CHAPTER 28

WAVE SHOALING OF FINITE AMPLITUDE WAVES

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ABSTRACT

Wave shoaling is calculated, based on finite amplitude wave theories on a uniform depth which were extended by the authors using the Stokes second definition for wave celerity. Change of wave characteristics with decrease in water depth is discussed from numerical computation, and difference in shoaling characteristics is considered in comparison with results obtained from usual wave theories by the first definition. Theoretical results are compared with preliminary experimental ones on the wave transformation in shoaling water conducted by the experimental facilities satisfying the physical conditions corresponding to the definition for wave celerity as well as possible, and the validity of each theory for practical purposes is investigated.

INTRODUCTION

There appears to be some problems in calculating wave shoaling by finite amplitude wave theories on a uniform depth under the assumption that energy flux of waves and wave period are invariable with change in water depth. One of them is the fact that as already pointed out by Stokes (1880), the physical definition is necessary to determine the wave celerity in the extension to the higher order approximate solution of finite amplitude wave theory. The one is the Stokes first definition for wave celerity, which means that the average horizontal water particle velocity over a wave length vanishes, and the other is the Stokes second definition for wave celerity, which means that the average momentum over a wave length vanishes by addition of a uniform motion.

The authors (1972) already calculated finite amplitude wave theories such as Stokes waves and cnoidal waves using the Stokes second definition for wave celerity and investigated the applicability by comparison with experimental results for wave celerity and water particle velocities.

In the calculation of wave shoaling, Le Méhauté and others (1964 and 1966) used usual Stokes wave theories of the third and the fifth orders by the first definition. Iwagaki and Sakai (1967) also used the hyperbolic wave theory of the second approximation. The hyperbolic wave theory was derived by Iwagaki (1968) from Laitone's cnoidal wave theory (1961), in which the wave celerity is calcu-

lated by the second definition.

In this paper, change of wave height and wave length with decrease in water depth is calculated using the above-mentioned theories by the second definition, and difference in shoaling characteristics is considered in comparison with results obtained from the usual wave theories by the first definition. Nextly, the applicability of each theoretical result is briefly discussed by the comparison with a preliminary experimental one.

ENERGY FLUX OF FINITE AMPLITUDE WAVES

According to Whitham (1961), mean energy flux of waves over a wave period $\bar{W}$ is defined as

$$\bar{W}=\frac{1}{T}\int_{-T/2}^{T/2}\int_{-h}^{\eta}\left\{\frac{1}{2}\rho(u^2+w^2)+p+\rho gz\right\}u\,dz\,dt \quad \cdots\cdots (1)$$

in which T is the wave period, h the depth of water, η the surface displacement from the still water level, p the wave pressure, ρ the density of fluid, g the acceleration of gravity and u and w are the horizontal and vertical water particle velocities respectively. Eq. (1) is transformed into Eq. (2) from Bernoulli's theorem, in which φ is the velocity potential.

$$\bar{W}=-\frac{1}{T}\int_{-T/2}^{T/2}\int_{-h}^{\eta}\rho\frac{\partial\varphi}{\partial t}u\,dz\,dt \quad \cdots\cdots (2)$$

Mean energy flux of waves calculated from the Stokes wave theory of the fourth order by the second definition $\bar{W}_{s2}$ is given as

$$\bar{W}_{s2}=\frac{\rho c^2\lambda^2}{8k}\left[A_{11}^2(2kh+\sinh 2kh)+8\lambda^2\left\{kh\left(A_{12}^2+\frac{1}{2}A_{11}A_{13}+A_{23}+\frac{A_{11}^2B_{22}}{8}+A_{02}A_{11}\cosh kh+\frac{A_{11}A_{22}}{2}\right.\right.\right.$$
$$\cosh kh+\frac{A_{11}A_{13}}{4}\sinh 2kh+\frac{3}{16}A_{11}^3\sinh 2kh+\frac{A_{11}^2B_{22}}{8}\cosh 2kh+\frac{A_{11}A_{22}}{2}\cosh 3kh$$
$$\left.\left.\left.+\frac{A_{22}^2}{4}\sinh 4kh\right\}\right]+O(\lambda^6) \quad \cdots (3)$$

in which c is the wave celerity, λ the small expansion parameter corresponding to the wave steepness, k the wave number and A_{ij} and B_{ij} are the function of kh respectively. In the calculation, the contribution of the higher order terms than the fourth order term to the result is neglected. Eq. (3) agrees exactly with Le Méhauté's result by the usual Stokes wave theory if $A_{02}=0$.

On the other hand, mean energy flux of Chappelear's cnoidal wave theory (1962) by the second definition $\bar{W}_{c2}$ is expressed by Eq. (4), based on the second order approximation.

$$\bar{W}_{c2}=\rho gh^2\sqrt{gh}\left[\frac{L_1^2}{3}\left\{\kappa^2-1+2(2-\kappa^2)\left(\frac{E}{K}\right)-3\left(\frac{E}{K}\right)^2\right\}+\frac{13}{3}L_1^2L_3\left\{\kappa^2-1+4\left(\frac{E}{K}\right)-2\kappa^2\left(\frac{E}{K}\right)-3\left(\frac{E}{K}\right)^2\right\}\right. \quad \cdots\cdots (4)$$
$$\left.+\frac{L_1^3}{15}\left\{18\kappa^4+11\kappa^2-29+(-36\kappa^4+6\kappa^2+124)\left(\frac{E}{K}\right)-5(7\kappa^4+25)\left(\frac{E}{K}\right)^2+30\left(\frac{E}{K}\right)^3\right\}\right]+O(L_1^mL_3^n)_{m+n=4}$$

In Eq. (4), κ is the modulus of elliptic function, K and E are the complete elliptic integrals of the first and second kinds respectively and L_1 and L_3 the small expansion parameters given in Eq. (15).

Mean energy flux of waves obtained from cnoidal wave theory by the first definition $\bar{W}_{c1}$ is calculated as the following equation.

$$\bar{W}_{c1}=\rho g h^2\sqrt{gh}\left[\frac{L_1^2}{3}\left\{\kappa^2-1+2(2-\kappa^2)\left(\frac{E}{K}\right)-3\left(\frac{E}{K}\right)^2\right\}+5L_1^2L_3\left\{\kappa^2-1+4\left(\frac{E}{K}\right)-2\kappa^2\left(\frac{E}{K}\right)-3\left(\frac{E}{K}\right)^2\right\}\right.$$
$$\left.+\frac{L_1^3}{15}\left\{23\kappa^4+6\kappa^2-29+(-46\kappa^4+31\kappa^2+119)\left(\frac{E}{K}\right)-15(4\kappa^2+7)\left(\frac{E}{K}\right)^2+15\left(\frac{E}{K}\right)^3\right\}\right]+0\left(\sum_{m+n=4}L_1^mL_3^n\right) \quad \cdots\cdots (5)$$

Substituting the relation between expansion parameters in Chappelear's first order approximate theory L_1 and L_3 given as

$$2L_3+L_1\left(\kappa^2+\frac{E}{K}\right)=0 \quad \cdots\cdots (6)$$

it is found that Eq. (5) coincides with Eq. (4) within the order of this approximation.

Based on Laitone's cnoidal wave theory expressed by the mean water depth, energy flux $\bar{W}_{L2}$ is given as

$$\bar{W}_{L2}=\rho g h^2\sqrt{gh}\left[\frac{1}{3\kappa^4}\left\{\kappa^2-1-2(\kappa^2-2)\left(\frac{E}{K}\right)-3\left(\frac{E}{K}\right)^2\right\}\left(\frac{H}{h}\right)^2+\frac{1}{30\kappa^6}\left\{4(-\kappa^4+3\kappa^2-2)+(8\kappa^4-53\kappa^2+53)\left(\frac{E}{K}\right)\right.\right.$$
$$\left.\left.+60(\kappa^2-2)\left(\frac{E}{K}\right)^2+75\left(\frac{E}{K}\right)^3\right\}\left(\frac{H}{h}\right)^3+0\left(\left(\frac{H}{h}\right)^4\right)\right] \quad \cdots\cdots (7)$$

Eq. (7) agrees exactly with that obtained from the second order approximate solution by the first definition within this approximation, and moreover if the parameters L_1 and L_3 are expanded into power series of the ratio of water depth to wave height H/h, energy flux by Chappelear's theory, Eqs. (4) and (5) yields Eq. (7). This is a self-evident truth, because expanding the parameters L_1 and L_3 in Chappelear's theory into the power series of H/h, Laitone's theory coincides with Chappelear's theory to the second order of H/h, as the authors (1974) already proved.

WAVE SHOALING OF FINITE AMPLITUDE WAVES

In general, the calculation of wave shoaling using energy flux from a theory on a uniform depth is based on the two assumption that energy flux and wave period are invariable with change of water depth. Then, under the assumption that energy flux of waves in deep water may be given by the results from the Stokes wave theory of the fourth order, in case which wave theories by both the definitions coincide each other, equations to calculate wave shoaling by Stokes wave theory are formulated by the following ones.

$$\left(\frac{h}{L}\right)^4\lambda_0^2\left(1+\frac{3}{4}\lambda_0^2\right)=\left(\frac{h}{L_0}\right)^4\lambda^2\left[\frac{A_{11}^2}{2}(2kh+\sinh 2kh)+\lambda^2\left\{4kh\left(A_{02}^2+\frac{A_{11}A_{13}}{2}+A_{22}^2\right)+\frac{A_{11}^2B_{22}}{2}\right.\right.$$
$$\left.+4A_{02}A_{11}\cosh kh+2A_{11}A_{22}\cosh kh+A_{11}A_{13}\sinh 2kh+\frac{3}{4}A_{11}^2\sinh 2kh+\frac{A_{11}^2B_{22}}{2}\cosh 2kh\right.$$
$$\left.\left.+2A_{11}A_{22}\cosh 3kh+A_{11}^2\sinh 4kh\right\}\right] \quad \cdots\cdots (8)$$

$$2\pi(1+\lambda_0^2)\left(\frac{h}{L_0}\right)=kh(1+C_1\lambda^2)\tanh kh \quad \cdots\cdots (9)$$

in which C_1 is the function of kh. The parameters λ and λ_0 are given in Eqs. (10) and (11) respectively.

$$\frac{\pi H_0}{L_0}=\frac{3}{8}\lambda_0^3+\lambda_0 \quad \cdots\cdots (10)$$

$$\frac{\pi H}{L}=B_{33}\lambda^3+\lambda \quad \cdots\cdots (11)$$

Under the similar assumption as the case of Stokes waves, equations to determine shoaling characteristics by the cnoidal wave theory are expressed as

$$\lambda_0^2\left(1+\frac{3}{4}\lambda_0^2\right)\left\{\frac{1}{2\pi}(1+\lambda_0^2)\left(\frac{h}{L_0}\right)\right\}^{3/2}=8\pi\left(\frac{h}{L_0}\right)^4\left(\frac{\bar{W}_0}{\rho g h^2\sqrt{gh}}\right) \quad \cdots\cdots (12)$$

$$\frac{1}{2\pi}(1+\lambda_0^2)\left(\frac{h}{L_0}\right)\left(\frac{L}{h}\right)^2=\left(\frac{c}{\sqrt{gh}}\right)^2 \quad \cdots\cdots (13)$$

in which $\bar{W}$ is the energy flux by each cnoidal wave theory mentioned above. Eqs. (14) and (15) are further added to calculate wave shoaling by Chappelear's theory.

$$\frac{H}{h}=\kappa^2L_1\left\{1+\frac{1}{4}L_1(7\kappa^2+10)+6L_3\right\} \quad \cdots\cdots (14)$$

$$2L_3+L_1\left(\kappa^2+\frac{E}{K}\right)+L_1^2\left\{-\frac{1}{5}(-9\kappa^4-6\kappa^2+1)+2(\kappa^2+1)\left(\frac{E}{K}\right)\right\}+6L_1L_3\left(\kappa^2+\frac{E}{K}\right)+L_3^2=0 \quad \cdots\cdots (15)$$

Numerical computation was done by use of iterative technique composed of the combination with the Regula-Falsi method and the Newton method.

NUMERICAL RESULTS AND CONSIDERATIONS

Fig. 1 shows change of wave height calculated from Stokes wave theories by both the definitions, in which the breaking inception is obtained from the Stokes criterion that the wave celerity equals to the horizontal water particle velocity at the water surface. In addition to the well-known fact that change of wave height calculated from Stokes wave theory is more evident than that by Airy wave theory with increase in deep water wave steepness H_o/L_o, the wave height calculated from Stokes wave theory by the second definition is larger than that by the first definition by about 7 % at most. It is also noted that the ratio H/H_o is greater than unity in the range of relatively large value of the ratio h/L_o, in

case when the Stokes wave theory by the second definition is used.

Fig. 2 is the results based on Chappelear's cnoidal wave theory as well as the Stokes wave theories mentioned above. Wave height increases rapidly with decrease in h/L_o compared with the results by the Airy wave theory, as extended by Iwagaki and Sakai, using the hyperbolic wave theory. Results from cnoidal wave theories by both definitions do not differ from each other in relatively small ratio of h/L_o. This is expected from the fact that the cnoidal wave theories by both the definitions coincide each other in the case of a solitary wave.

The comparison between change of wave height computed from Chappelear's cnoidal wave theory by the second definition and that from Laitone's theory is given in Fig. 3, in which the ratio from Chappelear's theory becomes greater than that

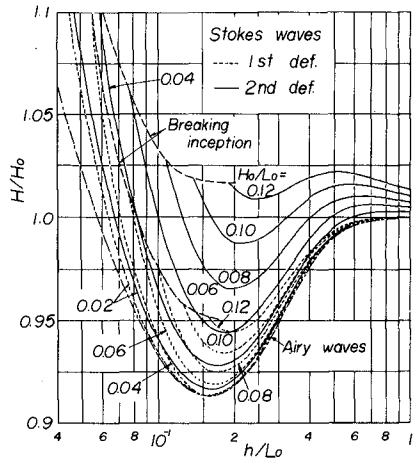

Fig. 1 Change of wave height calculated from Stokes wave theories by both definitions

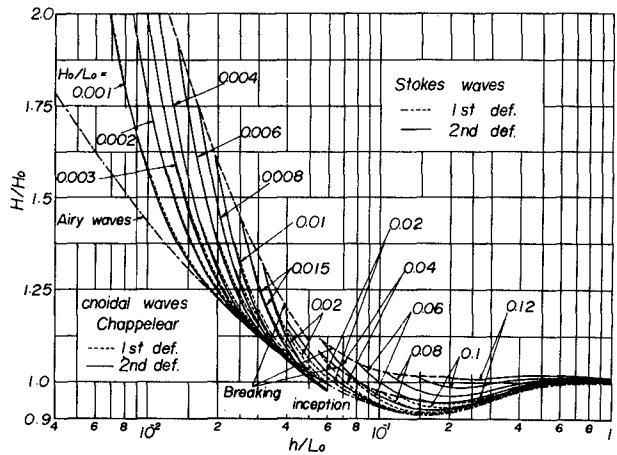

Fig. 2 Change of wave height calculated from Stokes and cnoidal wave theories by both definitions

from Laitone's theory with increase in deep water wave height H_o/L_o. Since, moreover, the horizontal water particle velocity at phase of wave crest at the water surface by Laitone's theory greatly increses compared to that by other finite amplitude wave theories in the vicinity of breaking point, the ratio H/H_o at breaking point calculated from the Stokes criterion by Laitone's theory becomes considerably smaller than that by Chappelear's theory.

Fig. 4 shows change of wave length computed from the Stokes wave and cnoidal wave theories by Chappelear. As shown in the figure, the wave length calculated from Stokes wave theory by the first definition as well as the cnoidal wave theories by both definitions increases with H_o/L_o for the same ratio of h/L_o, whereas the result from the Stokes wave theory by the second definition decreases with H_o/L_o within a range of the ratio of h/L_o.

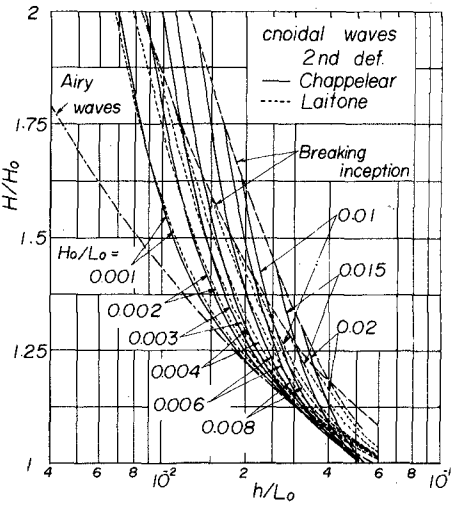

Fig. 3 Change of wave height calculated from Chappelear's theory and Laitone's one

The comparison between result for change of wave length L/L_o computed from Chappelear's cnoidal wave theory by the second definition and that from Laitone's theory is shown in Fig. 5. There is little difference between them as well as the result for wave height shown previously except for a region near the breaking point and a range of relatively large values of h/L_o and H_o/L_o, where the applicability of cnoidal wave theory is questionable.

COMPARISON WITH EXPERIMENTAL RESULTS AND CONSIDERATIONS

A preliminary experiment of wave shoaling was conducted at Ujigawa Hydraulic Laboratory, Disaster Prevention Research Institute. The wave tank used in the experiment is 25 m long, 0.5 m wide and 0.65 m deep. Various devices were made by Tsuchiya and Yasuda in order to keep the uniformity of mass transport induced by waves. That is to say, the wave tank of which both the ends are opened was installed in the wide semi-cicular basin of which diameter is 35 m. Nextly, attaching a big float to the usual piston-type wave generator, the generation of waves with opposing direction was suppressed, and a special wave absorber was set adjacent to the end of wave tank and at the sidewall of wave basin. These are owing to prevent the return flow in relation with the hydrostatic pressure gradient caused by mass transport of waves.

The experiment was carried out in the two cases. In the first case, a sloping model beach which gradient is 1/50 was installed in the wave tank so as

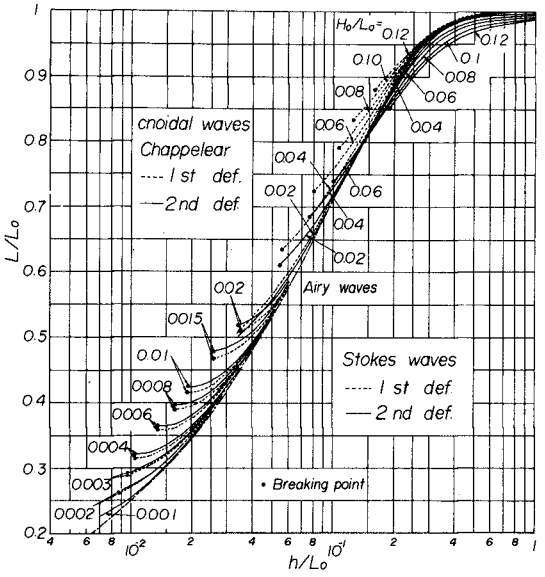

Fig. 4 Change of wave length calculated from Stokes and cnoidal wave theories by both definitions

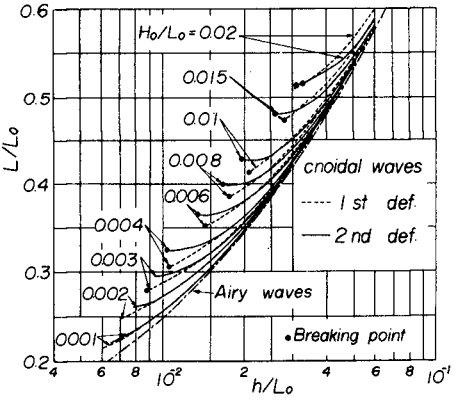

Fig. 5 Change of wave length calculated from Chappelear's theory and Laitone's one

to keep the water depth at the end of wave tank finite. This is corresponding to the experiment to realize the physical condition for the Stokes first definition to some extent. In the experiment, the care not to make the waves break on the sloping beach was taken, because wave breaking at the end of wave tank induces excess mass transport.

In the second case, the sloping beach model was installed under a situation that the outflow and inflow of water through the end of wave tank do not exist. Consequently, the dry bed on which waves are run up was ensured. This is corresponding to the experiment to realize the physical condition for the second definition.

Surface displacement was measured by six resistance-type wave gauges and wave celerity was estimated from propagation time between each gauge.

Since wave height and wave length in deep water are not uniquely determined from the given wave height and wave period at an arbitrary water depth when the wave theories by both the definitions are used, change of wave height and wave length starting from those at the most offshore wave gauge were treated for comparison.

The effect of wave damping due to bottom and sidewall friction on change of wave height with decrease in water depth was estimated by applying the following formula of wave damping step by step. The formula by Iwagaki et al. (1967) is expressed as

$$\frac{H_2}{H_1}=\exp\left(\frac{-\alpha\epsilon_{b+w}x}{L}\right),$$

$$\epsilon_{b+w}=\left(\frac{4\pi^2}{\beta L}\right)\left(1+\frac{1}{\psi}\right)\frac{1}{\sinh 2kh+2kh},\quad \psi=\frac{kB}{\sinh 2kh},\quad \beta=\left(\frac{\pi}{\nu T}\right)^{\frac{1}{2}} \qquad \cdots\cdots\cdots\cdots (16)$$

which was derived from laminar wave boundary layer theory on a uniform depth, in which x is the distance of wave propagation, B the width of wave tank and ν the kimematic viscosity. The value of α was adopted as 1.4 from comparison with their experiment on wave damping in order to take into account the effect of wave nonlineality and water surface contamination on wave damping. Consequently, it was found that the effect of wave damping on change of wave height is very important to be about 15 % for 1/50 slope.

Fig. 6 is one of examples for change of wave height with decrease in water depth, in which h_1 and H_1 are the water depth and wave height measured at the most offshore site respectively. In spite of wide scatter of experimental results, it appears that the qualitative tendencies of experimental results may be explained by each theory.

Fig. 7 shows change of wave length estimated from the experimental results for wave celerity. Wave celerity measured was regarded as the one at the middle point of two wave gauges. The experimental results agree relatively well with the theoretical ones for each experiment.

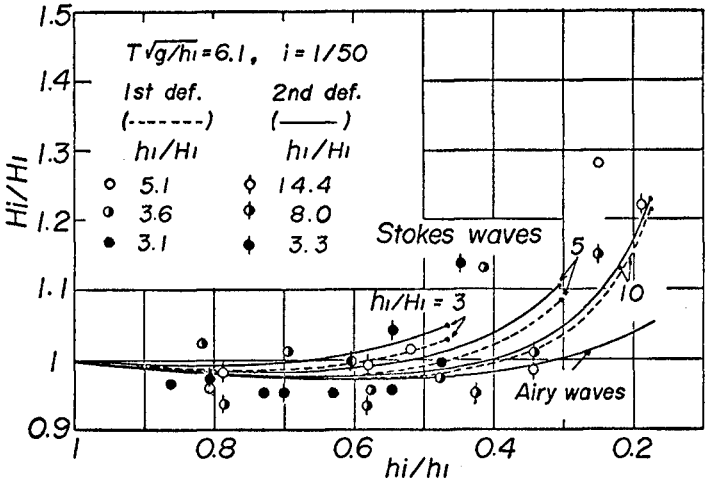

Fig. 6 Comparison between theoretical results and experimental ones for change of wave height

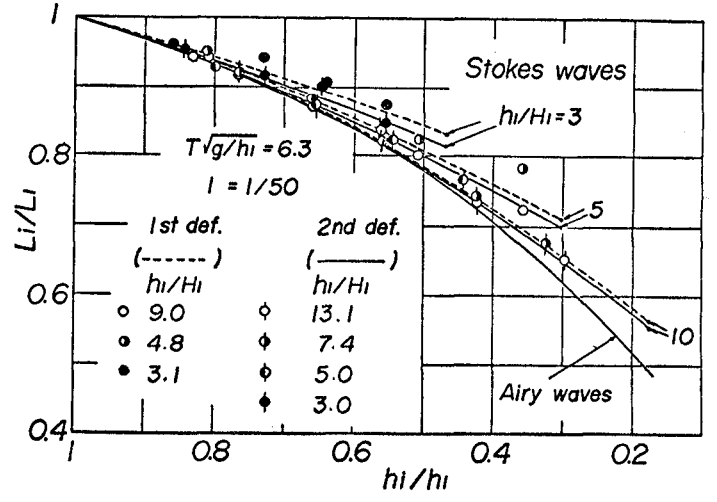

Fig. 7 Comparison between theoretical results and experimental results for change of wave length

CONCLUSIONS

Wave shoaling was calculated, based on finite amplitude wave theories on a uniform depth which were extended by the authors using the Stokes second definition for wave celerity and difference in shoaling characteristics was considered in comparison with results obtained from the usual wave theories by the first definition. It was found that the difference is considerably significant for a region of applicability to Stokes waves.

From comparison with the experimental results made under the considerations to satisfy the physical condition corresponding to each definition for wave celerity as much as possible, the tendency of shoaling characteristics obtained from each wave theory was qualitatively confirmed.

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