CHAPTER 192

DISPERSIVE TRANSPORT IN RIVER AND TIDAL FLOWS

by

R. B. Taylor[1]

ABSTRACT

Analytical results are presented which describe the mechanisms of longitudinal dispersive mass transport in rectangular channels of finite and infinite widths for both unidirectional (river) and oscillatory (tidal) flow regimes. Emphasis is placed upon the discussion of results and the characteristics of longitudinal dispersive mass transport revealed by the analytical treatment. Expressions presented for the dispersion coefficient were obtained from solutions to four sets of boundary value problems for the velocity and concentration variation components u" and c". Examination of these expressions reveals that in oscillatory flow the dispersive mass transport is described by a type of resonant interaction between the period of oscillation and the time scales of vertical and lateral mixing. The analysis also shows that for oscillatory flow regimes the effect of lateral shear becomes negligible for very wide channels and the three dimensional solution collapses to the two dimensional case in which vertical shear and mixing effects dominate. It is shown analytically that this is not the case in unidirectional flows. For this case the lateral shear and mixing effects dominate the corresponding vertical effects and dispersive mass transport increases without bound with increasing channel widths.

INTRODUCTION

Previous efforts (Bowden 1965; Holley et al. 1970; Fukuoka 1973) to describe the mechanism of dispersive mass transport in oscillatory flows have generally assumed that it is reasonable to describe this process through comparison by analogy to a similar process in a steady unidirectional flow. As will be seen from the results presented here, this assumption is misleading and can lead to erroneous results and conclusions. This is due to the fact that the dispersion process in oscillatory flow behaves quite differently from the corresponding process in unidirectional flow. Thus, unless the basic characteristics of dispersion are clearly understood for both types of flows, the interpretation of results can vary greatly depending upon the basis used for the comparison.

[1]Florida Coastal Engineers, Inc., Jacksonville, Florida

The mechanisms of dispersion in unidirectional flow are better understood. The most useful analytical treatment of this problem and the one which appears to provide the best results was developed by Fischer (1967). However, to apply Fischer's results one must have a detailed knowledge of the variations in the velocity field within the flow cross section. Moreover, his analysis assumes that the vertical effects of viscosity and diffusivity are small when compared to the same effects acting laterally across the channel. While this seems reasonable and has provided good results, it does not provide an analytical basis for examining those conditions under which either the vertical or lateral effects would play a dominant role in the dispersion process. Results presented in this paper allow for this comparison and are generally supportive of Fischer's work.

STATEMENT OF THE PROBLEM

The approach used to obtain expressions for the longitudinal dispersion coefficient generally follows the methodology devised by Taylor (1953) and applied later by others which defines the dispersive flux as the correlation over space and/or time of the velocity and concentration cross-sectional variation components by means of the following relationships:

$$-E_L \frac{\partial \bar{c}}{\partial \xi} = \overline{u'' c''} \qquad \text{(unidirectional flow)} \quad (1)$$

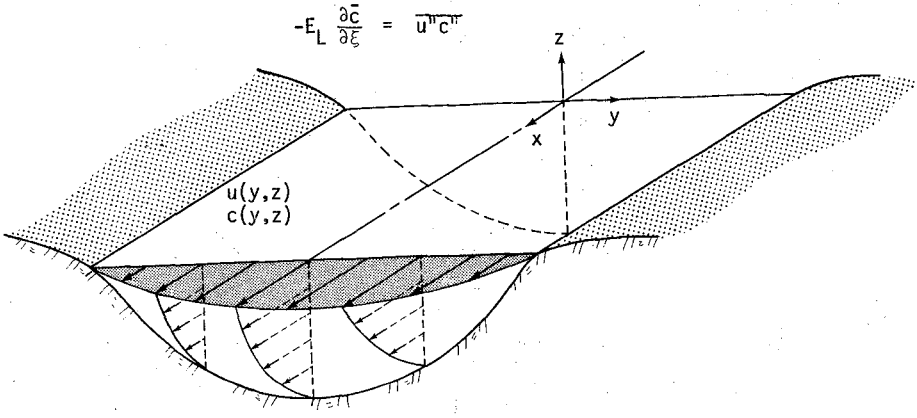
$$-\langle E_x \rangle_T \frac{\partial \bar{c}}{\partial \xi} = \langle \overline{u'' c''} \rangle_T \qquad \text{(oscillatory flow)} \quad (2)$$

where the overbar denotes the spatial average of the variable over the flow cross section, and the notation $\langle \ \rangle_T$ denotes an average over the period of oscillation, T. The variables u" and c" are defined by:

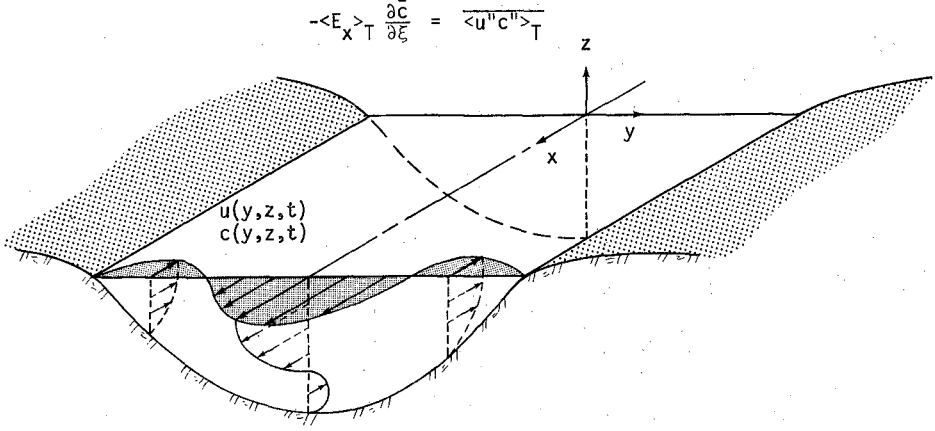
$$u'' = u - \bar{u} \qquad (3)$$

$$c'' = c - \bar{c} \qquad (4)$$

where u and c are the velocity and concentration variables, and $\bar{u}$ and $\bar{c}$ are the corresponding cross-sectional mean values. In carrying out the analysis, strict adherence to the use of companion solutions to the equation of motion and the transport diffusion equation was followed to obtain solutions for u" and c" which most nearly reflected the kinematic structure of the flow field of the particular problem being investigated. This is particularly important for the case of oscillatory flow regimes where, as illustrated by Figure 1b, flow reversals occur in the lower momentum regions of the channel which significantly affect u" during portions of the tidal cycle. These phenomena have been preserved here and represent a departure from the work of previous investigators in which temporal phase differences in the velocity field over the flow cross section have not been included (Holly and Harleman 1965; Okubo 1967; Holley, et al. 1970; Fukuoka 1973).



(a) RIVER



(b) TIDAL

FIGURE 1 - DESCRIPTIVE FLOW REGIMES AND ASSOCIATED DISPERSIVE MASS TRANSPORT RELATIONSHIPS

Expressions for the longitudinal dispersion coefficient were developed for the four combinations of flow and channel geometries previously mentioned. The analytical procedure was begun by solving the prescribed equation of motion for the velocity u for the particular flow regime and channel geometry under consideration. From this, an expression for u" was obtained which in turn was used to force the companion form of the transport diffusion equation. Solutions to the transport diffusion equation then yielded expressions for the concentration variation component, c". Expressions for the longitudinal dispersion coefficient were obtained from Equations (1) and (2) by averaging the product $Re(u") \cdot Re(c")$ over the flow cross section, and for oscillatory flow by averaging again over the period of oscillation.

Boundary Value Problem Formulation

Several assumptions must be introduced to obtain forms of the governing equations that can be solved in a reasonable manner. These are described as follows:

(a) The Boussinesq approximation is applied to the viscous terms in the equation of motion and the eddy viscosity coefficients, ε_y and ε_z, are introduced. These coefficients are considered to be constant and equal to their spatial and temporal mean where appropriate.

(b) Eddy diffusivity coefficients, $\overline{K}_z$ and $\overline{K}_y$, are used to formulate the turbulent diffusion terms in the transport diffusion equation in a manner similar to that described for the viscous terms.

(c) For purposes of the simplification of results and physical interpretation it is assumed that the eddy coefficients of viscosity and diffusivity are of equal magnitude. Thus, $\overline{K}_z = \overline{\varepsilon}_z$, and $\overline{K}_y = \overline{\varepsilon}_y$.

(d) For steady unidirectional flow the pressure field is independent of time. If the flow is uniform then it can be assumed that

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = K \qquad (5)$$

where K is constant.

(e) For oscillatory flow the pressure field is assumed to be temporally periodic, thus

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = Ke^{i\sigma t} \qquad (6)$$

where σ is the angular frequency of oscillation and t is time.

(f) The velocity and concentration variables in oscillatory flow are assumed to be temporally periodic and of the form

$$u = u_s e^{i\sigma t} \qquad (7)$$

$$c'' = c''_s e^{i\sigma t} \qquad (8)$$

where u_s and c''_s are complex variables of the spatial coordinates defining position within the flow cross section. This formulation allows for temporal phase differences of u" and c" within the flow cross section.

The forms of the transport diffusion equations presented in all four boundary value problems are consistent with those used by previous investigators (Taylor 1953, 1954; Fischer 1967; Bowden 1965; Holley, et al., 1970) in which the concentration variable is viewed from a Lagrangian frame of reference traveling with the cross-sectional mean velocity. This transformation is accomplished by the introduction of the variable ξ defined as follows:

$$\xi = x - \bar{u}t \qquad \text{(unidirectional flow)} \qquad (9)$$

$$\xi = x - \int_0^t u(t')dt' \qquad \text{(oscillatory flow)} \qquad (10)$$

The first two sets of boundary value problems (BVP #1 and BVP #2) describe the dynamics of fluid motion and the transport diffusion of a substance in which only the vertical effects of shear and eddy diffusivity are considered. As shown by Figure 2, the x coordinate has been chosen to act along the principal flow axis while the z coordinate is defined positive upward from the channel bed. The two dimensional shear flow cases examined by these two boundary value problems incorporate essentially the same boundary conditions. For the equations of fluid motion the conditions of no-slip at the channel bed and zero shear at the free surface are applied; whereas for the transport diffusion equation the conditions of zero Fickian flux across the free surface and channel bed are used.

The boundary value problem formulations for three dimensional steady unidirectional and oscillatory flow regimes are presented by boundary value problems 3 and 4 (BVP #3 and BVP # 4). Figure 3 illustrates the coordinate system used in these formulations. The origin has been located at the center of a rectangular region of height 2h and width w. The open rectangular channel is mathematically represented by the lower half of the full section shown in Figure 3. Selection of the coordinate system in this manner preserves the symmetry of the problem about the origin and, as will be shown later, correctly predicts the same dispersive mass transport for equal degrees of skewness in channel geometry in either the vertical or lateral directions. Equations (15) through (18) are of the same form as those presented for the two dimensional cases in BVP #1 and BVP #2 with additional terms included to account for the lateral effects of turbulent shear and diffusivity. Boundary conditions

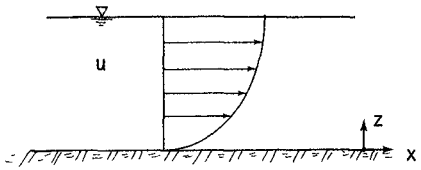


FIGURE 2 - DEFINITION SKETCH FOR TWO DIMENSIONAL SHEAR FLOW

BVP#1 : Two Dimensional Steady Unidirectional Flow

EOM : $$\frac{d^2u}{dz^2} = \frac{K}{\bar{\epsilon}_z} \qquad (11)$$

$$u = 0;\ z = 0$$

$$\frac{du}{dz} = 0;\ z = h$$

TDE : $$\bar{K}_z \frac{\partial^2 c''}{\partial z^2} = u'' \frac{\partial \bar{c}}{\partial \xi} \qquad (12)$$

$$\frac{\partial c''}{\partial z} = 0;\ z = 0,h$$

BVP#2 : Two Dimensional Oscillatory Flow

EOM : $$\bar{\epsilon}_z \frac{\partial^2 u_s}{\partial z^2} - i\sigma u_s = K \qquad (13)$$

$$u_s = 0;\ z = 0$$

$$\frac{\partial u_s}{\partial z} = 0;\ z = h$$

TDE : $$\bar{K}_z \frac{\partial^2 c_s''}{\partial z^2} - i\sigma c_s'' = u_s'' \frac{\partial \bar{c}}{\partial \xi} \qquad (14)$$

$$\frac{\partial c_s''}{\partial z} = 0;\ z = 0,h$$

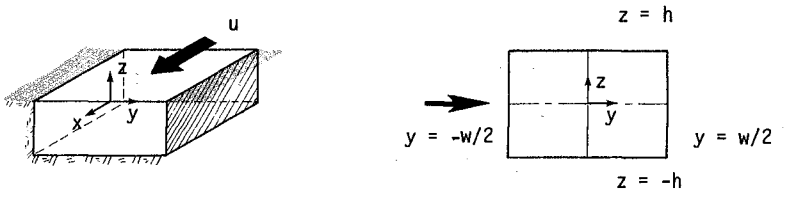


FIGURE 3 - DEFINITION SKETCH FOR THREE DIMENSIONAL SHEAR FLOW

BVP#3 : Three Dimensional Steady Unidirectional Flow

$$\text{EOM}: \quad \bar{\varepsilon}_z \frac{\partial^2 u}{\partial z^2} + \bar{\varepsilon}_y \frac{\partial^2 u}{\partial y^2} = K \qquad (15)$$

$$u = 0;\ z = \pm h,\ y = \pm w/2$$

$$\text{TDE}: \quad \bar{K}_z \frac{\partial^2 c''}{\partial z^2} + \bar{K}_y \frac{\partial^2 c''}{\partial y^2} = u'' \frac{\partial \bar{c}}{\partial \xi} \qquad (16)$$

$$\frac{\partial c''}{\partial z} = 0;\ z = \pm h$$

$$\frac{\partial c''}{\partial y} = 0;\ y = \pm w/2$$

BVP#4 : Three Dimensional Oscillatory Flow

$$\text{EOM}: \quad \bar{\varepsilon}_z \frac{\partial^2 u_s}{\partial z^2} + \bar{\varepsilon}_y \frac{\partial^2 u_s}{\partial y^2} - i\sigma u_s = K \qquad (17)$$

$$u = 0;\ z = \pm h,\ y = \pm w/2$$

$$\text{TDE}: \quad \bar{K}_z \frac{\partial^2 c_s''}{\partial z^2} + \bar{K}_y \frac{\partial^2 c_s''}{\partial y^2} - i\sigma c_s'' = u_s'' \frac{\partial \bar{c}}{\partial \xi} \qquad (18)$$

$$\frac{\partial c_s''}{\partial z} = 0;\ z = \pm h$$

$$\frac{\partial c_s''}{\partial y} = 0;\ y = \pm w/2$$

used include the conditions of no slip and zero Fickian flux at the sides of the full rectangular section.

Dispersion Coefficient Solutions

Solutions obtained for u" and c" from BVP #1 through BVP #4 were used as previously discussed to develop the corresponding expressions for the longitudinal dispersion coefficient. The reader is referred to Taylor (1974) for the detailed development of these expressions.

To simplify the expressions obtained, several mixing time scales were introduced. These are defined as follows:

a. Vertical Mixing Time, $T_{cz} = h^2/\overline{K}_z$

b. Lateral Mixing Time, $T_{cy} = w^2/4\overline{K}_y$

c. Non-Dimensional Vertical Mixing Time, $T'_z = T_{cz}/T$

d. Non-Dimensional Lateral Mixing Time, $T'_y = T_{cy}/T$

e. Relative Mixing Time, $T'_c = T_{cz}/T_{cy}$

where T is defined as the period of flow oscillation.

The solution of Equations (11) and (12) for the case of two dimensional steady unidirectional flow is straightforward. The expression obtained for the longitudinal dispersion coefficient is given as

$$E_L = \frac{8u^2_{max}T_{cz}}{945} \qquad (19)$$

where u_{max} denotes the maximum cross-sectional velocity. Equation (19) has been expressed as a function of u_{max} by maximizing the solution for u as a function of the pressure gradient modulus, K. This provides a relationship between u_{max} and K that can be used to express the dispersion coefficient solutions in terms of the more useful parameter u_{max}.

Except for the numerical constants, the expression given by Equation (19) is identical to that obtained by Taylor (1953) for steady flow in a circular tube. By itself it is of passing interest only. However, for comparative purposes later in this paper, it is noted from Equation (19) that in two dimensional unidirectional flow, the dispersive mass transport increases proportionally with increasing vertical mixing time.

For the case of oscillatory two dimensional shear flow, the solution of Equation (13) for the velocity has been known and documented for some time (Lamb 1945; Segall 1971). To obtain a solution for c''_s, the velocity variation

component was expanded in a Fourier cosine series, and a Fourier cosine series form of the solution for c_s'' in Equation (14) was correspondingly assumed. The assumed form for c_s'' and the expanded series form for u_s'' were then substituted into Equation (14) and the Fourier coefficients for c_s'' were solved for. Expressions obtained for u" and c" were then integrated over the flow cross section, and time-averaged over the period of oscillation to obtain the dispersive mass transport. This produced the following expression for the longitudinal dispersion coefficient as a function of u_{max}, the temporal and spatial maximum of the velocity:

$$\langle E_x \rangle_T = \pi u_{max}^2 T \, T_z'^2 \, \frac{(\cosh 2\sqrt{\pi T_z'} - \cos 2\sqrt{\pi T_z'})}{(\cosh \sqrt{\pi T_z'} - \cos \sqrt{\pi T_z'})^2} \sum_{n=1}^{\infty} \frac{(n\pi)^2}{[(n\pi)^4 + (2\pi T_z')^2]^2} \qquad (20)$$

It should be noted that Equation (20) incorporates the effects of an oscillatory velocity shear profile which allows for temporal phase differences of the flow over the water column. It also demonstrates the dependence of the dispersion mechanism on both the vertical mixing time scale and the period of oscillation. Although this dual dependence has been pointed out by previous investigators (Okubo 1967; Holley et al., 1970; Segall and Gidlund 1972; Fukuoka, 1973) the fundamental way in which these two time scales govern dispersive mass transport has not been identified. This will be discussed in the following section.

Equation (20) has also been expressed as a function of the pressure gradient modulus, K, and the excursion length of a surface particle during one-half of a period of oscillation. The reader is referred to Taylor (1974) for the development of these expressions. Results obtained from these, however, will be used later for discussion purposes.

Solutions for the longitudinal dispersion coefficient in three dimensional unidirectional and oscillatory flows were obtained by assuming double Fourier cosine series forms for u and c". Arguments of the cosine functions were selected to satisfy both the boundary conditions stated for BVP #3 and BVP #4 and the physical requirements of no-slip, zero shear stress and zero diffusive flux at the appropriate boundaries and points of the rectangular half-section corresponding to the open channel cross section. The remaining portions of the solution techniques used are similar to those described previously for the two dimensional shear flow cases. To obtain the dispersive mass transport, the product of the real parts of u" and c" were integrated over the full rectangular section shown in Figure 3 and the result divided by 2 to provide the actual dispersive mass transport in the rectangular open channel.

The expression thus obtained for the longitudinal dispersion coefficient as a function of u_{max} for the three dimensional steady unidirectional flow case is stated in nondimensional form as

$$E_L' = \frac{E_L}{u_{max}^2 T_{cz}} \qquad (21)$$

$$E_L' = \frac{64}{\pi^4 \phi'^2} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\mu_{\ell k}}{\delta(\ell,k,m,n,p,q)\, R_{mn}\, R_{pq}\, R_{\ell k}} \qquad (22)$$

where,

$$\delta(\ell,k,m,n,p,q) = [(2p+1)^2-(2\ell)^2][(2q+1)^2-(2k)^2][(2m+1)^2-(2\ell)^2][(2n+1)^2-(2k)^2]$$

$$R_{mn} = \left[\frac{(2m+1)\pi}{2}\right]^2 + T_c' \left[\frac{(2n+1)\pi}{2}\right]^2$$

R_{pq} = R_{mn} with p replacing m and q replacing n

$$R_{\ell k} = [(\ell\pi)^2 + T_c' (k\pi)^2]$$

$$\phi' = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{(2m+1)(2n+1)R_{mn}}$$

$$\mu_{\ell k} = \begin{cases} 1 & k \neq 0,\ \ell \neq 0 \\ \frac{1}{2} & k = 0, \text{ or } \ell = 0 \\ 0 & k = \ell = 0 \end{cases}$$

The non-dimensional form of the longitudinal dispersion coefficient obtained for the three dimensional oscillatory flow case (BVP #4) is presented on the following page as Equation (23). It is noted from an inspection of Equations (22) and (23) that:

(a) Dispersive mass transport in channels of finite width for both oscillatory and unidirectional flows is a function of both the vertical and lateral time scales of turbulent mixing through the variable T_c' .

Solution for Three Dimensional Oscillatory Flow (BVP #4)

$$E'_x = \frac{\langle E_x \rangle_T}{u^2_{max} T} = \frac{32T'_z}{\pi^4 \psi^2} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\mu_{\ell k}\{[r_2(\ell,k)-(2\pi T'_z)^2]^{\frac{1}{2}} \cos\lambda + (2\pi T'_z)\sin\lambda\}}{\delta'(\ell,k,m,n,p,q)\, r_1^{\frac{1}{2}}(m,n) r_1^{\frac{1}{2}}(p,q) r_2(\ell,k)} \tag{23}$$

where,

$$\delta'(\ell,k,m,n,p,q) = [(2p+1)^2-(2\ell)^2][(2q+1)^2-(2k)^2][(2m+1)^2-(2\ell)^2][(2n+1)^2-(2k)^2]$$

$$r_1(m,n) = \{[\frac{(2m+1)\pi}{2}]^2 + T'_c[\frac{(2n+1)\pi}{2}]^2\}^2 + (2\pi T'_z)^2$$

$r_1(p,q)$ = $r_1(m,n)$ with p replacing m and q replacing n

$$r_2(\ell,k) = \{(\ell\pi)^2 + T'_c(k\pi)^2\}^2 + (2\pi T'_z)^2$$

$$\lambda = \alpha_1(m,n) - \alpha_1(p,q)$$

$$\mu_{\ell k} = \begin{cases} 1 , & k \neq 0, \quad \ell \neq 0 \\ \frac{1}{2} , & k = 0 \text{ or } \ell = 0 \\ 0 , & k = \ell = 0 \end{cases}$$

$$\alpha_1(m,n) = \tan^{-1} \frac{2\pi T'_z}{[\frac{(2m+1)\pi}{2}]^2 + T'_c[\frac{(2n+1)\pi}{2}]^2}$$

$\alpha_1(p,q)$ = $\alpha_1(m,n)$ with p replacing m and q replacing n

$$\psi^2 = \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}[r_1(m,n)-(2\pi T'_z)^2]^{\frac{1}{2}}}{(2m+1)(2n+1) r_1(m,n)} \right\}^2 + \left\{ 2\pi T'_z \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{(2m+1)(2n+1) r_1(m,n)} \right\}^2$$

(b) The dispersive mass transport for the three dimensional oscillatory flow case is also a function of the period of oscillation through the variables T and T'_z.

DISCUSSION OF RESULTS

Two Dimensional Oscillatory Flow

Bowden (1965) was the first investigator to look at longitudinal dispersive mass transport in oscillatory flow. He found that for the case in which the period of oscillation is infinitely long the longitudinal dispersion coefficient is one-half the value of the same coefficient for a corresponding unidirectional flow having the same surface velocity, the same shear profile, and the same vertical eddy diffusivity. Okubo (1967) and Holley, et al. (1970) extended Bowden's work and demonstrated that the dispersion process in oscillatory flows was functionally dependent on both the time scale of vertical mixing and the period of oscillation. The manner in which Holley, et al. interpreted their results stimulated much of the interest in the work presented here, and will therefore be discussed for illustrative and comparative purposes.

Holley, et al. assumed a linear oscillatory profile for the spatial component of velocity of the form

$$u'' = \alpha z \sin \sigma t \qquad (24)$$

where α is a constant, z is the vertical coordinate, and σ is the angular frequency of oscillation, $2\pi/T$. Proceeding in the same manner as described here, they then used this expression for u" to obtain solutions for c" and the longitudinal dispersion coefficient corresponding to the general case of two dimensional oscillatory flow, and the special case in which the period of oscillation is infinitely long. The dispersion coefficient for the infinitely long period of oscillation, E_∞, was then used as the basis for comparison of the dispersion processes in oscillatory and unidirectional flows. This was accomplished by plotting the ratio of the dispersion coefficient for oscillatory flow divided by E_∞ versus the non-dimensional time T' defined as $1/T'_z$ as used here. This produced the results as shown by curve ① in Figure 4. The ordinate variable used in Figure 4 is normalized by the corresponding dispersion coefficient for unidirectional flow, E_L, which is exactly double the value of E_∞. The results of Holley, et al. were adjusted accordingly.

For comparison purposes Figure 4 includes results obtained from the present work as shown by curves ② and ③ . Curve ③ plots the ratio of Equation (20) divided by Equation (19) while curve ② plots the ratio of the corresponding forms of Equations (20) and (19) expressed as functions of the pressure gradient modulus, K.

The results depicted in Figure 4 demonstrate the fact that the relative magnitudes of the dispersion coefficients for oscillatory and unidirectional

flow vary considerably depending upon how one chooses to make the comparison. All three curves shown approach the limiting value of one-half for large periods of oscillation which agrees with the findings of Okubo (1967) and Bowden (1965). However, as T' decreases the curves begin to diverge considerably. The agreement between the results of Holley, et al. (1970) and the u_{max} normalization of the results presented here is good for large values of T'. However, it becomes increasingly worse as T' decreases. The close agreement for large values of T' is not surprising since both curves ① and ③ were developed by requiring the surface amplitude of the velocity in the oscillatory flow field to be equal to the velocity at the surface in the unidirectional flow field. The divergence of the two curves with decreasing values of T' results from the use of a truly oscillatory velocity profile in the present work as compared to the one assumed by Holley, et al. in which the velocity remains temporally in phase over the water column. Thus, as the period of oscillation decreases the effect of inertial forces increases thereby increasing the effects of flow reversals over the water column on the dispersive mass transport.

Curve ② in Figure 4 serves to illustrate the dramatic difference in behavior of the dispersion process when one chooses to compare the oscillatory and unidirectional cases by requiring both flow regimes to have the same pressure gradient modulus, K. For unidirectional flow the pressure gradient and viscous forces are in equilibrium so as to produce a constant unidirectional shear flow for the dispersive transport of substance. In oscillatory flow, however, the pressure gradient is in constant balance with the time varying inertial and friction forces. As the period of oscillation decreases, the inertial effects become very large such that in the limiting case little or no flow would be induced. This results in little or no dispersive transport.

The dramatic difference in the behavior of the velocity and pressure gradient normalized solutions raises the question of whether or not a plot such as Figure 4 is the most meaningful method of illustrating the characteristics of dispersive mass transport in oscillatory flow. It also suggests that the non-dimensionalization of the oscillatory dispersion coefficient by the corresponding unidirectional flow coefficient may in fact mask the fundamental behavior characteristics of the dispersion process in oscillatory flow. This is shown to be true by plotting separately the expressions obtained for E_L and $\langle E_x \rangle_T$ versus T_{cz}, the vertical mixing time. Figure 5 presents such a plot. In this figure, the longitudinal dispersion coefficient, $\langle E_x \rangle_T$, as given by Equation (20) is plotted versus the vertical mixing time, T_{cz}, for four periods of oscillation ranging from 22,526 to 223,560 seconds and a u_{max} of 1 ft./sec. Also plotted is the solution for the unidirectional flow coefficient, E_L, as given by Equation (19). This figure clearly shows the significantly different behavior of E_L and $\langle E_x \rangle_T$. The unidirectional flow coefficient varies directly with the vertical mixing time which for little or no turbulent mixing over the water column allows the shear flow to transport higher concentrations of substance far downstream. The behavior of the oscillatory flow coefficient, however, is governed by a type of resonant interaction between the period of oscillation and the vertical mixing time. As the period

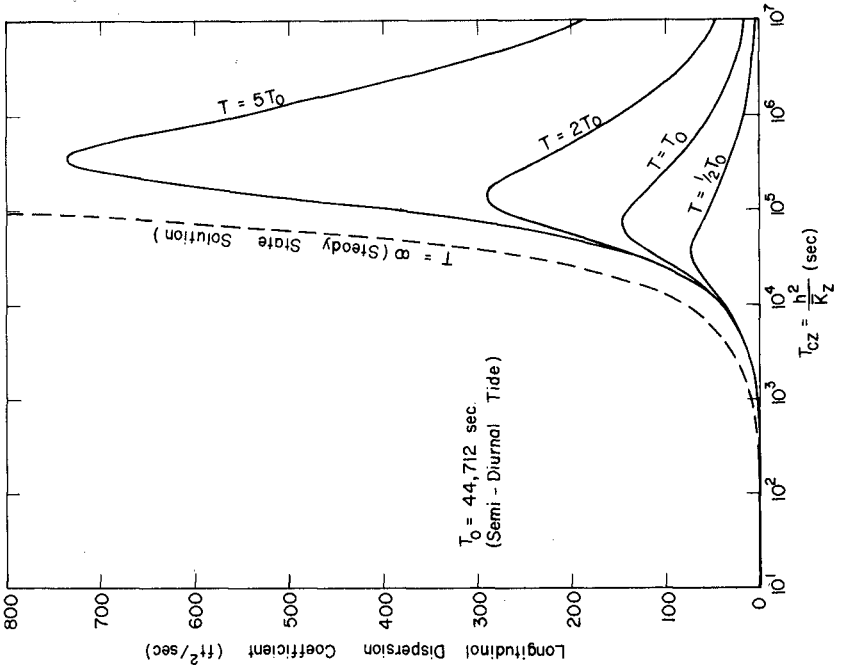


FIGURE 5 - COMPARISON OF DISPERSION PROCESSES AS A FUNCTION OF T_{cz}

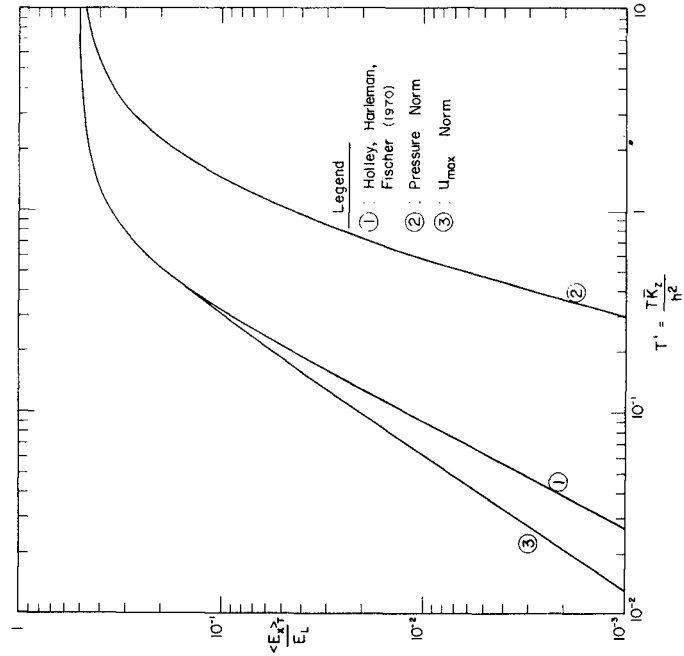


FIGURE 4 - NON-DIMENSIONAL DISPERSION COEFFICIENT $\langle E_x \rangle T/E_L$, AS A FUNCTION OF T'

of oscillation is increased, the resonant peak shifts to the right and likewise increases until the limiting case is reached where the peak is infinitely large and the values predicted for $\langle E_x \rangle_T$ are exactly one-half those predicted for the unidirectional flow coefficient, E_L. The resonant characteristic of $\langle E_x \rangle_T$ also explains the abrupt decrease in the ratio of $\langle E_x \rangle_T / E_L$ as exhibited by the curves in Figure 4.

The physical reasoning behind the resonant behavior of $\langle E_x \rangle_T$ is surprisingly simple. Consider first the case where the period of oscillation is much greater than the vertical mixing time. In this situation, the rate of vertical mixing is so rapid that there is no time for the velocity shear profile to transport the substance longitudinally before it loses its identity through vertical mixing. Thus, the oscillatory and unidirectional flow dispersive processes behave in a similar manner and both are small. This corresponds to conditions found in the extreme left regions in Figure 5. If T is held constant and T_{cz} is allowed to increase, the oscillatory shear profile is then able to transport the substance farther downstream before excessive mixing occurs; thereby increasing the dispersive transport. Thus, the oscillatory case continues to behave in the same manner as the unidirectional case. This continues to be true until the optimum ratio between the period of oscillation and the vertical mixing time is reached. At this point, the dispersive mass transport in the oscillatory flow regime has reached its maximum. If, however, the vertical mixing time is increased beyond this point, the oscillatory nature of the flow begins to become a factor and the longitudinal dispersion is decreased. This trend continues as T_{cz} is increased further until the limiting case is reached in which there is little or no vertical mixing taking place during one or more periods of oscillation. For this situation an elemental volume initially residing in the water column at elevation z, and containing an initial concentration c, would remain at this elevation thus being transported over the closed pathline of flow, and returned to its initial position with no longitudinal dispersion having occurred. By comparison, in a unidirectional flow with little or no vertical mixing, the longitudinal dispersion would be very large.

Thus, it is seen that for an oscillatory flow regime, the longitudinal dispersive mass transport becomes small for <u>both</u> $T \ll T_{cz}$ <u>and</u> $T \gg T_{cz}$; whereas for a unidirectional flow the longitudinal dispersive mass transport varies directly with T_{cz}.

The family of curves implied by Figure 5 was collapsed into one summarizing curve by expressing $\langle E_x \rangle_T$ as given by Equation (20) in the following non-dimensional form:

$$E'_x = \frac{\langle E_x \rangle_T}{u^2_{max} T} \qquad (25)$$

It is noted that the form of E'_x follows naturally from Equation (20). A

plot of E'_x versus the non-dimensional vertical mixing time T'_z is shown by Figure 6. As indicated, the maximum value of the longitudinal dispersion coefficient occurs when the vertical mixing time, T_{cz}, is 1.58 times the period of oscillation, T.

Three Dimensional Oscillatory Flow

The non dimensional form of the solution for the longitudinal dispersion coefficient in three dimensional oscillatory flow, as given by Equation (23), is plotted versus T'_z in Figure 7 for several values of the relative mixing time, T'_c. As stated earlier, the relative mixing time is defined as the ratio of the vertical mixing time scale to the lateral mixing time scale. Thus, small values of T'_c are indicative of a very wide shallow channel, whereas $T'_c = 1$ would represent a channel whose width is twice its depth provided $\overline{K}_z = \overline{K}_y$. T'_c can therefore be considered as a measure of the relative effects of vertical and lateral shear for a given width-to-depth ratio, and vertical and lateral eddy diffusivities. Values of T'_c ranging from 1×10^{-6} to 1 were used to develop the curves shown in Figure 7.

It is seen from Figure 7 that when both the vertical and lateral effects of shear and turbulent mixing are considered, the dispersion process in oscillatory flow is described by an infinite number of resonant curves as compared to the single curve presented in Figure 6 when only vertical effects are considered. The effect of varying T'_c causes changes in the shape of the curve, the maximum value of E'_x achieved, and the value of T'_z at which this maximum occurs.

The innermost curve shown in Figure 7 is identified by $T'_c \leq \times 10^{-6}$ on the left hand side of the peak and by $T'_c \leq 0.001$ on the right. A comparison of this curve and the one shown in Figure 6 shows that the general shapes of the two curves are nearly identical with both peaks occurring at $T'_z = 1.58$. The peak value of E'_x in Figure 6 is 3.27×10^{-3} whereas for the limiting curve in Figure 7 it is 3.07×10^{-3} or 6 percent below the value shown for the two dimensional case. Differences in these two curves are attributed to the number of terms used to generate Figure 7. Because the solution for E'_x as given by Equation (23) contains six infinite summations nested in series, some limitations were necessary in carrying out the required computations. By varying the upper bound on each of the sums, it was demonstrated that the solution converges upward to its limiting value. The convergence occurs reasonably rapidly; however, above an upper limit of 10 terms for each sum the rate of convergence is slowed considerably. The results presented here were computed using an upper bound of 10 on each sum; therefore, each value of E'_x includes approximately 1.2×10^6 terms. Notwithstanding this obvious

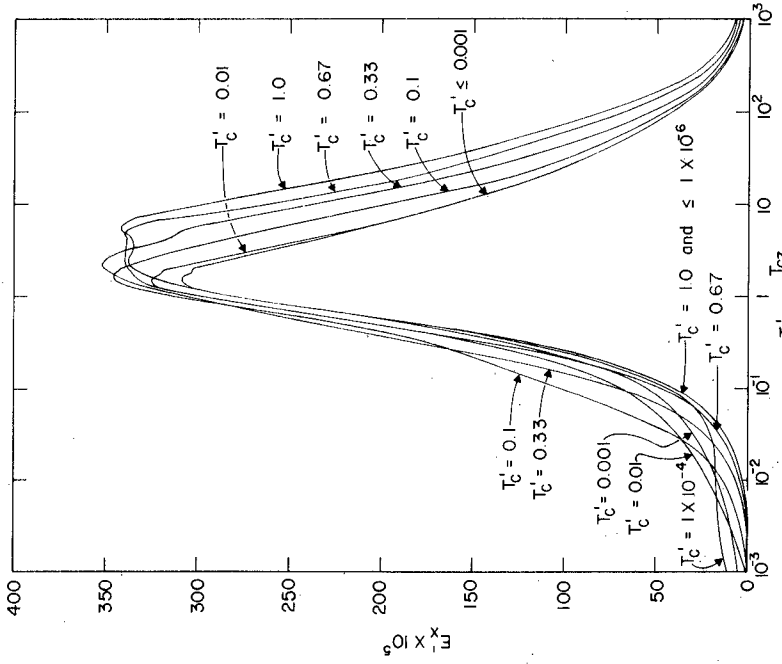


FIGURE 6 - E'_x AS A FUNCTION OF T'_z FOR TWO DIMENSIONAL OSCILLATORY SHEAR FLOW

FIGURE 7 - E'_x AS A FUNCTION OF T'_z AND T'_c FOR THREE DIMENSIONAL OSCILLATORY SHEAR FLOW

limitation, it appears that the three dimensional solution for E'_x approaches the two dimensional solution for E'_x in the limit as $T'_c \to 0$.

Before passing, it is worthy to note the mathematical and physical symmetry of the dispersion process described by Equation (23). If T'_c had been inversely defined as T_{cy}/T_{cz}, Equation (23) would be of the same form with T'_y replacing T'_z everywhere, and the newly defined T'_c multiplying all complementary terms in r_1, r_2, and α_1. This form of the solution would predict the same value of E'_x as the original formulation provided that u_{max} and T were the same, and that the new T'_y equaled the old T'_z and the new T'_c equaled the old T'_c. Stated another way, for the same u_{max} and T, a channel whose half-width was twice its depth would produce the same dispersive mass transport as a channel whose depth was twice its half-width provided that the ratio $\overline{K}_y/\overline{K}_z$ for the first case equaled $\overline{K}_z/\overline{K}_y$ in the second case. One way to show the symmentry of the solution about $T'_c = 1$ is by means of a plot such as Figure 8 in which isolines of E'_x are shown for channel geometries skewed in both width and depth. The lower half of this figure corresponds to geometries in which $w/2h > 1$ whereas the upper half corresponds to geometries in which $w/2h < 1$, provided that $\overline{K}_z = \overline{K}_y$.

Three Dimensional Steady Unidirectional Flow

Values for E'_L obtained from Equation (22) are plotted versus T'_c in Figure 9. In performing the computations, a summation limit of 12 was used on each of the six infinite summations included in the solution. By adjusting this value in a manner similar to that discussed for the oscillatory case, it was concluded that the solution as presented in Figure 9 is near that given by the infinite summations.

It is seen from Figure 9 that the longitudinal dispersion coefficient becomes very large with increasing channel widths. By comparison, if the definition of E'_L given by Equation (21) is applied to the two dimensional solution for E_L given by Equation (19) the result is

$$E_L = \frac{8}{945} = \text{constant} \qquad (26)$$

This is shown in Figure 9 by the dotted line. The difference between the behavior of the two solutions is dramatic and for unidirectional flows supports Fischer's theory regarding the dominant effect of lateral shear on longitudinal dispersion in wide channels. It is apparent that the longitudinal dispersion for the three dimensional case increases without bound as the width of the

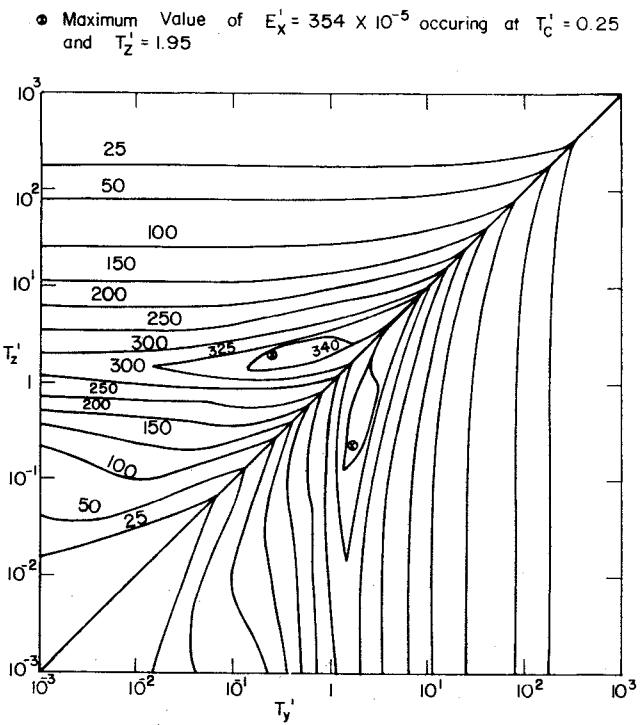


FIGURE 8 - SYMMETRIC DISPLAY OF ISOLINES OF $E_x' \times 10^5$ IN THREE DIMENSIONAL OSCILLATORY SHEAR FLOW

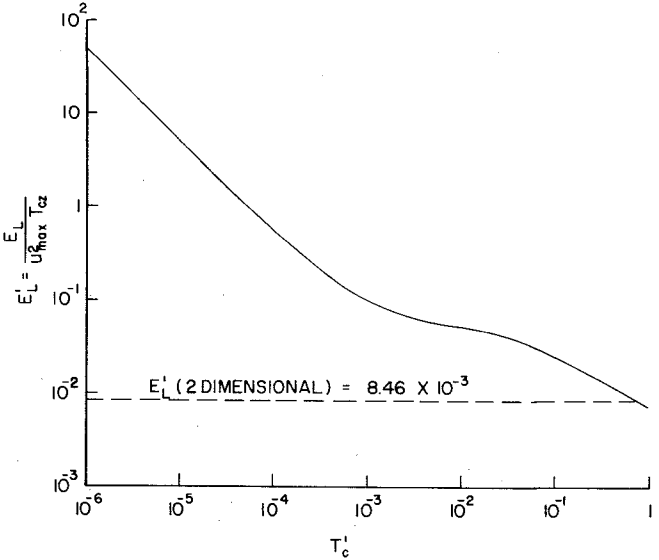


FIGURE 9 - E_L AS A FUNCTION OF T_c' FOR TWO AND THREE DIMENSIONAL UNIDIRECTIONAL SHEAR FLOW

channel is increased correspondingly, whereas the longitudinal dispersion predicted for the two dimensional case remains independent of changes in the channel width.

The fact that the three dimensional solution does not approach the two dimensional solution in the limit as $T_c' \to 0$ is not surprising if one considers the physical characteristics of the unidirectional flow regime and the mathematical formulations of the two cases examined. The three dimensional problem has by definition a lateral shear effect which, no matter how wide the channel, is forever present. Conversely, the two dimensional problem by definition has no such effect. Thus, the presence of this shear coupled with a unidirectional flow and a long time scale of lateral mixing must produce a significantly higher longitudinal dispersive mass transport than would be produced in the case where the lateral effect is non-existent. This phenomena does not apply to the oscillatory flow case because of the completely different nature of the flow regime. In that situation, the oscillatory characteristics of the flow negate the effects of a large lateral mixing time by periodically transporting the water mass across a fixed point of reference rather than transporting it far downstream as in unidirectional flow.

It is noted that the solution given by Equation (22) is symmetrical about $T_c = 1$ in exactly the same manner as discussed for the three dimensional oscillatory flow case. Thus, the discussion presented applies equally to those channels whose geometries are skewed in depth rather than width provided the conditions previously discussed are satisfied. This, of course, would not include the period of oscillation, T.

SUMMARY AND CONCLUSIONS

Analytical expressions for the longitudinal dispersion coefficient have been presented for four cases of shear flows, namely:

a. Unidirectional flow in an infinitely wide rectangular channel

b. Oscillatory flow in an infinitely wide rectangular channel

c. Unidirectional flow in a rectangular channel of finite width

d. Oscillatory flow in a rectangular channel of finite width

Examination of these results shows that in oscillatory flow regimes the dispersion process is governed by a type of resonant interaction between the period of oscillation and the time scales of transverse mixing over the flow cross section. For two dimensional oscillatory flows in which lateral effects are not considered, this interaction is confined to the period of oscillation and the time scale of vertical mixing. The dispersion coefficient was found to reach its maximum value for this case when the vertical mixing time is 1.58 times the period of oscillation. For the case of three dimensional oscillatory flows in which both the vertical and lateral effects of shear and

turbulent mixing are considered, it was found that the inclusion of the lateral effects affected the shape of the resonant curve, the maximum value of the dispersion coefficient achieved, and the value of T_z' at which the maximum occurred. It was also shown that the three dimensional solution for the dispersion coefficient in oscillatory flow approaches the two dimensional solution in the limit for wide channels.

The solution presented for the dispersion coefficient in three dimensional unidirectional flow was shown to predict increasing dispersive mass transport as the channel width was increased correspondingly. Thus, the three dimensional solution does not collapse in the limit to the two dimensional case as was found for oscillatory flow. This behavior supports Fischer's (1967) theory regarding the dominant effect of lateral shear on longitudinal dispersion in wide channels.

ACKNOWLEDGEMENTS

The author would like to thank Robert G. Dean who, as his doctoral studies advisor, provided many hours of encouragement and scientific insight for the work presented.

REFERENCES

Bowden, K.F., "Horizontal Mixing in the Sea Due to a Shearing Current," J. Fluid Mech., Vol. 21, 1965, pp. 83-95.

Fischer, H.B., "The Mechanics of Dispersion in Natural Streams," J. Hyd. Div., ASCE, Vol. 93, No. HY6, 1967, pp.187-216.

Fukouka, S. "Longitudinal Dispersion of Matter in Alternating Shear Flows," Res. Bull. No. C9, Dept. of Eng., James Cook University of North Queensland, 1973.

Holley, E. R. and D. R. F. Harleman, "Dispersion of Pollutants in Estuary Type Flows," M.I.T. Hydro Lab Tech. Rpt. No. 74, 1965.

Holley, E. R., Harleman, D. R. F. and H. B. Fischer, "Dispersion in Homogeneous Estuary Flow," J. Hyd. Div., ASCE, No. HY8, 1970, pp. 1691-1709.

Lamb, H., Hydrodynamics, 6th ed., New York: Dover Publications, 1945.

Okubo, A. "The Effect of Shear in an Oscillatory Current on Horizontal Diffusion from an Instantaneous Source," Int. J. of Oceanology and Limnology, Vol. 1, No. 3, 1967, pp. 194-204.

Segall, B. A. Velocity Gradients and Their Influence on Dispersion in Oscillating Shear Flow, Ph.D. dissertation, New York University, at New York, June, 1971.

Segall, B.A. and E. R. Gidlund, "Velocity Profiles and Dispersion in Estuarine Flow," J. San. Eng. Div., ASCE, Vol. 98, No. SA4, 1972.

Taylor, G. I., "Dispersion of Soluble Matter in Solvent Flowing Slowly Through a Tube," Proc. Roy. Soc., A, Vol. 219, 1953, pp. 186-203.

Taylor, G. I., "The Dispersion of Matter in Turbulent Flow Through a Pipe," Proc. Roy. Soc., A, Vol 223, 1954, pp. 446-468.

Taylor, R. B., "Dispersive Mass Transport in Oscillatory and Unidirectional Flows," Coastal and Oceanographic Laboratory Tech. Rpt. No. 24, University of Florida, August, 1974.