CHAPTER 195

Finite Element Model for Estuaries with Inter-Tidal Flats

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Abstract

This paper deals with finite element formulations for the numerical computation of two-dimensional incompressible long-period shallow water waves. The described mathematical model is used to reproduce the dynamic situation occuring at the tidal propagation in estuaries. Areas which fall dry and wet again within a tidal cycle - so called inter-tidal flats - are taken into account.

Introduction

Since more than ten years the finite element method has been applied with considerable success in structural mechanics. In the last years the method was also used in fluid mechanics. Investigators like Grotkop [1], Connor, Wang [2], Davis, Taylor [3] and other made use of it to compute shallow water waves. But up to now inter-tidal flats - that are areas which fall dry and wet again within a tidal cycle - were not considered in these models. Only some investigators like for example Ramming [4] and Apelt, Gout, Szewczyk [5] took these areas into account using the finite difference method.

In the south eastern part of the North Sea in Europe many inter-tidal flats extend in front of the coast line. Without considering a natural phenomenon like that the hydrodynamic situation near these areas could be predicted only incompletely by a mathematical model.
Therefore calculation algorithms for inter-tidal flat elements and for normal ones were developed, which are coupled in one mathematical model. The coupling is useful for saving computing time because only for elements of the first type much effort has to be made to include the permanent changing geometry of the area.

Basic Differential Equations

For the present problem two vertically averaged, horizontal velocities v_i (i=0,1) defined in a Cartesian coordinate system and the height of water level h (see Fig.1) are introduced as unknown parameters. The differential equations for this two-

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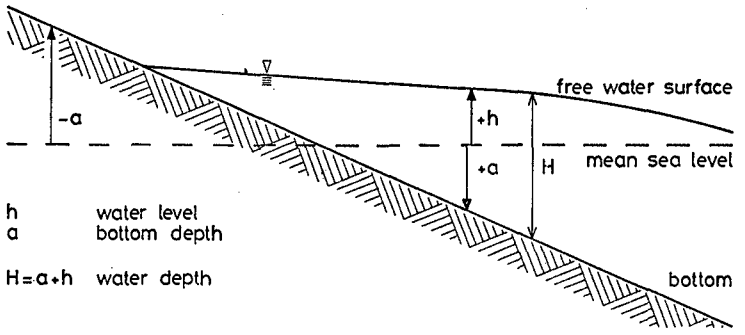

Fig.1 Definition Sketch Described in a Vertical Section

dimensional problem are obtained from the three-dimensional form of the continuity equation and the equations of motion by integrating over the depth (see e.g. Dronkers [6]).
The vertically integrated equations received in this way are the continuity equation

$$h_{,t} + q_{i,i} - q^* = 0 \tag{1}$$

and the equations of motion (2)

$$v_{i,t} + v_j v_{i,j} + g h_{,i} + \frac{\lambda \sqrt{v_j v_j}}{a+h} v_i - \Omega \varepsilon_{ij} v_j + \frac{1}{\rho} p_{o,i} - \frac{\mu_w \sqrt{W_j W_j}}{a+h} W_i = 0$$

with $i,j = 0,1$ to be summed. In the continuity equation the flux per length q_i will be replaced by the relation

$$q_i = (a+h)\, v_i \tag{3}$$

later. q^* is the inflow in the area of computation and will be specified afterwards.
In the equations of motion there are special terms for bottom friction, Coriolis force, atmospheric pressure and wind force as usual - g signifies the acceleration due to the gravity, λ a dimensionless friction parameter, Ω the Coriolis parameter, which is a function of the latitude, ε_{ij} an alternating tensor, ρ the density of the water, p_o the atmospheric pressure, μ_w a dimensionless wind friction parameter and W_i are components of the wind velocity.
In the equations $(\,)_{,i}$ and $(\,)_{,t}$ mean partial differentiations with respect to the coordinates x_i and to the time t.
Besides the friction term there are two further non-linear terms in the basic equations. For these the following linea-

rizations are realized:

$$q_i = (a + \bar{h})\, v_i + (h - \bar{h})\, \bar{v}_i \tag{4}$$

$$v_j v_{i,j} = \bar{v}_j v_{i,j} + v_j \bar{v}_{i,j} - \bar{v}_j \bar{v}_{i,j} \tag{5}$$

$\bar{h}$ and $\bar{v}_i$ are e.g. in time extrapolated values or values from the last step of iteration, when an iteration within each time step is used, or initial values of the time step when no better values are available.
Boundary conditions are the hydrographs of the water level or of the flux across the boundaries (see Fig.2):

$$h - \hat{h} = 0 \quad \text{on} \quad S_h \tag{6}$$

$$q_i n_i + \hat{q} = 0 \quad \text{on} \quad S_q \tag{7}$$

$\hat{h}$ signifies a prescribed water level and $\hat{q}$ a flux normal to the boundaries; $\hat{q}$ is positive when the water flows in the area of computation. On closed boundaries $\hat{q}$ is equal to zero.

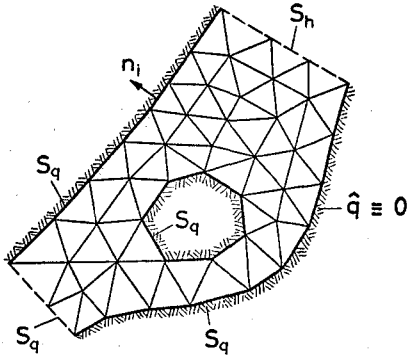

Fig.2 Designation of Boundaries

Finite Element Analysis

Because there exists no functional approach for the problem the method of weighted residuals is used and is the basic for

the application of the finite element method (see e.g. Zienkiewicz [7]).
The domain of computation is subdivided into finite elements. For the integration of the weighted differential equations elements in space and time are chosen with linear shape functions Θ_E and τ_T as described in Fig.3. In time the differential

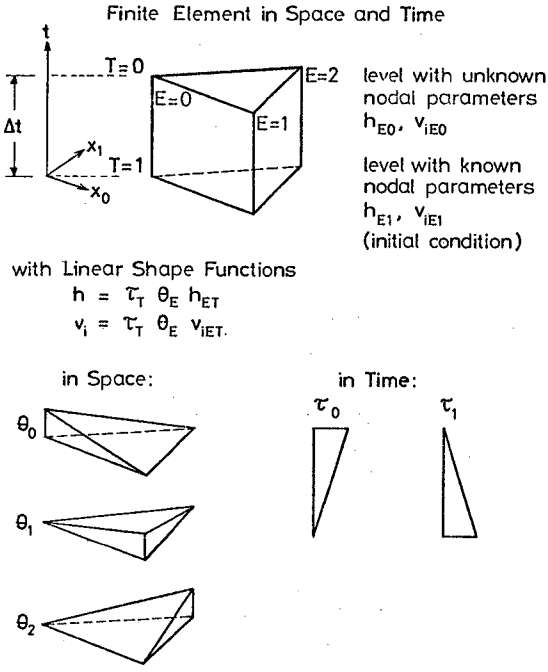

Fig.3 Discretization and Shape Functions

equations are solved in a stepwise manner as usual.
The weighting functions

$$\delta h = \bar{\tau}_0 \, \Theta_E \, \delta h_{E0}$$

and

$$\delta v_i = \bar{\tau}_0 \, \Theta_E \, \delta v_{iE0}$$

are functions of space and time; δh_{E0} and δv_{iE0} are arbitrary values. In the present model the functions in space Θ_E are identical with the shape functions (Galerkin method) though

in conformity with a new publication from Gärtner [8] better ones could be used. In time special weighting functions $\bar{\tau}_o$ (see Fig.4) are chosen to reduce the numerical damping. With

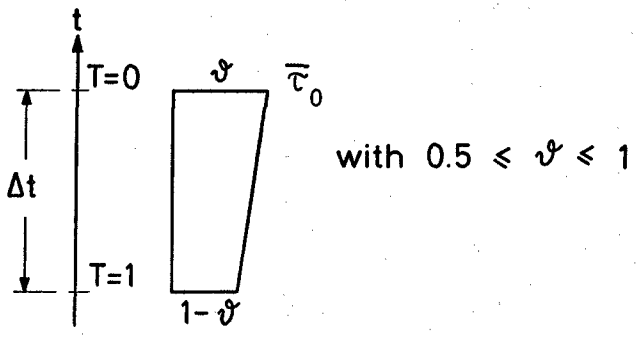

Fig.4 Weighting Function in Time

$\vartheta = 0.5$ the Crank-Nicolson time integrating factors are received and with $\vartheta = 1$ the method of Galerkin is again employed. The best value is near 0.5.
The continuity equation (1) is weighted with respect to δh

$$\int\int_{At} \delta h\,(h_{,t} + q_{i,i})\,dt\,dA - \int\int_{At} \delta h\,q\,dt\,dA - \int_{t} \delta h_K\,Q_K\,dt = 0 \quad . \tag{8}$$

Instead of q^* an areal distributed mass inflow q and a point source of mass inflow Q_K at node K of the system are introduced. After integrating by parts and insertion of the boundary conditions (7) on S_q equation (8) gets the form (9)

$$\int\int_{At} \delta h\,h_{,t}\,dt\,dA - \int\int_{At} \delta h_{,i}\,q_i\,dt\,dA = \int\int_{S_q t} \delta h\,\hat{q}\,dt\,ds + \int\int_{At} \delta h\,q\,dt\,dA + \int_{t} \delta h_K Q_K\,dt$$

and finally after substitution of q_i and linearization according to (4)

$$\int\int_{At} [\delta h\,h_{,t} - \delta h_{,i}\,([a+\bar{h}]v_i + [h-\bar{h}]\bar{v}_i)]\,dt\,dA = \int\int_{S_q t} \delta h\,\hat{q}\,dt\,ds + \int\int_{At} \delta h\,q\,dt\,dA + \int_{t} \delta h_K Q_K\,dt \quad . \tag{10}$$

The equations of motion (2) are weighted with respect to δv_i

$$\iint_{At} \delta v_i \left[v_{i,t} + v_j v_{i,j} + g h_{,i} + \frac{\lambda \sqrt{v_j v_j}}{a+h} v_i - \Omega \varepsilon_{ij} v_j + \frac{1}{\rho} p_{o,i} - \frac{\mu_w \sqrt{W_j W_j}}{a+h} W_i \right] dt\, dA = 0 . \quad (11)$$

After the linearization (5) these equations receive the shape

$$\iint_{At} \delta v_i \left[v_{i,t} + \bar{v}_j v_{i,j} + v_j \bar{v}_{i,j} - \bar{v}_j \bar{v}_{i,j} + g h_{,i} + \frac{\lambda \sqrt{\bar{v}_j \bar{v}_j}}{a+h} v_i - \Omega \varepsilon_{ij} v_j + \frac{1}{\rho} p_{o,i} - \frac{\mu_w \sqrt{W_j W_j}}{a+h} W_i \right] dt\, dA = 0 . \quad (12)$$

The integration of the weighted equations (10) and (12) is carried out in time t over a time interval Δt because of the stepwise solution algorithm and in space strictly speaking over the whole area A of the computation. As the finite element technique is used the integration in space can be reduced to an element area A_e. This leads to matrix equations of an element exsisting of nine equations with nine unknown parameters. As the integration have to be performed over the whole area A the matrix equations of all elements are added. So the big set of equations is received which have to be calculated in each time step.
The boundary condition (6) is inserted in the whole set of equations by erasing of lines and columns as usual in the finite element technique.

Area of Inter-Tidal Flats

Two basic problems appear when computing areas of inter-tidal flats in a mathematical model: First the difficulty to describe the physical situation near the changing water-land boundary (water boundary) and second the organizing problem when the boundaries of the mathematical model start to wander because some areas fall dry.
The author proposes the following procedure in principle for the solution of these problems:

- The discretization in space remains constant.
- Elements with at least one dry node in the end of a time step are approximately removed from the area of computation.
- In the partly flooded elements only the remaining volume of water is considered and is fixed as a function of the water level in the flooded nodes.

By these simplifications the dynamic is not exactly represented in the direct reach of the water boundary but the continuity is guaranteed. As this area is very small in comparison with the remaining model and the water depth mostly low the defect will be unimportant. On the other hand thus the whole organization of the program seems to be mastered in a

reasonable time.
To perform the proposed solution some postulated conditions have to be fulfilled:

- The boundaries of the flooded area are described by boundary integrals.
- In special nodes of the system an exactly defined mass inflow or outflow of water can be realized.
- An iteration is practicable within each time step to correct the actual boundary of computation and to improve the assumed volume of water in the partly flooded elements.
- A relaxation can be used to accelerate the iteration.
- For a better starting-point of the iteration the height of water level is extrapolated in the time direction with application of the least squares method.

In the following the procedure will be explained in detail. For the computation of a new time step there exist defined initial conditions of all parameters from the last time step. By an extrapolation in time with consideration of several known time levels the water level can be roughly predicted in all nodes.
Thus elements can be found which have three dry nodes at the end of a time step; these are dry elements (Fig.5). Further

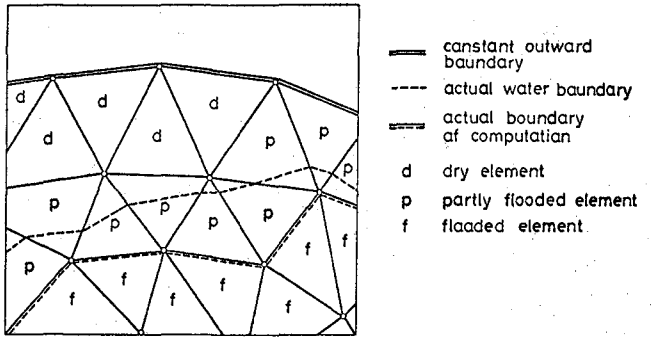

Fig.5 Various Boundaries in the Area of Inter-Tidal Flats

there are elements with one or two dry nodes, the so called partly flooded elements. All the other ones are flooded elements and these constitute the remaining mathematical model. The boundary between the flooded and the partly flooded elements - the actual boundary of computation - is described by boundary integrals. The actual water boundary is found by a horizontal extrapolation of the water level in adjacent flooded elements and is used to compute the remaining water volume (Fig.6). The difference of the water volume between

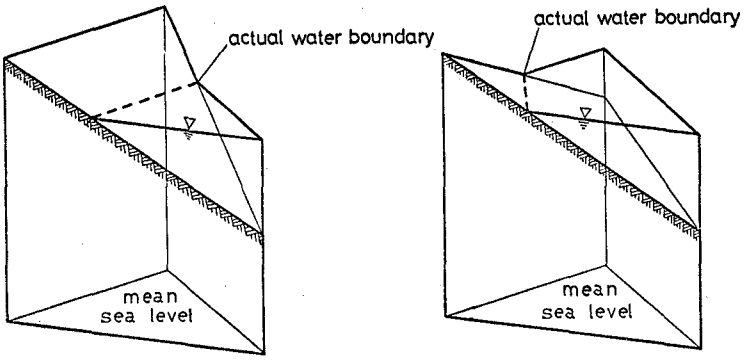

Fig.6 Partly Flooded Elements

the initial and the new state in the partly flooded and dry elements is given as a point source of mass inflow or outflow in or out of the remaining dynamic model in the same time step. By this the partly flooded elements effect the dynamic behaviour and correct the continuity in the flooded area. After computing the element matrices of the flooded elements and solving the set of equations the new water levels and velocities are received. These differ from the extrapolated values in general. By an iteration within the same time step the water levels are improved and by this the actual boundary of computation and the assumed volume of water in the partly flooded elements. The iteration is accelerated by a relaxation.

Numerical Computation

The program system MECCA (Modular Element Concept for Continuum Analysis) is used for the numerical computation. The modular constructed system (Fig.7) performs the always recurrent operations as being found in the finite element technique for boundary value problems or like in this case for initial and boundary value problems. MECCA manages organization problems such as input and output, data storage on disk packs, assembling and solving the set of equations and a graphic representation of the results. The input and partly the control over the sequence of program modules is managed by a problem oriented language (Fig.8). A special data organization (Fig.9) is prepared for the storage of the big and complex data sets.

The differential equations describing a physical process and the characteristics of an element are specified in a separate element program (Fig.7,8) which is linked to special modules

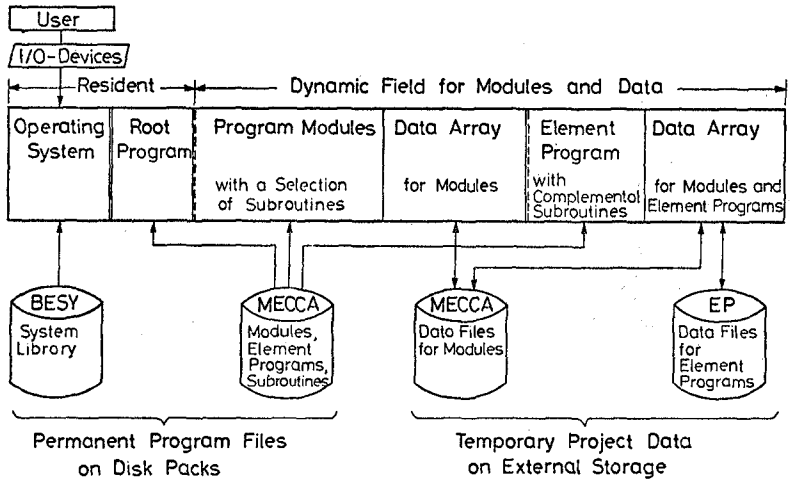

Fig.7 Allocation of the Computer by the Program System MECCA

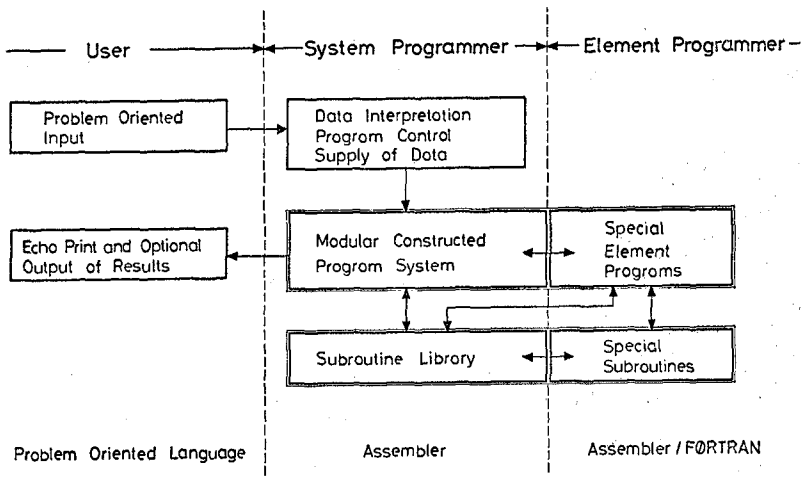

Fig.8 Schematic Organization of the Program System MECCA

of MECCA. Thus the possibility exists to couple different types of elements - in this case inter-tidal flat elements and normal ones - using common parameters in the connection nodes.
A first publication of MECCA happened by Beyer [9] and Herrling, Pfeiffer [10].

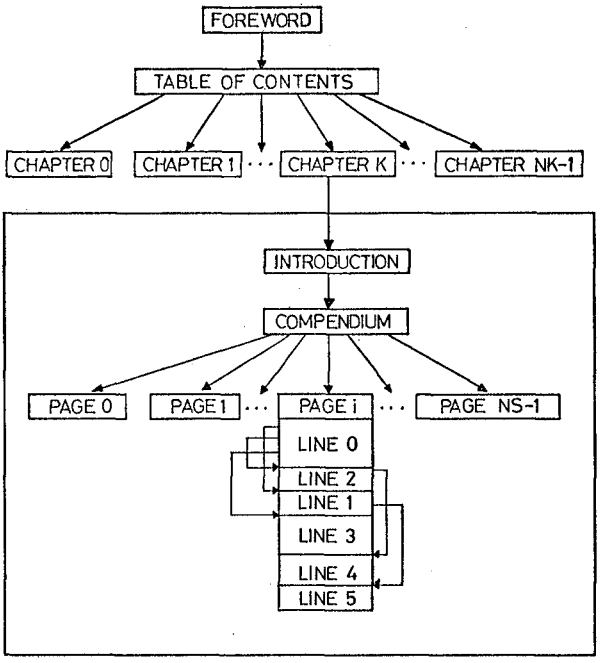

Fig.9 Set-up of a Data File

Numerical Results

The usefulness of the presented method is demonstrated by two examples: First a test of continuity is carried out in a basin falling dry partly, and in a second test the program is employed to the tide situation in the exterior Jade.
For the test of continuity a sloping basin has been chosen with a finite element network and a distribution of depth as shown in Fig.10. Three boundaries of the basin are closed and the other one is open. On this boundary the time dependent flux $\hat{q}$ is described (see Fig.11). The discharge is so propor-

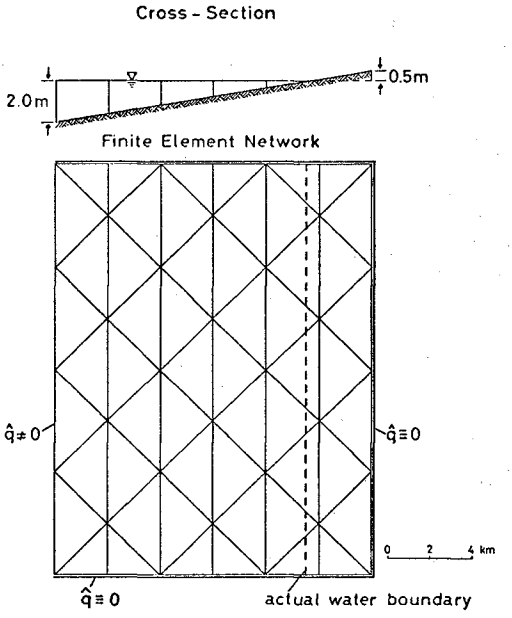

Fig.10 FE Network and Distribution of Depth

tioned that the water volume goes down for 1 m in the whole basin.

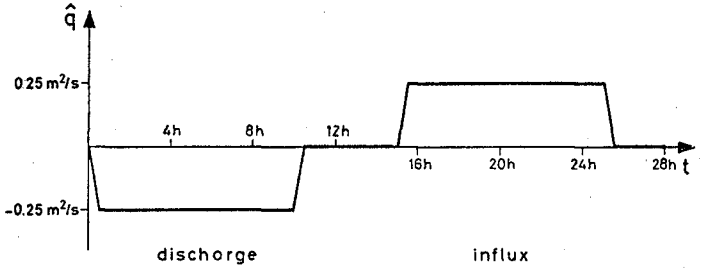

Fig.11 Prescribed Flux across the Boundary

In Fig.12 to 15 the numerical results are shown at different times. The time step Δt amounts to 30 minutes and 0.003 is the value of the friction coefficient.

Fig.13 State at 8 Hours

Fig.12 State at 4 Hours

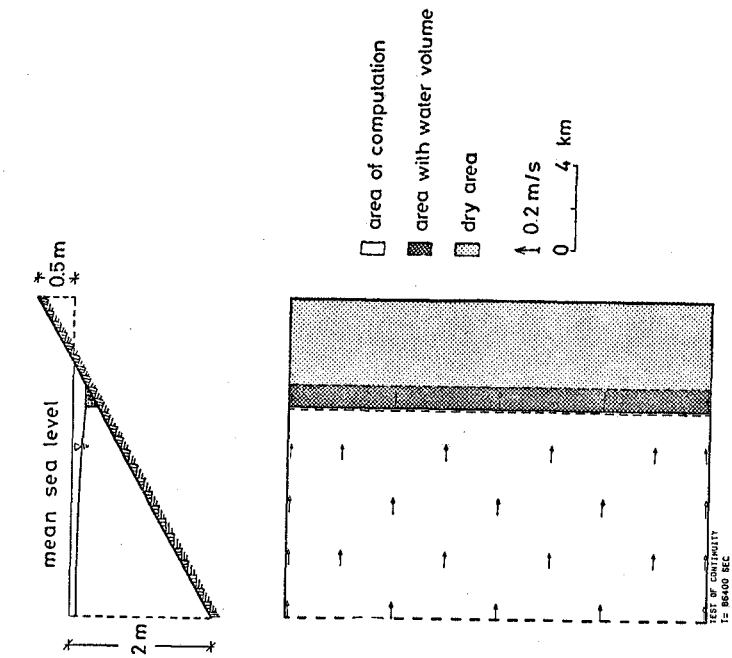

Fig.15 State at 24 Hours

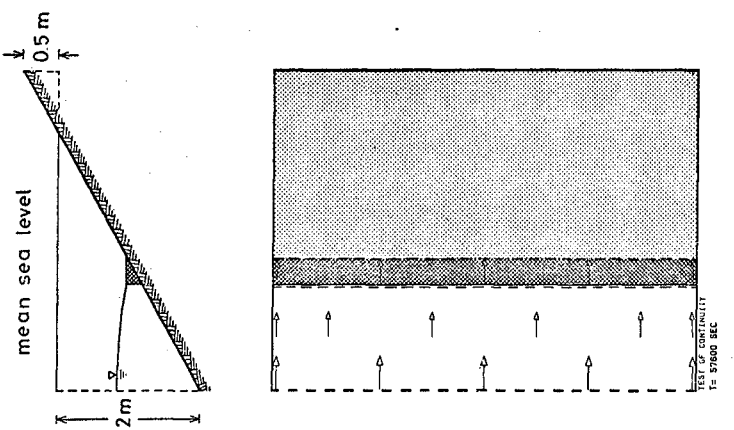

Fig.14 State at 16 Hours

As a second example the time dependend distribution of water levels and velocities in the exterior Jade (Fig.16) has been computed. The Jade is a German estuary in the south eastern part of the North Sea.

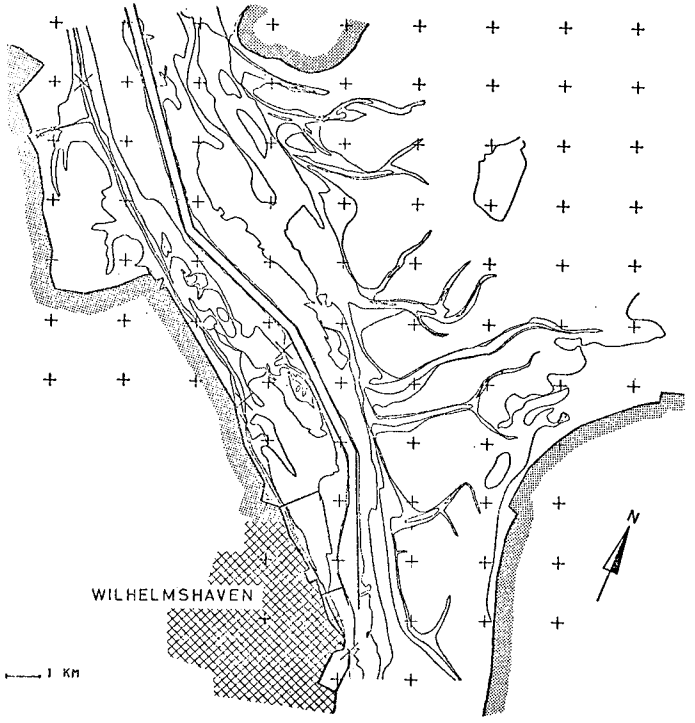

Fig.16 Topology of the Exterior Jade

Fig.17 shows the very coarse network of elements and Fig.18 the distribution of depth in the model. In this mathematical model two different types of elements are used (Fig.19). Boundary conditions are prescribed water levels at the open ends of the model (Fig.20).
The friction coefficient has the value 0.003 and the time step amounts to 10 minutes.
Fig.21 and 22 demonstrate the distribution of velocities and the areas of dry and partly flooded elements.

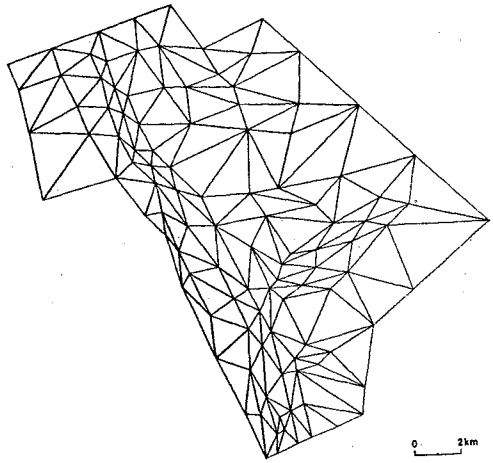

Fig.17 Network of Elements

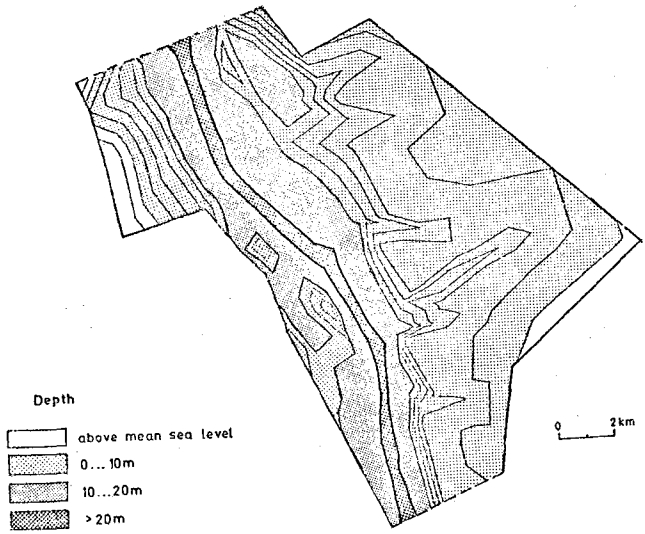

Fig.18 Distribution of Depth in the Model

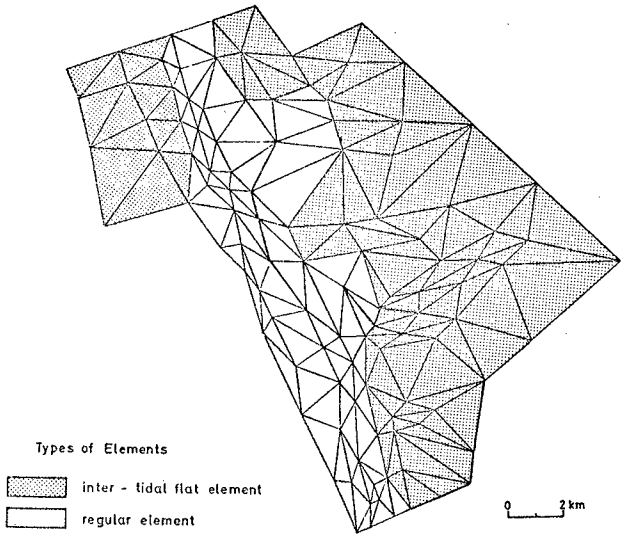

Fig.19 Used Types of Elements

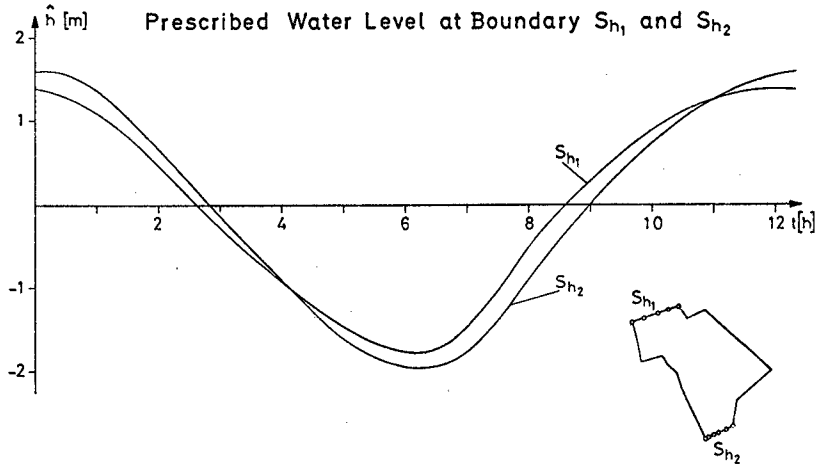

Fig.20 Boundary Conditions

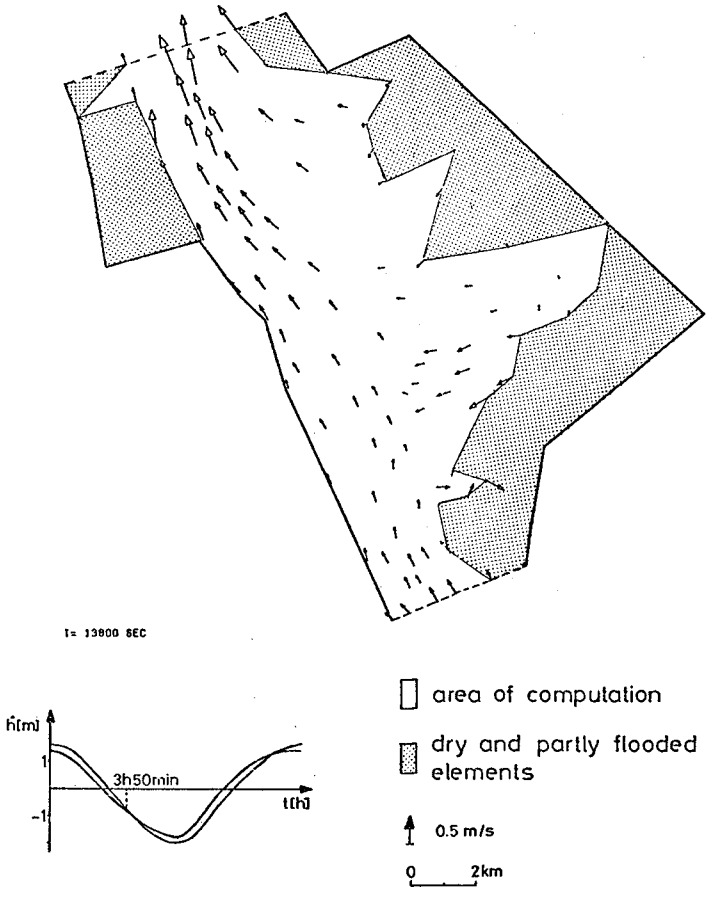

Fig.21 State at 3 h 50 min

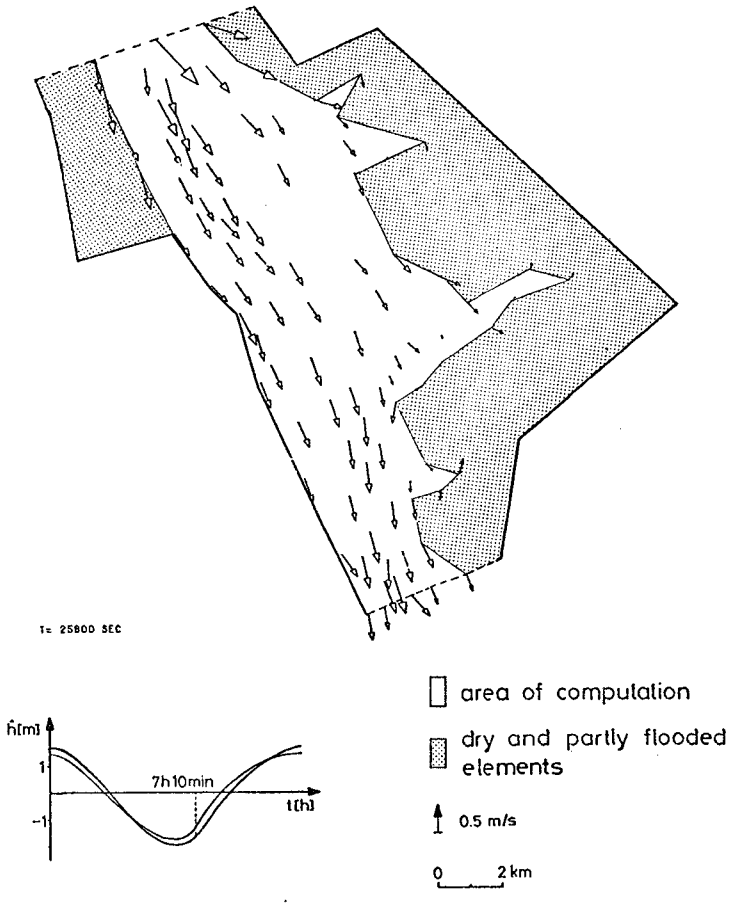

Fig.22 State at 7 h 10 min

Conclusion

The numerical results have shown that the new method is qualified to analyze the dynamic of estuaries with inter-tidal flats. In the mathematical model the dynamic behaviour is correct and the continuity of the water masses is guaranteed. The author hopes that he has provided a further contribution for a wide future application of the in hydrodynamics more and more advanced method of finite elements by the new opportunity to consider areas of inter-tidal flats in a mathematical model.

Acknowledgement

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