CHAPTER 47

RIP CURRENT SPACING AS AN EIGENVALUE

by
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Abstract

Mass, momentum and wave action conservation laws, including the radiation stress, are used to obtain a rip current spacing as an eigenvalue. A coastal region is divided into two parts: offshore region and surf zone separated by a breaker line. Only the case of normal incidence of the waves is considered. From the matching conditions of the two horizontal velocity components at the breaker line, we can obtain rip current spacing as a function of a nondimensional parameter characterizing the surf zone, for an arbitrary value of a parameter indicating the strength of horizontal mixing.

Introduction

A coastal region with a linear bottom slope is divided into two parts: offshore region and surfzone separated by a breaker line. Wave set-up, wave energy and mean current are assumed to be composed of basic state, which is a function of only the distance from the shore and of superposed two dimensional perturbations.

In the case of normal incidence of the waves, basic steady current system vanishes and perturbations are found to be cellular. When we take into account the horizontal mixing in the surf zone, as LONGUET-HIGGINS(1970) did in the case of the steady longshore current, stream functions of the perturbed motions, which satisfy the boundary conditions at the coastline, can be represented by series expansion, wherein only one of the four roots of the indicial equation is selected. This selected one reduces to the value of the confluent hypergeometric function when the horizontal mixing is ignored(IWATA 1976):i. e. we have in the surf zone chosen a solution corresponding to the confluent hypergeometric equation modified by horizontal mixing, and the other solutions of the differential equation of fourth order are disregarded, partly because of the boundary conditions at

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the coast, partly to take account of the effect of the horizontal eddy viscosity due to wave breaking upon the solution of the inviscid model.

In the offshore region, where wave breaking does not occur, we disregard from the outset horizontal mixing so that the stream function can be represented by the modified Bessel function(IWATA 1976).

The matching conditions at the breaker line must be continuity of the two horizontal velocity components as well as surface elevation and energy. The continuity of energy follows immediately from the energy equation, but the elevation becomes discontinuous, because our model includes the effect of horizontal mixing only in the surf zone mainly due to the wave breaking, so that the elevations do not coincide with each other at the breaker line.

From the matching conditions of the two velocity components at the breaker line, we can obtain rip current spacing numerically as a function of a nondimensional parameter composed from bottom friction coefficient, bottom slope and the ratio of the breaker height to the depth for an arbitrary value of a parameter indicating a ratio of the horizontal mixing to the bottom stress as used by LONGUET-HIGGINS(1970).

§1. Basic equations

Mass, momentum and wave action conservation equations are given as follows,

$$\rho\frac{\partial\zeta}{\partial t} + \nabla\cdot\mathrm{M} = 0, \tag{1.1}$$

$$\frac{\partial\mathrm{M}}{\partial t} + \nabla\cdot(\mathrm{MU}+S) + \rho gh\nabla\zeta + \tau - \nabla\cdot(\mu h\nabla\mathrm{U}) = 0, \tag{1.2}$$

$$\frac{\partial}{\partial t}\left(\frac{E}{\sigma}\right) + \nabla\cdot\left[(\mathrm{U}+\mathrm{C}g)\frac{E}{\sigma}\right] + \frac{D}{\sigma} = 0. \tag{1.3}$$

Here ρ is the density of water assumed constant, $h = d+\zeta$ where d is local still water depth and ζ is the mean water level perturbation due to the presence of current and waves, M and U denotes the total mean momentum and the mean current respectively

$$\mathrm{M} = \tilde{\mathrm{M}} + \rho h\tilde{\mathrm{U}}, \qquad \mathrm{U} = \frac{\mathrm{M}}{\rho h},$$

where $\tilde{\mathrm{U}}$ shows the basic current, supposed independent of the depth and M denotes the mean wave momentum

$$\tilde{\mathrm{M}} = \langle\int_{d}^{\zeta}\rho u dz\rangle$$

where $\mathbf{u}$ is the velocity vector associated with the wave motion. $\nabla(=i\partial/\partial x + j\partial/\partial y)$ is the two dimensional gradient operator. μ denotes the eddy viscosity coefficient, D is the rate of energy dissipation mainly due to wave breaking. E shows the wave energy

$$E = \frac{1}{2}\rho g a^2 ,$$

where a is the local wave amplitude.

$$\sigma = \tilde{\sigma} - \mathbf{k}\cdot\mathbf{U}$$

σ is the frequency relative to a coordinate sytem moving with the current velocity $\mathbf{U}$ and $\tilde{\sigma}$ is the frequency relative to a fixed coordinate system. $\mathbf{k}$ shows wave number. Cg denotes group velocity and its magnitude is

$$Cg = \frac{\partial\sigma}{\partial k} = \frac{1}{2}c\left[1+\frac{2kh_0}{\sinh 2kh_0}\right]$$

$$c^2 = \frac{g}{k}\tanh kh_0$$

where $h_0=d+\zeta_0$ and ζ_0 is, as later shown, steady wave set-up.

The radiation stress tensor S and its components referred to the coordinate system shown in Fig.1 are

$$S = \begin{pmatrix} E\left[\frac{Cg}{c}(1+\cos^2\theta)-\frac{1}{2}\right] & E\frac{Cg}{c}\frac{\sin 2\theta}{2} \\ E\frac{Cg}{c}\frac{\sin 2\theta}{2} & E\left[\frac{Cg}{c}(1+\sin^2\theta)-\frac{1}{2}\right] \end{pmatrix}$$

The bottom stress averaged over one cycle τ is given by (IWATA 1976)

$$\tau = f\begin{pmatrix} U(1+\cos^2\theta)+V\frac{\sin 2\theta}{2} \\ V(1+\sin^2\theta)+U\frac{\sin 2\theta}{2} \end{pmatrix}$$

where $f = \frac{2}{\pi}\rho K v$, $v = \frac{a}{h_0}c$.

The bottom friction coefficient K is givem by(KAJIURA 1968)

$$K = \alpha\left(\frac{v}{\sigma z_0}\right)^{-2/3} , \quad \alpha = \text{const} ,$$

where z_0 is the roughness length of the bottom material.

For the coastal water considered now, we make the

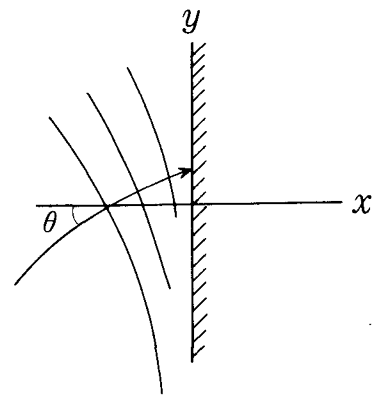

Fig.1 Coordinate axis and incident angle.

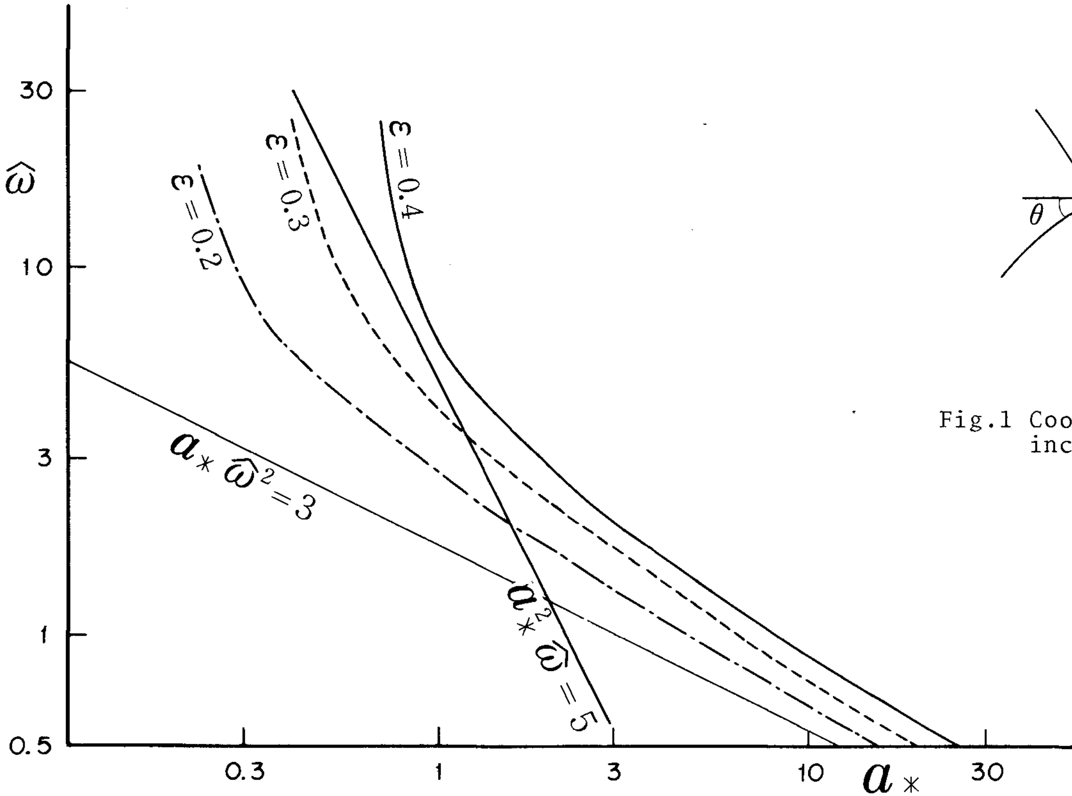

Fig.2 Nondimensional rip current wave number $\hat{\omega}$ as functions of a characteristic parameter a_*.

longwave approximation so that $c^2=gh_0$, moreover in the surf zone we assume $a=\gamma h_0$, $\gamma=0.4$. From the above approximations we get(IWATA 1976)

$$f = \begin{cases} f_0\sqrt{z} & ; \; z<1 \\ f_0 & ; \; z>1 \end{cases}$$

where $z = \frac{h_0}{h_*}$, $f_0 = \frac{2}{\pi}\rho\gamma K_*\sqrt{gh_*}$, $K_* = \alpha(\frac{\gamma}{kz_0})^{-2/3}$,

where h_* is the water depth at the breaker line. K_* is the bottom friction coefficient in the surf zone and becomes constant. μ denotes the horizontal mixing coefficient and we assume(LONGUET-HIGGINS1970)

$$\mu = \begin{cases} \rho N|x|\sqrt{gh_0} & ; \; h_0 \leqslant h_* \\ \rho N|x|\sqrt{gh_*} & ; \; h_0 \geqslant h_* \end{cases}$$

where N is constant.

From now on we consider a straight coastline where the local still water depth is linear: $d = -sx$, and assume that the field variables are composed of basic steady state as well as superposed perturbations,

$$\mathrm{U} = \mathrm{U}_0(x) + \delta\mathrm{U}_1(x,y,t),$$

$$\zeta = \zeta_0(x) + \delta\zeta_1(x,y,t),$$

$$E = E_0(x) + \delta E_1(x,y,t), \text{ etc.}$$

For the steady state we have

$$\nabla\cdot\mathrm{M}_0 = 0, \tag{1.4}$$

$$\nabla\cdot S_0 + \rho g h_0\nabla\zeta_0 + \tau_0 - \nabla\cdot(\mu h_0\nabla\mathrm{U}_0) = 0, \tag{1.5}$$

$$\nabla\cdot[(\mathrm{U}_0+\mathrm{C}g)\frac{E_0}{\sigma_0}] + \frac{D_0}{\sigma_0} = 0. \tag{1.6}$$

To study the perturbations we consider conveniently a case of normal incidence of the waves ($\mathrm{U}_0=0$), as well as steady state ($\partial/\partial t = 0$),

$$\nabla\cdot\mathrm{M}_1 = 0, \tag{1.7}$$

$$\nabla\cdot S_1 + \rho g h_0\nabla\zeta_1 + \tau_1 - \nabla\cdot(\mu h_0\nabla\mathrm{U}_1) = 0, \tag{1.8}$$

$$\nabla\cdot[\mathrm{C}g(E_1+E_0\frac{\mathrm{k}\cdot\mathrm{U}_1}{\sigma_0}) + \mathrm{U}_1E_0] + \frac{\mathrm{k}\cdot\mathrm{U}_1}{\sigma_0}D_0 = 0. \tag{1.9}$$

To derive the above equations we have assumed,

$$\frac{\mathrm{k}\cdot\mathrm{U}_1}{\sigma_0}D_0 \gg D_1, \quad h_0\nabla\zeta_1 \gg \zeta_1\nabla\zeta_0$$

where $D = D_0 + \delta D_1$.

Before we proceed to the rip current system, let us consider the basic state. For the case of normal incidence there is no basic current, $U_0 = 0$, so that eqs(1.5) and(1.6) can be transformed as follows,

$$\rho g h_0 \frac{\partial \zeta_0}{\partial z} + \frac{3\partial E_0}{2\partial z} = 0, \tag{1.10}$$

$$\frac{\partial}{\partial z}(E_0 Cg) + x_* D_0 = 0, \tag{1.11}$$

where $h_* = -sx_*$.

(i) In the surf zone $(h_0 \leqslant h_*)$ we assume as usual

$$E_0 = E_* z^2, \quad E_* = \frac{1}{2}\rho g \gamma^2 h_*^2,$$

then eq.(1.10) gives the basic wave set-up,

$$\zeta_0 = \zeta_* + \frac{3}{2}\gamma^2 h_* (1-z), \tag{1.12}$$

where ζ_* shows the value at the breaker line. In this case eq.(1.11) shows simply the rate of energy dissipation mainly due to the wave breaking, because the energy dissipation by the bottom stress is given by

$$\hat{D}_0 = \frac{8}{3\pi} K \frac{v}{h_0} E_0,$$

whereas(1.11) gives

$$D_0 = \frac{5s}{2h_0} E_0 \sqrt{g h_0} \quad ,$$

that is, energy dissipation by the bottom stress is much smaller than that due to wave breaking.

(ii) Outside the breaker zone$(h_0 \geqslant h_*)$ we can neglect energy dissipation, $D_0 = 0$, so that the energy flux becomes constant, $E_0 Cg = const.$ It follows then

$$E_0 = E_* z^{-0.5} \quad , \tag{1.13}$$

introducing (1.13) into (1.10) we have at once,

$$\zeta_0 = -\frac{\gamma^2 h_*}{4} z^{-1.5} \quad , \quad (z \geqslant 1) \tag{1.14}$$

From now on we take a new coordinate system with origin at the basic wave set-up shore line.

§2. Rip current system

From eq.(1.7) we get stream function as follows,

$$hU_1 = \frac{\partial\psi}{\partial y}, \quad hV_1 = -\frac{\partial\psi}{\partial x} .$$

From eqs.(1.8) and (1.9) we can derive an equation for the stream function. We can put in general $\psi=\psi\cos my$, $E_1=E_1\sin my$ and $\zeta_1=\zeta_1\sin my$, where m denotes the wave number of the rip current circulation cells.

(i) In the surf zone we have,

$$\tau_1 = f_0\sqrt{z}\left(\frac{2}{h_0}\frac{\partial\psi}{\partial y}, \quad -\frac{1}{h_0}\frac{\partial\psi}{\partial x}\right), \quad \mu h_0 = f_0\varepsilon|x_*|^2 z^{2.5},$$

where $\varepsilon = \dfrac{\pi sN}{2\gamma K_*}$.

This particular parameter ε denotes essentially the ratio of horizontal mixing to the bottom friction and plays an important role in the rip current system.

$$\eta\psi'' - \left(\frac{1}{4}+\eta\right)\psi' - \left(\frac{1}{2a_*} - \frac{1}{4}\right)\psi = \varepsilon P, \tag{2.1}$$

$$P = 4\eta^3\psi''''+9\eta^2\psi'''-\left(1+\frac{2\eta}{a_*}\right)\eta\psi''-\frac{1\eta}{4a_*}\psi'+\frac{1}{8a_*}\left(1+\frac{2}{a_*}\eta\right)\psi ,$$

where $a_* = \dfrac{\pi s\gamma}{8K_*}$, $\eta = \dfrac{1}{2}a_*(\hat{\omega}z)^2$, $\hat{\omega} = \sqrt{2}m|x_*|$.

When we ignore horizontal mixing(ε=0) eq.(2.1) turns out to be confluent hypergeometric equation (IWATA 1976).

The solution of eq.(2.1) can be put in the series,

$$\psi = A\eta^{\rho}\Sigma g_n\eta^n, \tag{2.2}$$

where $g_0 = 1$,

$$g_n h_0(\rho+n) + g_{n-1}h_1(\rho+n-1) + \cdots\cdots + g_0 h_n(\rho) = 0 ,$$

$$h_0(\rho) = \rho\left[(\rho-1)\left\{(\rho-2)\left(\rho-\frac{3}{4}\right)-\frac{1}{4}\right\}+\frac{1}{4\varepsilon}\left(\frac{5}{4}-\rho\right)\right] ,$$

$$h_1(\rho) = \frac{1}{2a_*}\left(\frac{1}{16}+\frac{7}{8}\rho-\rho^2\right) + \frac{1}{\varepsilon}\left\{\frac{1}{8}\left(\frac{1}{a_*} - \frac{1}{2}\right) + \frac{1}{4}\rho\right\},$$

$$h_2(\rho) = \frac{1}{16a_*^2} .$$

In general the indicial equation $h_0(\rho)=0$ gives four roots. For the case of no horizontal mixing $\rho=0$ and $\rho=5/4$ are obtained, however from the boundary conditions at the coast only $\rho=5/4$ was selected.

In order to see the effect of horizontal mixing upon the inviscid solution, we choose in this study one particular root of the indicial equation, which reduces to $\rho = 5/4$ for the case of no horizontal mixing, and the other roots are all omitted.

Energy in the surf zone can be obtained from eq.(1.9),

$$E_1 = \frac{1}{\sqrt{z}}\int_0^z Q\,dz \qquad (2.3)$$

where $\quad Q = \beta\hat{\omega}E_* z\left(\frac{\partial\psi}{\partial z} - \frac{1}{2}\frac{\psi}{z}\right) \; ; \; \beta = \frac{s}{h_*^2\sqrt{2gh_*}}$.

Perturbed wave set-up can also be expressed by the stream-function as follows,

$$\zeta_1 = \frac{-f_0}{\rho g h_*^2}\left[\frac{1}{2}\frac{h_*}{f_0}E_1 + \frac{1}{\hat{\omega}}\left(\frac{2}{z}\right)^{0.5}\frac{\partial\psi}{\partial z} - \varepsilon L\left(\frac{1}{z}\frac{\partial\psi}{\partial z}\right)\right] \qquad (2.4)$$

where $\quad L(\phi) = \left(z^{1.5}\frac{\partial^2}{\partial z^2} + \frac{1}{2}z^{0.5}\frac{\partial}{\partial z} - \frac{1}{2}\frac{1}{\sqrt{z}} - \frac{\hat{\omega}^2}{2}z^{1.5}\right)\phi$

(ii) Offshore region.
Outside the breaker zone we can put $E_0 = E_* z^{-0.5}$ and $f = f_0$. Moreover in this region we can neglect horizontal mixing compared to the surf zone, where the mixing is vigorous owing to breaking of the waves.

$$\xi^2\frac{\partial^2\psi}{\partial\xi^2} - (2+a_*\hat{\omega}^2)\xi\frac{\partial\psi}{\partial\xi} - (\xi^2 - 3a_*\hat{\omega}^2)\psi = 0 \qquad (2.5)$$

where $\quad \xi = \hat{\omega}z$.

The solution of the above equation is expressed by the modified Bessel function,

$$\psi = B\xi^{\alpha}K_{\nu}(\xi), \qquad (2.6)$$

where $\quad \alpha = \frac{1}{2}(3+a_*\hat{\omega}^2), \quad \nu = \pm\frac{1}{2}(3-a_*\hat{\omega}^2)$.

The energy in the offshore region is given by

$$E_1 = \frac{1}{\sqrt{z}}\left[\int_1^z R\,dz + E_1^*\right] \qquad (2.7)$$

where $\quad R = \beta\hat{\omega}E_* z^{-1.5}\left(\frac{\partial\psi}{\partial z} - 3\frac{\psi}{z}\right); \quad E_1^* = \int_0^1 Q\,dz$.

Perturbed wave set-up can be expressed,

$$\zeta_1 = -\frac{f_0}{\rho g h_*^2}\left[\frac{1 h_*}{2 f_0}E_1 + \frac{1}{\hat{\omega}}\left(\frac{2}{z}\right)^{0.5}\frac{\partial\psi}{\partial z}\right] \tag{2.8}$$

The matching conditions at the breaker line are the continuity of two horizontal velocity components as well as perturbed wave set-up. As we can see from eq.(2.7), energy is always continuous. Perturbed wave set-up outside the breaker zone is obtained under the assumption of no horizontal mixing, so that it becomes discontinuous on the breaker line even if two horizontal velocities are continuous on it.

At the breaker line $z = 1$, $\eta = \eta_0 = 1/2a_*\hat{\omega}^2$ and $\xi = \hat{\omega}$, accordingly the continuity condition of velocity components are

$$\begin{vmatrix} \Sigma g_n \eta_0^n & 1 \\ 2\Sigma g_n(\rho+n)\eta_0^n & \alpha+\nu-\hat{\omega}\dfrac{K_{\nu+1}(\hat{\omega})}{K_\nu(\hat{\omega})} \end{vmatrix} = 0 \ . \tag{2.9}$$

The eigenvalues $\hat{\omega}$ are numerically obtained and represented in Fig.2 as functions of a_* from eq.(2.9) for $\varepsilon=0.2 \sim 0.4$. In Fig.2 is also shown an empirical formula $\hat{\omega} = 5a_*^{-2}$. Another curve $\hat{\omega} = \sqrt{3}a_*^{-0.5}$ is reproduced from the previous paper(IWATA 1976), which is obtained for the limiting case of no horizontal mixing $\varepsilon=0$. Fig.3 shows experimental results(HORIKAWA and MIZUGUCHI 1975)from which the above experimental formula is derived.

From Fig.2 we can see remarkably good agreement with the empirical formula is obtained for the case $\varepsilon = 0.3$. It must be noticed this value $\varepsilon=0.3$ is also obtained for the basic longshore current, when we consider obliquely incident waves(LONGUET-HIGGINS 1970). For a practical application we can put for the range $\varepsilon = 0.3 \sim 0.4$,

$$\hat{\omega} = \begin{cases} 5.0a_*^{-2} & ; \quad a_* < 1.5, \\ 2.7a_*^{-0.5} & ; \quad a_* > 1.5. \end{cases}$$

In Fig.4 and Fig.5 we show an example of calculated stream function for $\varepsilon = 0.3$, $a_* = 0.6$ and $\hat{\omega} = 8.4$.

Conclusion

Some of the numerous studies of the near shore circulations insist that irregular bottom topography always exerts complete control over the water circulation, that is,

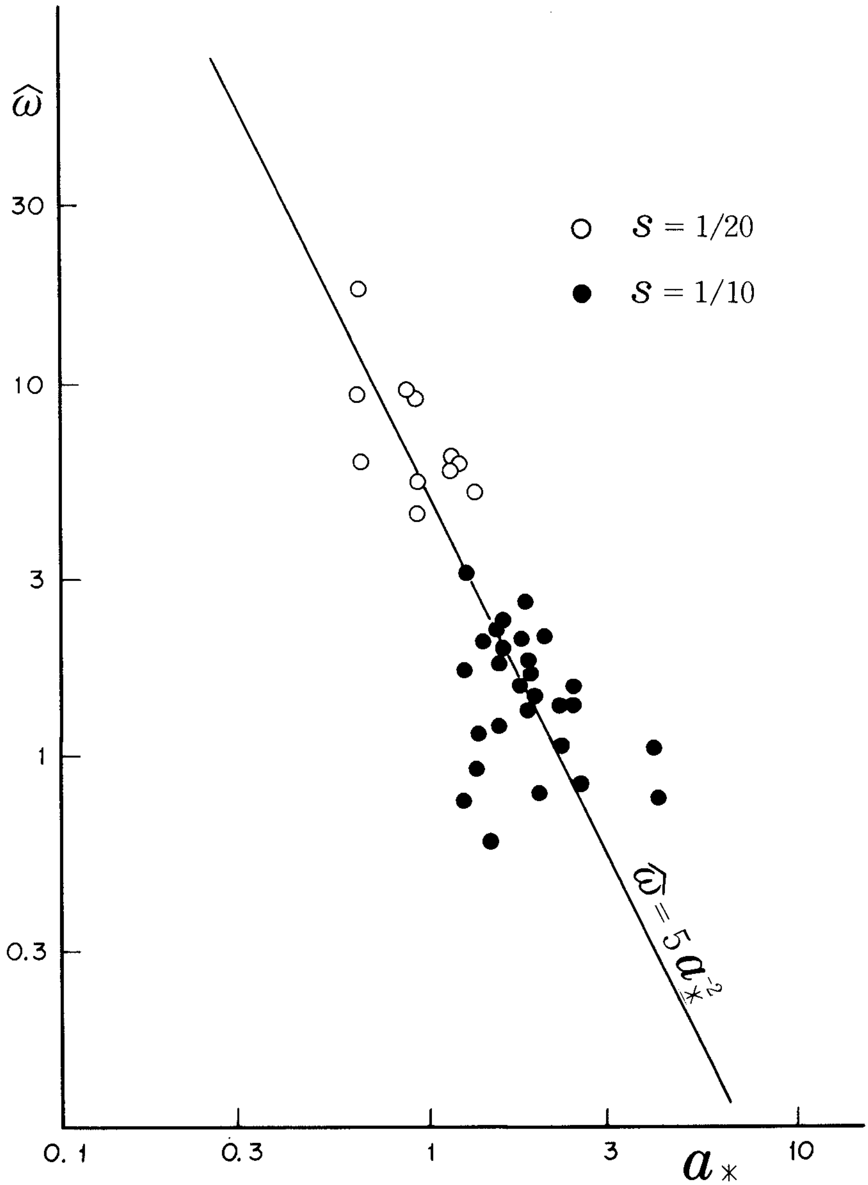

Fig.3 Experimental data and empirical relationship between rip current wave number $\hat{\omega}$ and a_*, where $z_0=1/30z_1$, $z_1=0.005$ cm are assumed to compute K_*.

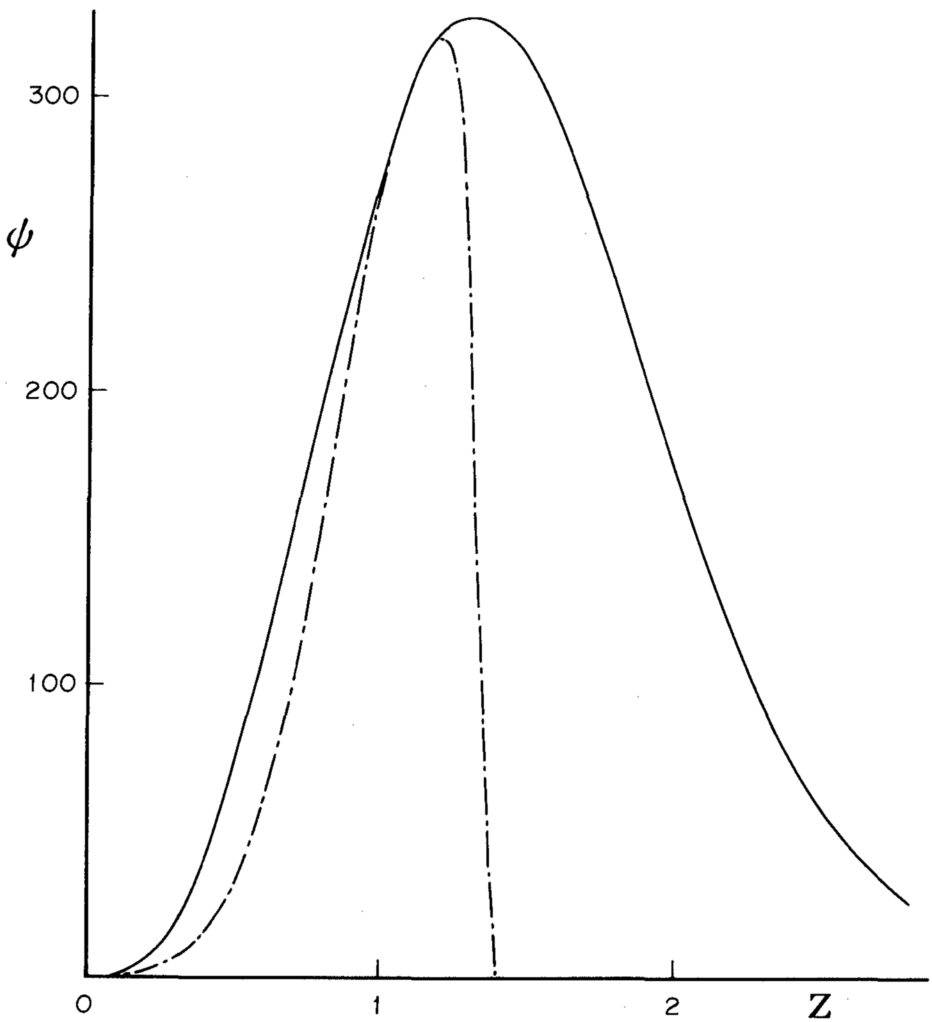

Fig.4 An example of stream functions on either side of breaker line z=1.
—— Obtained from offshore equation.
- - - - Obtained from surfzone equation.
α_*=0.6, ε=0.3 and $\hat{\omega}$=8.4.

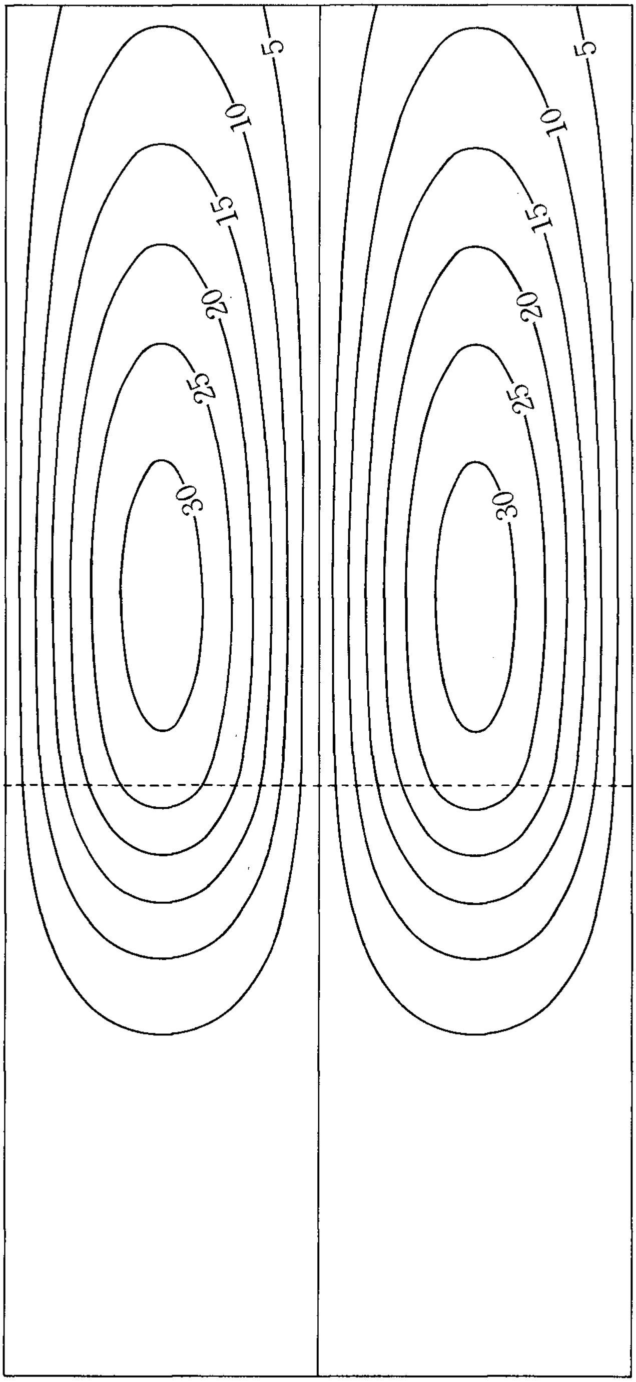

Fig.5 An example of rip current pattern for $a_*=0.6$, $\varepsilon=0.3$ and $\hat{\omega}=8.4$.

the topography is a primary cause to generate nearshore circulation. But there remains the basic question of how the original bottom deformation was formed if not by a cell circulation. Rip current can develop on smooth beaches without any bottom irregularities, presumably cause sediment transport and produce bottom topography. At some later stage the deformed bottom topography may be eventually effective to control the nearshore circulation pattern. Based on these reasonings attempt has been made to clarify the rip current spacing as an eigenvalue of a vorticity equation.

Our model is not self-consistent as far as we do not consider the horizontal mixing in the offshore region. When we take into account the effect of this horizontal mixing, right-hand side of eq.(2.5) is not zero but must be

$$\varepsilon[\xi^4\psi''''-2\xi^3\psi'''+(2-\xi^2)\xi^2\psi''-(2-\xi^2)\xi\psi'+\tfrac{1}{4}\xi^4\psi]$$

To find the solution which tends to zero at infinity and reduces to eq.(2.6) in the limiting case of $\varepsilon=0$ is a future problem.

Acknowledgment: Many thanks are due I.Watabe for his helpful programmings of the numerical computations.

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