenty last years, tidal modelling has been intensively developed. Following the growth of engineering needs in coastal areas, more and more accurate models have been established, and this constant research of better accuracy in the representation of real phenomena brings us to very expansive models. One way of reducing these costs is to use variable grids in space, in order to concentrate refined meshes in areas of interest. But the finite difference schemes are not well adapted to this kind of procedure : this is why several attempts have been made recently to use finite element technics : C. TAYLOR and J.M. DAVIS in 1975 [11] , C.A. BREBBIA and P.W. PARTRIDGE in 1976 [2] , ... But these applications are not easy.

During the same period, since 1975, more complex tentative have been made using Fourier transform of the equations, previously to any kind of numerical integration : tides are effectively quasi periodic phenomena, and their spectra are well known. Two important points arise in doing this :

- time variable is eliminated from the hyperbolic problem of propagation, transformed into a set of elliptic problems.
- for each elliptic problem, a variational formulation is available.

It becomes thus possible to look at the various components of the real tides, and to use finite element technic to integrate numerically these problems in real basins. In this way, B.M.JAMART and D.F. WINTER have used recently a purely numerical procedure based upon the Fast Fourier Transform to carry their tidal computations in fjords, cf. [5], while A. ASKAR and A.S. CAKMAK introduced a perturbation technic to handle the non linearities, very important in such problems, cf. [1] . We have followed a similar approach to study the complete spectrum of the tides in shallow water areas for the european seas : North Sea and English Channel, cf. [9] . The aim of this paper is to illustrate the main ideas of our method applied on an academic one-dimensional problem.

I. THE EQUATIONS.

In the study of the dynamics of tidal waves in shallow waters, the long wave equations are classically used. They are obtained from the Navier Stokes equations by integration over the vertical coordinate, under the assumption that the characteristic vertical scale H is much smaller than the horizontal scale L ($H/L \ll 1$). With this assumption, it can be shown that the pressure is hydrostatic. Without any meteorological effect at the sea surface, and neglecting the horizontal eddy viscosity, the NS equations reduce to :

$$(1.1)\quad \begin{aligned} &\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} - \tilde{f}\tilde{v} + g\frac{\partial \tilde{\zeta}}{\partial \tilde{x}} + \frac{\tilde{c}}{\tilde{h}+\tilde{\zeta}}\sqrt{\tilde{u}^2+\tilde{v}^2}\,\tilde{u} = 0 \\ &\frac{\partial \tilde{v}}{\partial \tilde{t}} + \tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}} + \tilde{f}\tilde{u} + g\frac{\partial \tilde{\zeta}}{\partial \tilde{y}} + \frac{\tilde{c}}{\tilde{h}+\tilde{\zeta}}\sqrt{\tilde{u}^2+\tilde{v}^2}\,\tilde{v} = 0 \end{aligned}$$

Similarly, the continuity equation can be written :

$$(1.2)\quad \frac{\partial \tilde{\zeta}}{\partial \tilde{t}} + \frac{\partial (\tilde{h}+\tilde{\zeta})\tilde{u}}{\partial \tilde{x}} + \frac{\partial (\tilde{h}+\tilde{\zeta})\tilde{v}}{\partial \tilde{y}} = 0$$

with :

- $\tilde{x}$, $\tilde{y}$: horizontal cartesian coordinates in the plane of undisturbed sea surface
- $\tilde{z}$: vertical coordinate
- $\tilde{t}$: time
- $\tilde{h}$: undisturbed depth of water
- $\tilde{\zeta}$: elevation of the sea surface
- $\tilde{u}$, $\tilde{v}$: components of the depth averaged currents in the $\tilde{x}$, $\tilde{y}$ directions

$$(1.3)\quad \tilde{u} = \tilde{u}(\tilde{x},\tilde{y},t) = \frac{1}{\tilde{h}+\tilde{\zeta}}\int_{-\tilde{h}}^{\tilde{\zeta}} \tilde{u}(\tilde{x},\tilde{y},\tilde{z},\tilde{t})\,d\tilde{z},\quad \tilde{v} = \tilde{v}(\tilde{x},\tilde{y},t) = \frac{1}{\tilde{h}+\tilde{\zeta}}\int_{-\tilde{h}}^{\tilde{\zeta}} \tilde{v}(\tilde{x},\tilde{y},\tilde{z},\tilde{t})\,d\tilde{z}$$

- $\tilde{f}$: Coriolis parameter ($\tilde{f} = 2\tilde{\Omega}\sin\lambda$, with $\tilde{\Omega} = \frac{2\pi}{24h}$ and λ : latitude of point $(\tilde{x},\tilde{y})$)
- $\tilde{c}$: coefficient of quadratic bottom friction
- g : acceleration due to gravity.

Tidal problems are generally solved in areas $\mathcal{D}$ limited by coastal boundaries Γ_1 and open boundaries Γ_2. Along Γ_1 the classical boundary condition is $\vec{V}_n = 0$ (impermeability of coastal lines). Along Γ_2, several conditions are used :

$$(1.4)\quad \tilde{\zeta} = \zeta^*(\tilde{x},\tilde{y},\tilde{t}) \quad \text{or} \quad \tilde{V}_N = V_N^*(\tilde{x},\tilde{y},\tilde{t}),\ \text{normal velocity to } \Gamma_2$$

f^* being a given function on $(\tilde{x},\tilde{y}) \in \Gamma_2$, for all t .

It should be noted that, with formulation (1.1), tides are assumed to be induced in $\mathcal{D}$ by the open boundaries Γ_2. But the method here presented can be applied to the more general case of an oceanic basin influenced by the tide generating potential (cf. C. LE PROVOST and A. PONCET, 1977 [8]).

II. GENERAL PRESENTATION OF THE SPECTRAL METHOD.

II.1. Dimensionless equations.

In order to simplify, it is convenient to use non dimensional variables :

$$x = \frac{\tilde{x}}{\tilde{L}}\ ,\ y = \frac{\tilde{y}}{\tilde{L}}\ ,\ \zeta = \frac{\tilde{\zeta}}{\tilde{H}}\ ,\ h = \frac{\tilde{h}}{\tilde{H}}\ ,\ t = \frac{\tilde{t}}{\tilde{L}/\tilde{c}}\ ,\ u = \frac{\tilde{u}}{\tilde{c}}\ ,\ v = \frac{\tilde{v}}{\tilde{c}} \text{ with } \tilde{c} = \sqrt{gH}$$

$$\Omega = \frac{\tilde{\Omega}}{\tilde{c}/\tilde{L}}\ ,\ \omega = \frac{\tilde{\omega}}{\tilde{c}/\tilde{L}}\ ,\ A = \frac{\tilde{A}}{\tilde{c}}\ ,\ A' = \frac{\tilde{A}'}{\tilde{H}}\ ,\ k = \tilde{c}\,\frac{\tilde{L}}{\tilde{H}}$$

Thus equations (1) are :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - 2\Omega \sin\lambda\, v + \frac{\partial \zeta}{\partial x} + \frac{k}{h+\zeta}\sqrt{u^2+v^2}\ u = 0$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + 2\Omega \sin\lambda\, u + \frac{\partial \zeta}{\partial y} + \frac{k}{h+\zeta}\sqrt{u^2+v^2}\ v = 0 \qquad (2.2)$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} + \frac{\partial \zeta u}{\partial x} + \frac{\partial \zeta u}{\partial x} = 0$$

which can be written :

$$MS = B + E \qquad (2.3)$$

with

$$M = \begin{vmatrix} \frac{\partial}{\partial t} & -2\Omega\sin\lambda & h\frac{\partial}{\partial x} \\ 2\Omega\sin\lambda & \frac{\partial}{\partial t} & h\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial t} \end{vmatrix} \qquad S = \begin{vmatrix} hu \\ hv \\ \zeta \end{vmatrix}$$

$$B = \begin{vmatrix} -h\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) \\ -h\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) \\ -\frac{\partial}{\partial x}(\zeta u) - \frac{\partial}{\partial y}(\zeta v) \end{vmatrix} \qquad E = \begin{vmatrix} -\frac{kh}{h+\zeta}\sqrt{u^2+v^2}\ u \\ -\frac{kh}{h+\zeta}\sqrt{u^2+v^2}\ v \\ 0 \end{vmatrix} = \begin{vmatrix} f_x \\ f_y \\ 0 \end{vmatrix}$$

II.2. Introduction of small parameters procedure.

We know from the theory of oceanic tides, and from observations, the structure of the tidal spectrum at the open boundary Γ_2. We can thus suppose that (1.4) are of the form :

$$\tilde{U}_N = \sum_{i=1}^{N_p} \tilde{A}_i \tilde{U}_{N_i}^* \cos(\tilde{\omega}_i \tilde{t} + \tilde{\psi}_{N_i}^*) \qquad U_N = \sum_{i=1}^{N_p} A_i U_{N_i}^* \cos(\omega_i t + \psi_{N_i}^*)$$

(2.4) or $\tilde{\zeta} = \sum_{i=1}^{N_p} \tilde{A}_i' \tilde{\zeta}_i^* \cos(\tilde{\omega}_i \tilde{t} + \tilde{\varphi}_i^*)$ i.e. or $\zeta = \sum_{i=1}^{N_p} A_i' \zeta_i^* \cos(\omega_i t + \varphi_i^*)$

where $U_{N_i}^*, \psi_{N_i}^*$, and ζ_i^*, φ_i^* are given functions on Γ_2 corresponding to the N_p i tidal components of pulsation ω_i inducing the movement in $\mathcal{D}^p$ through the open boundary ; the orders of magnitude of each of these components are characterized by parameters A_i or A_i'.

When vectors B and E are neglected, resolution of (2.3) with boundary conditions (2.4) is not difficult (cf. HANSEN, 1962 [4]). But in coastal areas, non linearities are important. In order to handle the different orders of magnitude of these non linearities, we have introduced a perturbation method (cf. J. KRAVTCHENKO and C. LE PROVOST, 1977 [6]). Solutions of (2.3) are considered under the form :

$$S = A_i S_{i1} + A_i^p S_{ip} + A_i^q A_j^r S_{ijqr} + \ldots \tag{2.5}$$

where i, j = 1, 2, ... N_p and p, q, r = 1, 2, ... ∞
Taking (2.5) in (2.3), and gathering the terms of same power in A_i, we obtain for each order of magnitude in A_i a set of equations defining S_{i1}, S_{ip}, S_{ijqr}, ... :

$$\begin{aligned} M\, S_{i1} &= 0 \\ M\, S_{ip} &= B_{ip} + E_{ip} \\ M\, S_{ijqr} &= B_{ijqr} + E_{ijqr} \\ &\ldots\ldots \end{aligned} \tag{2.6}$$

with corresponding limit conditions coming from (2.4)

Following a classical procedure in forced vibration theory, solutions of (2.6) are expected under the form :

$$S_{i1} = \begin{vmatrix} h\, u_{i1}(x,y) \cos[\omega_i t + \psi_{i1}(x,y)] \\ h\, v_{i1}(x,y) \cos[\omega_i t + \chi_{i1}(x,y)] \\ \zeta_{i1}(x,y) \cos[\omega_i t + \varphi_{i1}(x,y)] \end{vmatrix}$$

$$S_{ip} = S_{ip}^{(a,b,c..)} = \begin{vmatrix} h\, u_{ip}^{(a,b,c..)} \cos[(a\omega_1 + b\omega_2 + c\omega_3 + ..)t + \psi_{ip}^{(a,b,..)}] \\ h\, v_{ip}^{(a,b,c..)} \cos[(a\omega_1 + b\omega_2 + c\omega_3 + ..)t + \chi_{ip}^{(a,b,..)}] \\ \zeta_{ip}^{(a,b,c..)} \cos[(a\omega_1 + b\omega_2 + c\omega_3 + ..)t + \varphi_{ip}^{(a,b,..)}] \end{vmatrix} \tag{2.7}$$

$$S_{ijqr} = \ldots\ldots$$

with a, b, c, .. = 0, $\pm$ 1, $\pm$ 2, ... ∞
The resolution of each system (2.6) is consequently splitted into a set of systems of the following form, of which time is eliminated :

$$\bar{M}_{il} \cdot \bar{S}_{il} = 0$$

(2.8)

$$\bar{M}_{ip}^{(a,b,c..)} \cdot \bar{S}_{ip}^{(a,b,c..)} = \bar{B}_{ip}^{(a,b,c..)} + \bar{E}_{ip}^{(a,b,c..)}$$

where S_{il} are vectors of 6 unknows of two variables x and y :

$hu_{il}\cos\psi_{il}$, $hu_{il}\sin\psi_{il}$, $hv_{il}\cos\chi_{il}$, $hv_{il}\sin\chi_{il}$, $\zeta_{il}\cos\varphi_{il}$, $\zeta_{il}\sin\varphi_{il}$
(and the same for $S_{ip}^{(a,b,c..)}$, ...).

The matrices $\bar{M}_{il}$, $\bar{M}_{ip}^{(a,b,c..)}$ are easy to deduce from (2.3), (2.6) and (2.7) ; it is the same for vectors $B_{ip}^{(a,b,c..)}$, $B_{iqr}^{(a,b,c..)}$, ... But an important difficulty arises for vector E, the terms of which are not analytical in the vicinity of small values of the parameters A_i, and impossible "a priori" to develop under a form $E_{ip}^{(a,b,c..)}$.

II.3. Development of the quadratic terms of friction.

We have established an approximate development of vector E, in the form of generalized Fourier series (cf. C. LE PROVOST, 1973 [7]), under the assumption of the existence of a "dominant" wave in the complete tidal spectrum over the studied area. This "dominant constituent" must have everywhere in $\mathcal{D}$ a maximum of velocity much bigger than the other constituents in the spectrum. As an example, the M_2 constituent of the tide is the "dominant" wave for the european seas. Taking index 1 for this "dominant" constituent, f_x and f_y are expanded as follows :

$$f_x = A_i A_j k \sum_{i=1}^{N} FX_i \cos(\omega_1 t + \phi X_i)$$

(2.9)

$$f_y = A_i A_j k \sum_{i=1}^{N} FY_i \cos(\omega_1 t + \phi Y_i)$$

where FX_i, FY_i, ϕX_i, ϕY_i are functions of the amplitude and the phase of the dominant constituent, and of the other constituents of the spectrum. N is theoretically infinite, but in fact it can be limited to a finite value N_F ($\geqslant N_p$).

The aim of this paper is not to present the details of this development. Let us notice only that two classes of terms can be distinguished in (2.9).

a. A first group corresponds to the damping effect played by friction for all the constituents in the tidal spectrum. Considering these terms, it appears that :

- For the dominant wave, this damping can be considered independently of the other constituents of the spectrum, as a first approximation.

- For the other constituents, this damping is strongly influenced by the local characteristics of the dominant wave.

b. A second group of terms corresponds to linear combinations of the pulsations of the different constituents of the complete tidal spectrum : they represent the generating effect of new constituents played by friction in shallow water areas.

II.4. The perturbation method.

In coastal areas, friction is so much important that a simple small parameter method applied as presented in II.2 to solve (2.2) is not rapidly converging towards the real solution within practical limits : this has been noticed in 1971 by B. GALLAGHER and W. MUNK. It is necessary to use a perturbation method in which the first approximation of the solution is already representative of the damped solution. The developments (2.9) show us the way : the first order solution must be the dominant wave studied in the presence of the damping effect of friction, and the other constituents of the tide will appear as perturbations, studied separatly as prescribed by a classical process of successive orders of approximation, cf. [6].

III. ILLUSTRATION OF THE PERTURBATION METHOD APPLIED TO A MONO DIMENSIONAL PROBLEM.

Let us consider a channel $\mathcal{C}$ of constant depth h, closed at one end by a vertical wall, and connected with the ocean at the other end. The problem is reduced to a monodimensional one, with the following equations :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial \zeta}{\partial x} + \frac{kh}{h+\zeta}\,|u|\,u = 0$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial \zeta u}{\partial x} = 0 \qquad (3.1)$$

The limit conditions are expressed in such a way that they correspond to a monoperiodic wave coming inside the channel, of pulsation ω, and that every non linear wave induced in $\mathcal{C}$ by non linear processes is coming out of this channel through the limit x = 0 without any reflexion. This kind of radiation condition has been established from the theory of characteristics and is formulated as follows :

$$\tilde{u} + 2\sqrt{g(\tilde{h}+\tilde{\zeta})} = 2\tilde{A}\cos\tilde{\omega}\tilde{t} + 2\sqrt{g\tilde{h}} \qquad (3.2)$$

$$\text{i.e. } u + 2\sqrt{1+\zeta} = 2A\cos\omega t + 2$$

III.1. Development of the friction term.

Let us assume that, limited to the second order of approximation, the solution can be written :

$$u = \sum A^p u_p = A\, u_{11} \cos(\omega t + \psi_{11}) +$$
$$+ A^2 \left[u_{o2} + u_{12} \cos(\omega t + \psi_{12}) + u_{22} \cos(2\omega t + \psi_{22}) + \right.$$
$$\left. + u_{32} \cos(3\omega t + \psi_{32}) + \ldots \right] + O(A^3)$$

$$(3.3) \quad \zeta = \sum A^p \zeta_p = A\, \zeta_{11} \cos(\omega t + \varphi_{11}) +$$
$$+ A^2 \left[\zeta_{o2} + \zeta_{12} \cos(\omega t + \varphi_{12}) + \zeta_{22} \cos(2\omega t + \varphi_{22}) + \right.$$
$$\left. + \zeta_{32} \cos(3\omega t + \varphi_{32}) + \ldots \right] + O(A^3)$$

As it was said in § III.3, the friction term :

$$F = \frac{k}{1+\zeta} |u|\, u$$

can be expanded into a Fourier serie. We do not present here the details of the analytical development (see. C. LE PROVOST and A. KABBAJ, 1978, [10]) ; let us write only the result of these computations :

$$f = |u|\, u =$$

second order	* damping	$A^2 \cdot \frac{8 u_{11}}{3\pi} u_{11} \cos(\omega t + \psi_{11})$
	* generation of non linear constituents	$A^2 \frac{8}{15\pi} u_{11}^2 \cos(3\omega t + 3\psi_{11}) - \frac{8}{105\pi} u_{11}^2 \cos(5\omega t + 5\psi_{11}) + \ldots$
third order	* damping	$A^3 \frac{4 u_{11}}{\pi} \left[u_{o2} + u_{12} \cos(\omega t + \psi_{12}) + u_{22} \cos(2\omega t + \psi_{22}) + u_{32} \cos(3\omega t + \psi_{32}) + \ldots \right]$
	* generation of non linear constituents	$A^3 \left\{ \frac{4}{\pi} u_{11} u_{22} + \frac{4}{3\pi} u_{11} \left[u_{12} \cos(\omega t + 2\psi_{11} - \psi_{12}) + u_{32} \cos(\omega t + 2\psi_{11} - \psi_{32}) \right] - \frac{4}{15\pi} u_{11} \left[u_{32} \cos(\omega t + 4\psi_{11} - \psi_{32}) + u_{52} \cos(\omega t + 6\psi_{11} - \psi_{52}) \right] + \frac{4}{35\pi} u_{11} \left[u_{52} \cos(\omega t + 6\psi_{11} - \psi_{52}) + \ldots \right] + \ldots \right\}$

The different terms considered as "damping" terms appear to be a kind of linearization of friction. Using the notations :

$$(3.4)\quad \lambda = \frac{8k}{3\pi} Au_{11} \quad , \quad \lambda' = \frac{4k}{3\pi} Au_{11} \quad , \quad \lambda_{32} = \frac{8k}{15\pi} \quad , \quad \lambda_{52} = -\frac{8k}{105\pi}$$

F can be written :

$$F = \frac{1}{1+\zeta}\left[\lambda Au_1 + \lambda' A^2 u_2 + \lambda_{32} A^2 u_{11}^2 \cos 3(\omega t + \psi_{11}) + \right.$$

$$\left. + \lambda_{52} u_{11}^2 \cos 5(\omega t + \psi_{11}) + \ldots + O(A^3)\right]$$

i.e., using development : $(1+\zeta)^{-1} = 1 - A\zeta_1 + O(A^2)$

$$(3.5)\quad F = \lambda Au_1 - \lambda A^2 u_1 \zeta_1 + \lambda' A^2 u_2 + \lambda_{32} A^2 u_{11}^2 \cos 3(\omega t + \psi_{11}) +$$

$$+ \lambda_{52} u_{11}^2 \cos 5(\omega t + \psi_{11}) + \ldots + O(A^3).$$

We must notice that coefficients λ and λ', which can be called "linearized friction coefficients" are not constants and depend on the solution Au_{11} itself.

III.2. Application of the perturbation method.

First order : Following the formulation (2.6), the system giving the first order solution is :

$$(3.6)\quad \begin{aligned} &\frac{\partial \zeta_1}{\partial t} + \frac{\partial u_1}{\partial x} = 0 \\ &\frac{\partial u_1}{\partial t} + \frac{\partial \zeta_1}{\partial x} + \lambda u_1 = 0 \end{aligned}$$

with the limit conditions deduced from (3.2),(using development $(1+\zeta)^{1/2} = 1 + \frac{\zeta}{2} + \ldots$),:

$$(3.7)\quad \begin{aligned} &u_1(1) = 0 \\ &u_1(0) + \zeta_1(0) = 2\cos\omega t \end{aligned}$$

Notice that, doing this, we introduce in the definition of the first order solution, the damping effect of friction λu_1 which is, strictly speaking, a term of second order.

Let us use the complex notations :

$$(3.8)\quad \begin{aligned} \alpha_{ki} &= \frac{1}{2} \zeta_{ki} e^{j\varphi ki} \\ \mu_{ki} &= \frac{1}{2} u_{ki} e^{j\psi ki} \end{aligned}$$

i.e. $$\zeta_{ki}\cos(k\omega t + \varphi_{ki}) = \alpha_{ki}\, e^{jk\omega t} + \alpha_{ki}^{*}\, e^{-jk\omega t}$$

$$u_{ki}\cos(k\omega t + \psi_{ki}) = \mu_{ki}\, e^{jk\omega t} + \mu_{ki}^{*}\, e^{-jk\omega t}$$

with f^{*} being the conjugate of the complex function f. Equations (3.6) and (3.7) reduce so to :

$$j\omega\alpha_{11} + \frac{d\mu_{11}}{dx} = 0 \qquad \mu_{11}(1) = 0$$

(3.9) with

$$j\omega\mu_{11} + \frac{d\alpha_{11}}{dx} + \lambda\mu_{11} = 0 \qquad \alpha_{11}(0) + \mu_{11}(0) = 1$$

which can be written :

$$\alpha_{11} = \frac{j}{\omega}\frac{d\mu_{11}}{dx} \qquad \mu_{11}(1) = 0$$

(3.10) with

$$\frac{d^2\mu_{11}}{dx^2} + \omega^2\left(1 - j\frac{\lambda}{\omega}\right)\mu_{11} = 0 \qquad \mu_{11}(0) + \frac{j}{\omega}\frac{d\mu_{11}(0)}{dx} = 1$$

Finally, we have to solve a second order differential equation of the complex function μ_{11} of one variable only : x. This equation is non linear, because of the presence of $\lambda = \lambda(u_{11})$. We have solved this equation by a numerical finite difference scheme, using a method of successive approximations for the non linearity : as a first approximation, λ is taken equal to zero, which corresponds to the linear solution of (3.1) without friction.

A numerical application has been done with the following numerical values :

h = 50 m, L = 495 km, T = 12 h. 25 mn, A = 1 m/s, c = 3.10^{-3} MKSA (which schematically corresponds to the semi-diurnal tidal wave in the English Channel). On figure 1, we have plotted the amplitude of the sea surface elevation and of the current at x = 0, x = L/2 and x = L, obtained at the different steps of the iterative process used for the integration of (3.10). As it can be seen, the solution is stable after five iterations.

In order to check our solution, we have integrated problem (3.1) under the same limit conditions, with the same numerical values by a classical Lax Wendroff finite difference scheme. The solutions u(x,t) and ζ(x,t) thus obtained have been expanded by Fourier analysis under the form :

$$u^{LW}(x,t) = u_o^{LW}(x) + \sum u_k^{LW}(x)\cos\left[k\omega t + \psi_k^{LW}(x)\right]$$

(3.11)

$$\zeta^{LW}(x,t) = \zeta_o^{LW}(x) + \sum \zeta_k^{LW}(x)\cos\left[k\omega t + \varphi_k^{LW}(x)\right]$$

We have plotted on figure 2 and 3 the results for U_1^{LW} and ζ_1^{LW} in order to compare these values with u_{11} and ζ_{11} deduced from the integration of (3.10). The results fit very well.

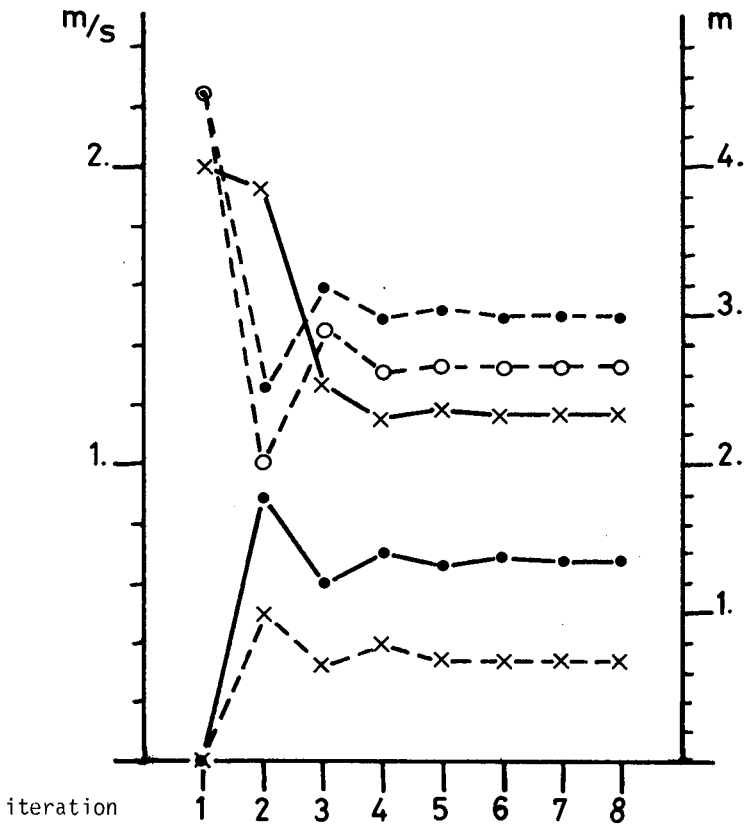

Fig. 1. Convergence of the iterative method

——— Velocity ----- Sea surface elevation
• mouth of the channel
x middle of the channel
o end of the channel

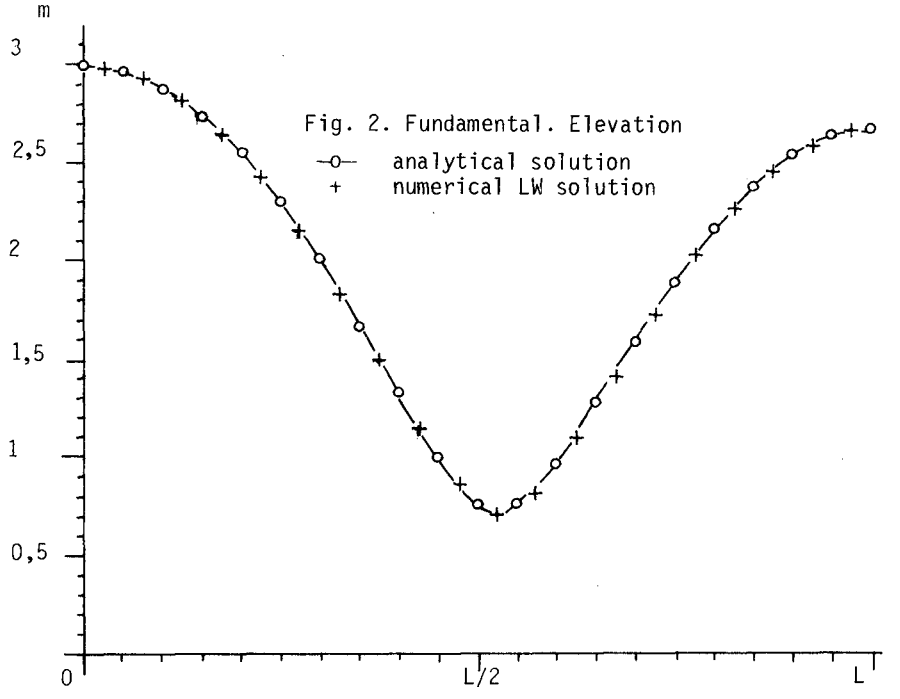

Fig. 2. Fundamental. Elevation
—o— analytical solution
\+ numerical LW solution

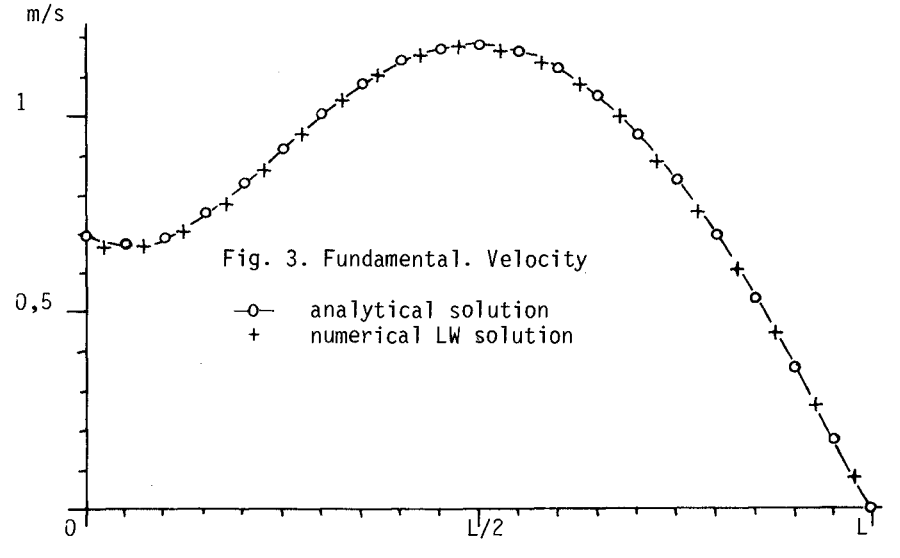

Fig. 3. Fundamental. Velocity
—o— analytical solution
\+ numerical LW solution

This numerical example illustrates the details of our method : the basic solution used for our perturbation procedure is the damped dominant wave ; it can be seen on figure 1 the important role played by the friction term : iteration 1 corresponds to no damping, and the corresponding solution is 1,8 the exact solution, for the maximum of amplitude of the velocity field. The agreement shown on figure 2 between our approximate solution and the Lax Wendroff solution, which can be considered the exact one, is sufficient to convince of the interest of the proposed method.

Knowing this dominant solution with a good accuracy, we calculate now the second order solution, following formulation (2.6).

Second order : The system giving the second order solutions is :

$$\frac{\partial \zeta_2}{\partial t} + \frac{\partial u_2}{\partial x} = -\frac{\partial (\zeta_1 u_1)}{\partial x}$$

$$\frac{\partial u_2}{\partial t} + \frac{\partial \zeta_2}{\partial x} = -\frac{1}{2}\frac{\partial u_1^{\,2}}{\partial x} - F_2 \qquad (3.12)$$

$$u_2(1) = 0$$

$$u_2(0) + \zeta_2(0) = \frac{\zeta_{1(0)}^{\,2}}{4}$$

with $$F_2 = \lambda' u_2 - \lambda u_1 \zeta_1 + \lambda_{32}\, u_{11}^{\,2} \cos 3(\omega t + \psi_{11}) + \lambda_{52}\, u_{11}^{\,2} \cos 5(\omega t + \psi_{11})$$

Using the complex notation defined in (3.8), the second members of (3.12) can be written :

$$\frac{\partial \zeta_1 u_1}{\partial x} = \frac{\partial}{\partial x}(\alpha_{11}\mu_{11}^{*} + \alpha_{11}^{*}\mu_{11}) + \left[\frac{\partial}{\partial x}(\alpha_{11}\mu_{11})e^{2j\omega t} + c.c.\right]$$

$$\frac{\partial u_1^{\,2}}{\partial x} = 2\frac{\partial \mu_{11}\mu_{11}^{*}}{\partial x} + \left[\frac{\partial \mu_{11}^{\,2}}{\partial x} e^{2j\omega t} + c.c.\right] \qquad (c.c. : \text{complex conjugate})$$

$$\zeta_1^{\,2} = 2\alpha_{11}\alpha_{11}^{*} + \left[\alpha_{11}^{\,2} e^{2j\omega t} + c.c.\right]$$

$$F_2 = \lambda' u_{o2} - \lambda\left[\alpha_{11}\mu_{11}^{*} + \alpha_{11}^{*}\mu_{11}\right] + \left[\lambda'\mu_{12} e^{j\omega t} + c.c.\right] \qquad (3.13)$$

$$+ \left[(\lambda'\mu_{22} - \lambda\,\mu_{11}\alpha_{11})e^{2j\omega t} + c.c.\right]$$

$$+ \left[(\lambda'\mu_{32} + 2\lambda_{32}\mu_{11}^{\,2} e^{j\psi_{11}})e^{3j\omega t} + c.c.\right]$$

$$+ \left[(\lambda'\mu_{52} + 2\lambda_{52}\mu_{11}^{\,2} e^{3j\psi_{11}})\, e^{5j\omega t} + c.c.\right]$$

(3.12) is a linear system of equations which therefore can be splitted up into differential systems of the variable x only :

Term of zero frequency H_o :

$$\frac{du_{o2}}{dx} = -\frac{d}{dx}\left(\alpha_{11}\mu_{11}^{*} + \alpha_{11}^{*}\mu_{11}\right)$$

$$\frac{d\zeta_{o2}}{dx} = -\left(\mu_{11}\mu_{11}^{*}\right)_x - \lambda' u_{o2} + \lambda\left(\alpha_{11}\mu_{11}^{*} + \alpha_{11}^{*}\mu_{11}\right) \tag{3.14}$$

$$u_{o2}(1) = 0$$

$$\zeta_{o2}(0) + u_{o2}(0) = \frac{1}{2}\alpha_{11}(0)\alpha_{11}^{*}(0)$$

Term of frequency 1 :

No forcing term occur in the corresponding equations deduced from (3.12) and (3.13). The corresponding solution is evidently 0. No correction of solutions u_1, ζ_1 occurs at the second order of approximation.

Term of frequency 2, H_2 :

$$\frac{d\mu_{22}}{dx} + 2j\omega\alpha_{22} = -\frac{d}{dx}\left(\alpha_{11}\mu_{11}\right)$$

$$\frac{d\alpha_{22}}{dx} + 2j\omega\mu_{22} = -\frac{1}{2}\frac{d}{dx}\left(\mu_{11}^{2}\right) - \lambda'\mu_{22} + \lambda\alpha_{11}\mu_{11} \tag{3.15}$$

$$\mu_{22}(1) = 0$$

$$\mu_{22}(0) + \alpha_{22}(0) = \frac{\alpha_{11}^{2}(0)}{4}$$

Term of frequency 3, H_3 :

$$\frac{d\mu_{32}}{dx} + 3j\omega\alpha_{32} = 0$$

$$\frac{d\alpha_{32}}{dx} + 3j\omega\mu_{32} = -\lambda'\mu_{32} - 2\lambda_{32}\mu_{11}^{2}\, e^{j\psi_{11}} \tag{3.16}$$

$$\mu_{32}(1) = 0$$

$$\mu_{32}(0) + \alpha_{32}(0) = 0$$

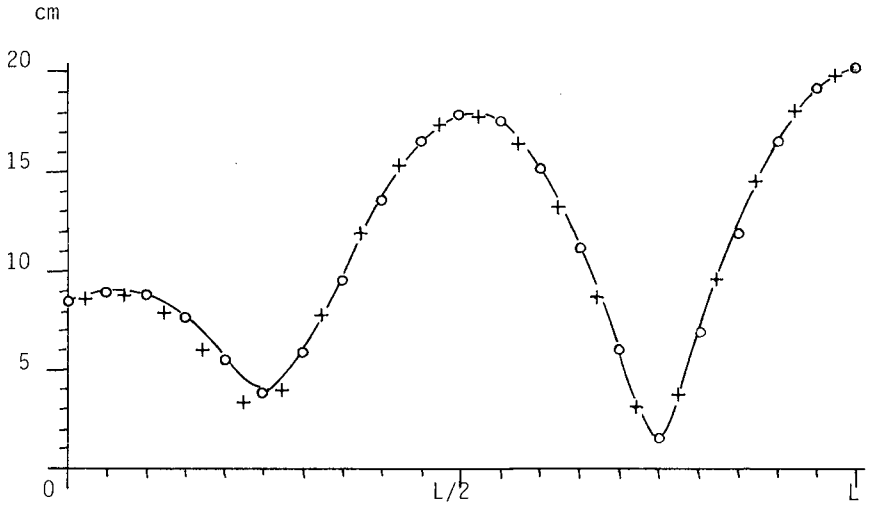

Fig. 4. Harmonic 2. Elevation

—o— analytical solution. 2nd order
\+ numerical LW solution

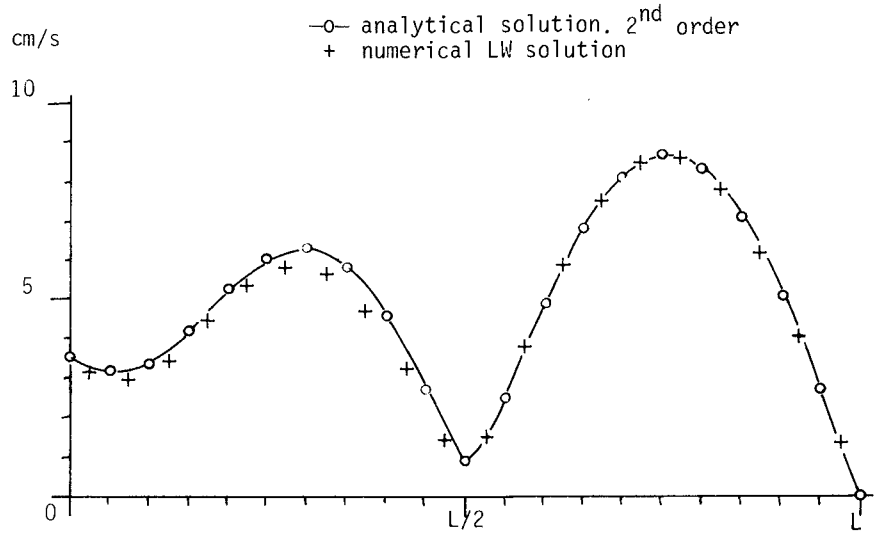

Fig. 5. Harmonic 2. Velocity

—o— analytical solution. 2nd order
\+ numerical LW solution

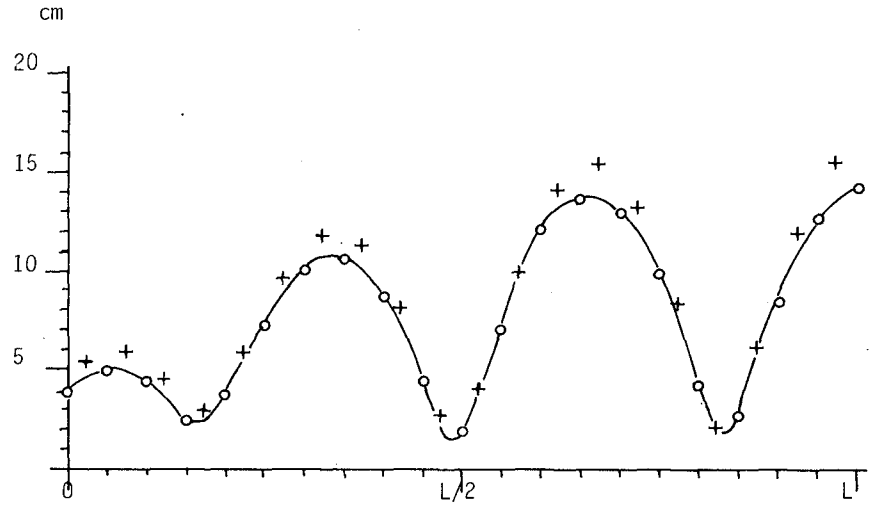

Fig. 6. Harmonic 3. Elevation

—o— analytical solution. 2nd order
\+ numerical LW solution

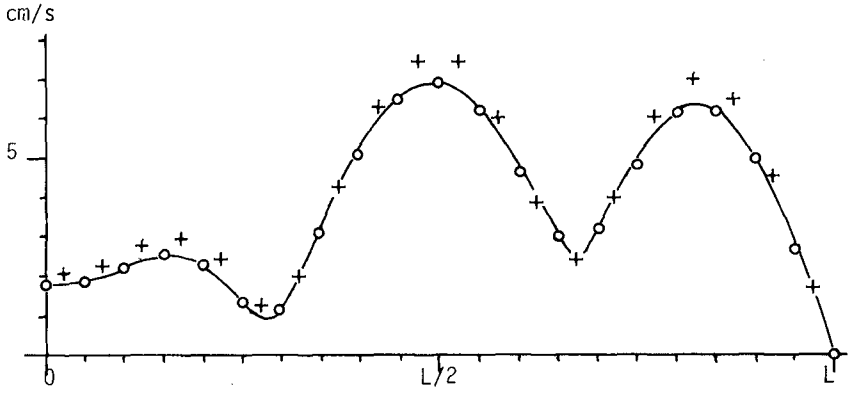

Fig. 7. Harmonic 3. Velocity

—o— analytical solution. 2nd order
\+ numerical LW solution

System (3.14) can be numerically integrated without any difficulty. Systems (3.15) and (3.16) are of the same kind as system (3.9), but with second members, and purely linear coefficients (λ' is function of the dominant solution u_{11} only). These systems can be numerically integrated without any difficulty.

With the numerical values already used to illustrate the computation of the dominant wave, we obtain solutions presented on figures 4 and 5 for the harmonic H_2, and on figures 6 and 7 for the harmonic H_3 (higher harmonics are too small to be considered). As comparison, on the same figures, the numerical solutions obtained from integration of the time dependent problem with the Lax Wendroff scheme are plotted. We can see that the correspondence is quite good. This agreement illustrates the ability of the method to reproduce the non linear harmonic constituents produced by sinusoïdal tidal waves propagating in coastal areas. Let us notice in (3.14), (3.15) and (3.16) the presence of damping terms $\lambda' u_{ij}$, with λ' function of the dominant velocity field u_{11} : as for the definition of the first order dominant wave, it is essential to take into account these damping terms for the computation of the non linear wave of second order.

In the numerical case here considered, the second order is sufficient to correctly represent the complete solution, but for smaller relative depth areas, the non linear contributions can be amplified, and higher order approximations may be necessary.

With this very simple mono-dimensional problem, the principal steps of the perturbation spectral method have been clearly illustrated :

- computation of the "dominant" solution : resolution of a non linear problem (damping coefficient being function of the solution itself) solved by an iterative process
- computation of the other components of the spectrum : resolution of linear problems (with damping coefficients fixed by the dominant solution).

IV. CONCLUSION. EXTENSION TO THE TWO DIMENSIONAL PROBLEM.

No important difficulty arises when the two dimensional problem is considered. Similarly to (3.10), the dominant wave is defined by a second order differential equation of the complex variable α_{11}, corresponding to the sea surface elevation ; because of the damping effect of friction, this equation is non linear. It can be shown that a variational formulation is available for this problem (C. LE PROVOST and A. PONCET, 1978 [9]), so that the natural way to realize numerical integrations in real basins is to use finite element methods. A first application has been done for the M_2 tide in the English Channel : the primilarly results published in [9] are satisfactory (cf. figures 8, 9). We are actually computing the M_4 constituent in the same area.

With such a procedure, computations are very cheap, because we have to solve a stationary problem for each important component of the tide in the studied area : except the dominant wave, for which an iterative process is necessary to take into account the non linear damping effect of bottom friction, the second order differential equation defining the amplitude of each constituent is solved only one time. It becomes thus

possible to realize a detailed study of all the components of the tidal spectrum in coastal areas, which has still not be realized in any case, because of excessive computing time necessities.

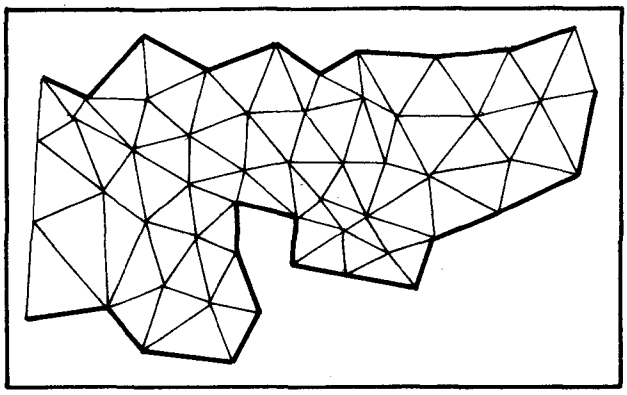

Fig. 8. Finite element grid for the English Channel

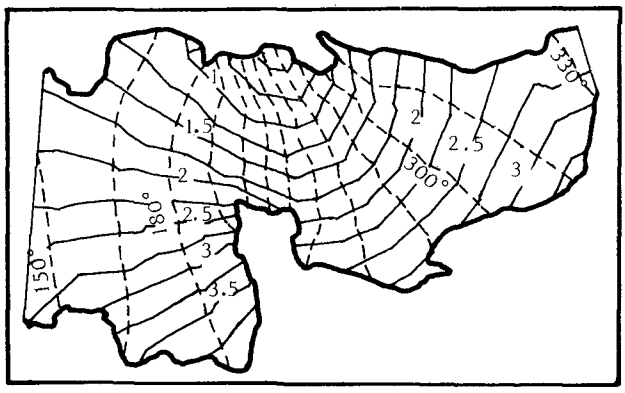

Fig. 9. Cotidal chart for M_2
from C. LE PROVOST and A. PONCET 9

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