CHAPTER 18

SIX-PARAMETER WAVE SPECTRA

by

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ABSTRACT

In an attempt to develop a systematic series of wave spectra covering a variety of spectral shapes observed in the ocean, this paper presents a newly developed series of wave spectra which involves six parameters. In the development of the six-parameter wave spectra, the spectra are decomposed into two parts. Each part is expressed by a mathematical formula with three parameters, and the total spectrum is expressed by the combination of two sets of three-parameter spectra. Results of analysis have shown that the six-parameter wave spectra thus derived appear to represent almost all stages of development of a sea during a storm. Then, from the statistical analysis of 800 spectra observed at the North Atlantic Ocean, the values of the six-parameters are expressed in terms of significant wave height so that a family of spectra for a desired sea severity can be generated.

INTRODUCTION

Recent progress in application of statistics to ocean and coastal engineering enables us to evaluate responses of ocean structures in a seaway at an early design stage by carrying out spectral analysis. In applying the linear superposition principle for prediction, however, wave spectra in desired seas have to be prepared in advance, and the magnitude of responses of ocean structure is significantly influenced by the shape of wave spectra for a given sea severity.

The shape of wave spectra observed in the ocean varies considerably (even though the significant wave heights are same) depending on the duration and fetch of wind, stage of growth and decay of a storm, and existence of swell. For example, Figure 1 shows a variety of shapes

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of wave spectra all of which have the same significant wave height of 3.5 m (11.48 ft) (±1%), and wind speeds are between 20 to 25 knots.

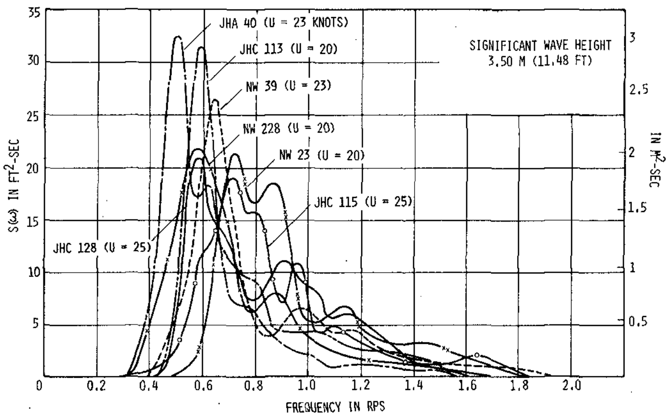

Figure 1 Variety of wave spectra for significant wave height of 3.5 m (11.48 ft)

As can be seen in the figure, the spectrum JHA 40 has a very sharp single peak at the lower frequencies, while some spectra (JHC 128, NW 23, NW 39) have double peaks. Furthermore, three spectra (JHC 113, NW 228, and JHC 128) have the same modal frequency of 0.58. Thus, even though two parameters (significant wave height and modal frequency) are the same, the shape of the spectra are significantly different, and this may result in a significant difference in the evaluated magnitude of responses of ocean structures. This indicates that additional parameters are required for more accurate representation of wave spectra to provide useful information for more rational design of ocean structures.

In an attempt to develop a systematic series of wave spectra covering a variety of spectral shapes, this paper presents a newly developed series of wave spectra which involves six parameters.

In the development of the six-parameter wave spectra, the wave spectra are decomposed into two parts--one which includes primarily the lower frequency components of the wave energy and the second which covers primarily the higher frequency components of the energy. Then, each wave spectrum is expressed in a mathematical formula with three parameters; i.e., significant wave height, modal period, and shape parameter, and the entire spectrum is expressed by a combination of two sets of three-parameter spectra. The parameters are determined numerically such that the difference between the theoretical six-parameter and observed spectra is minimal.

The six-parameter representation of ocean waves is made on a total of 800 observed spectra, and the results are classified into 10 groups depending on the severity. Then, for each group a statistical analysis is carried out on the parameters taking into account the correlation between them. Finally, the results are presented in a family of spectra including the most probable spectrum expected to occur for a specified sea state as well as the limiting spectral shapes which may occur with a confidence coefficient of 0.95. The values of the six-parameters for this set of mathematical spectra are expressed in terms of significant wave height so that a family of spectra for a desired sea severity can be generated.

DERIVATION OF SIX-PARAMETER WAVE SPECTRA

In the development of the six-parameter wave spectra, the spectra are decomposed into two parts as illustrated in Figure 2; one which includes primarily the lower frequency components of the wave energy and the second which covers primarily the higher frequency components of the energy. This concept of decomposing the wave spectrum into two parts was also proposed by Strekalov et. al. in 1972 in their analysis of measured wave spectra [1]. In the present analysis, the spectrum of each part is expressed by a mathematical formula with three parameters so that the total spectrum is expressed by the combination of two sets of three-parameter spectra.

It may be of interest to note here the shape of wave spectra which have been most frequently observed to date. Although numerous shapes of wave spectra have been observed, the results of analysis made on available data indicate that spectral shapes similar to those shown in Figure 3 through 5 have been most frequently observed. In a broad sense, the shape of these spectra have a peak at the lower frequencies which decreases exponentially to a plateau at the higher frequencies. Thus, it is very difficult to express the entire spectral shape in a simple mathematical formula. The lower frequency part of the spectra

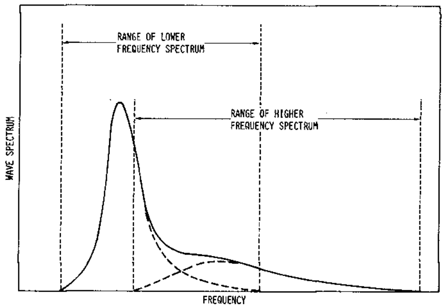

Figure 2 Decomposition of wave spectra

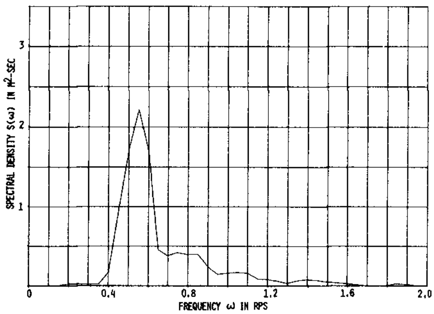

Figure 3 Example of wave spectrum (sig. height 2.89 m, 9.47 ft)

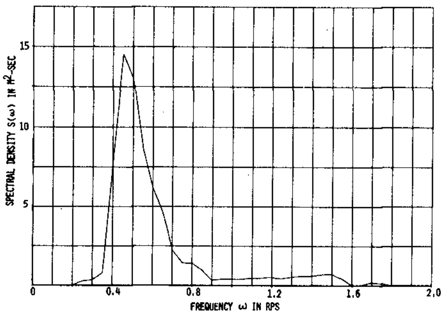

Figure 4 Example of wave spectrum (sig. height 7.47 m, 24.5 ft)

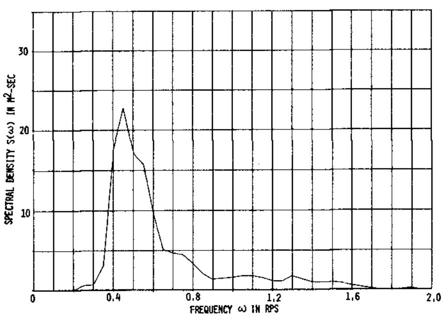

Figure 5 Example of wave spectrum (sig. height 10.09 m, 33.1 ft)

shown in these figures may be well expressed by some currently available spectral formulations; however, in the presently existing formula, the wave energy at the higher frequencies decreases exponentially with increase in frequency and hence the energy at the plateau is not represented. Although the wave energy at the higher frequencies is usually much less than that at the lower frequencies, its contribution to responses of marine vehicles and structures may be significant depending on their size and vehicle speed. Thus, it is highly desirable to represent the shape of the entire spectrum as closely as possible, and this may be achieved by separating the spectra into two parts.

First, we may derive the three-parameter representation for both the low and high frequency part of the spectrum as follows:

From results of dimensional analysis of ocean waves, Phillips has derived the following form of the spectrum of ocean waves over a range of frequencies between the modal frequency and that at which capillary waves become significant [2]:

$$S(\omega) = \alpha g^2 \omega^{-5} \tag{1}$$

where, α = equilibrium range constant
g = gravity acceleration

The validity of this spectral presentation for wind-generated waves has been shown by Kitaigorodskii [3] from an analysis of Burling's observed data, and many formulae currently available for spectral representation of ocean waves are expressed in the following form [4] [5] [6];

$$S(\omega) = \frac{A}{\omega^5} e^{-B/\omega^4} \tag{2}$$

Since it is commonly assumed that ocean waves comprise a Gaussian process with a narrow-band spectrum, it can be shown that the significant wave height, ζ, defined as the average of the highest one-third waves becomes,

$$\zeta = 4\sqrt{m_0} = \sqrt{\frac{4A}{B}} \tag{3}$$

where, $$m_0 = \int_0^\infty S(\omega)\, d\omega = \frac{A}{4B}$$

Next, let the spectrum, $S(\omega)$, be divided by the area, m_0, so that the spectrum has a unit area. That is,

$$S'(\omega) = \frac{4B}{\omega^5} e^{-B/\omega^4} \tag{4}$$

The unit area spectrum, denoted by $S'(\omega)$, can be considered as if it were a probability density function, since $S'(\omega)$ is a positive, continuous, and integrable function with unit area, and thereby satisfies all conditions required for the probability function. $S'(\omega)$, in fact, yields the exponential probability density function by letting $\omega^4 = 1/x$. Hence, it can be generalized in the form of a new probability density function with an additional parameter, λ, which yields the gamma probability function. That is,

$$S'(\omega) = \frac{4}{\Gamma(\lambda)} \frac{B^{\lambda}}{\omega^{4\lambda+1}} e^{-B/\omega^4} \tag{5}$$

Under the assumption that the spectrum is narrow-banded, $S'(\omega)$ may be converted to the dimensional wave spectrum $S(\omega)$ which satisfies the condition that the area under the spectrum is equal to $(\zeta/4)^2$ given in Equation (3). Then, we have,

$$S(\omega) = \frac{1}{4} \frac{B^{\lambda}}{\Gamma(\lambda)} \frac{\zeta^2}{\omega^{4\lambda+1}} e^{-B/\omega^4} \tag{6}$$

The constant, B, in Equation (6) can be expressed in terms of the modal value, ω_m, by setting the differentiation of $S(\omega)$ with respect to ω to be zero. That is,

$$B = \left(\frac{4\lambda+1}{4}\right) \omega_m^4 \tag{7}$$

Thus, from Equations (6) and (7), the following spectral formulation can be made:

$$S(\omega) = \frac{1}{4} \frac{\left(\frac{4\lambda+1}{4} \omega_m^4\right)^{\lambda}}{\Gamma(\lambda)} \frac{\zeta^2}{\omega^{4\lambda+1}} e^{-\left(\frac{4\lambda+1}{4}\right)\left(\frac{\omega_m}{\omega}\right)^4} \tag{8}$$

The spectrum given in Equation (8) has three parameters; namely, significant wave height, ζ, modal frequency, ω_m, and a parameter λ.

The parameter, λ, controls the shape (sharpness) of the spectrum when the other two parameters are held constant, and hence it may be called a spectral shape parameter. For example, Figure 6 shows computer printout of a family of wave spectra for various λ while the other two parameters, ζ and ω_m, are held constant at 3.0 m (9.8 ft.) and 0.6 rps, respectively. As can be seen in the figure, the spectral shape becomes sharper with increasing λ.

Let $\lambda = 1$ in Equation (8). Then, we have,

$$S(\omega) = \frac{1.25}{4} \zeta^2 \frac{\omega_m^4}{\omega^5} e^{-1.25\left(\frac{\omega_m}{\omega}\right)^4} \tag{9}$$

The spectrum given in Equation (9) is the Bretschneider two-parameter wave spectrum expressed in terms of significant wave height and modal frequency. Thus, the three-parameter spectrum given in Equation (8) includes the Bretschneider two-parameter spectrum as a special case ($\lambda = 1$), and this in turn also includes the Pierson-Moskowitz one-parameter spectrum ($\lambda = 1$ and $\omega_m = 0.4\sqrt{g/\zeta}$).

By combining two sets of three-parameter spectra, one representing the low frequency components and the other the high frequency components of the wave energy, the following six-parameter spectral representation can be derived:

$$S(\omega) = \frac{1}{4} \sum_j \frac{\left(\frac{4\lambda_j+1}{4}\omega_{mj}\right)^{\lambda_j}}{\Gamma(\lambda_j)} \frac{\zeta_j^2}{\omega^{4\lambda_j+1}} e^{-\left(\frac{4\lambda_j+1}{4}\right)\left(\frac{\omega_{mj}}{\omega}\right)^4} \tag{10}$$

where, $j = 1, 2$ stands for the lower and higher frequency components, respectively.

The six parameters, ζ_1, ζ_2, ω_{m1}, ω_{m2}, λ_1, and λ_2, involved in Equation (10) are determined numerically such that the difference between theoretical and observed spectra is minimal. For this, appropriate initial values are chosen for each parameter, and then the computation is carried out for various combinations of the parameters in order to determine the final values for which the difference between theoretical and observed spectra is minimal. An example of the computer output is shown in Figure 7. The heavy line in the figure is the measured spectrum and the line with the circles is the six-parameter spectral representation. The values of the parameters for this example are $\zeta_1 = 4.14$ m (13.59 ft.), $\zeta_2 = 3.27$ m (10.73 ft.), $\omega_{m1} = 0.58$, $\omega_{m2} = 1.00$, $\lambda_1 = 2.67$, and $\lambda_2 = 1.37$.

Recently, a computer program has been developed by applying a nonlinear least square fit technique by which the six parameters can be evaluated directly from the information on wave spectra stored on tape. This eliminates the need to generate individual spectra in visual form from the tape to which mathematical formula is fitted.

Examples of comparisons between observed spectra and mathematical six-parameter spectra are shown in Figures 8 through 11. For example, Figure 8 shows a comparison for the case when swell coexists with wind-generated waves and hence the spectrum has double peaks. Figure 9 shows an example of the case when the wind-generated waves are growing and are nearly fully-developed, and spectral shapes similar to this are frequently observed as mentioned earlier. On the other hand, Figure 10 shows an example for seas of mild severity with a relatively broad-band spectrum while Figure 11 shows a comparison for a very severe sea of significant wave height 14.5 m (47.7 ft) in which the sea is partially developed by strong wind, and has a very sharp peak at the lower frequencies in the spectrum. As can be seen in these examples, the six-parameter spectra derived in Equation (10) appear to represent almost all stages of development of a sea during a storm.

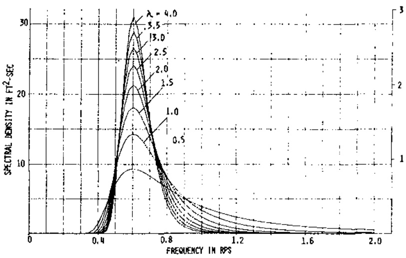

Figure 6 Three-parameter spectra for various λ-values (sig. height 3.0 m, 9.8 ft, modal frequency 0.6 rps)

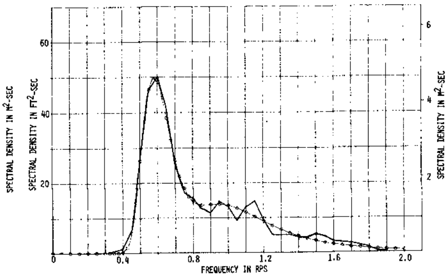

Figure 7 Comparison of measured and six-paramenter spectrum (sig. height 5.28 m, 17.3 ft)

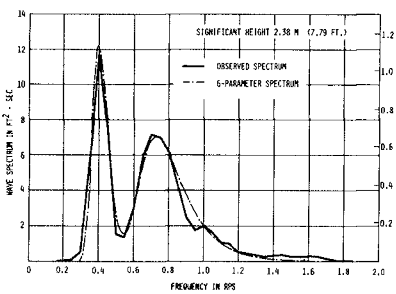

Figure 8 Comparison of measured and six-parameter spectrum (sig. height 2.38 m, 7.79 ft)

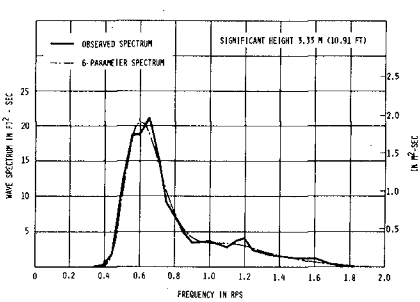

Figure 9 Comparison of measured and six-parameter spectrum (sig. height 3.33 m, 10.91 ft)

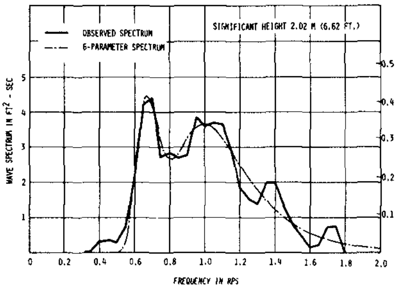

Figure 10 Comparison of measured and six-parameter spectrum (sig. height 2.02 m. 6.62 ft)

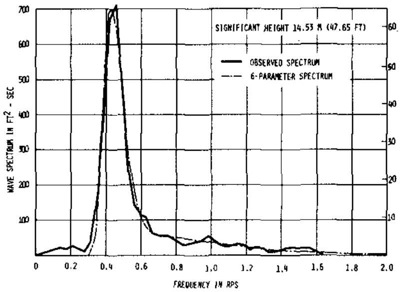

Figure 11 Comparison of measured and six-parameter spectrum (sig. height 14.54 m, 47.65 ft)

STATISTICAL ANALYSIS OF PARAMETERS

It was shown in the preceeding section that the six-parameter spectra represent fairly well a variety of spectral shapes observed in the ocean. Perhaps one of the most useful applications of the spectra is to carry out a statistical analysis on each parameter so that a family of spectra for a given sea severity will be generated with a preassigned probability of assurance. For this purpose, a total of 800 available spectra observed in the North Atlantic Ocean [7] [8] [9] are classified into ten groups depending on severity as given in Table 1. Then, for each group a statistical analysis is carried out on the parameters, except for the last two groups IX and X where the number of samples is small.

First, consider the statistical properties of significant wave heights ζ_1 and ζ_2, for the low and high frequency components, respectively, for a given severity. Consider the i-th group in Table 1 in which the wave spectra with significant heights ranging from $(\zeta_i)_{min}$ to $(\zeta_i)_{max}$ are included in the sample. Then, the significant heights for the low and high frequency components (ζ_1 and ζ_2, respectively) cannot exceed $(\zeta_i)_{max}$. In other words, the probability density function of ζ_1 and ζ_2 have to be truncated at $(\zeta_i)_{max}$. The results of the analysis have shown that the significant heights ζ_1 and ζ_2 both obey the normal probability law, and that the concept of truncation is necessary only for ζ_1, the significant height for the low frequency component.

Figure 12 shows a comparison between the histogram obtained from the data and the truncated normal probability distribution curve of ζ_1, for Group IV (a nominal significant height ζ = 4.57 m, 15 ft.). Figure 13 shows an example of the comparison for ζ_2 for the same Group IV. Still other examples, Figures 14 and 15 show similar comparisons made for more severe seas, Group VII (ζ = 9.15 m, 30 ft.). As can be seen in these figures, the significant wave heights ζ_1 and ζ_2 obey the normal probability law from which the confidence band of each parameter can be established. However, for a given sea severity, it is not necessary to consider the confidence band for ζ_1 and ζ_2 individually. This is because the two significant wave heights ζ_1 and ζ_2, and the significant wave height of the entire spectrum, ζ, are functionally related as discussed in the following:

If it is assumed that the low and high frequency component spectra are both narrow-banded and that their sum (namely, the entire spectrum) is also narrow-banded, then the following simple relationship holds:

$$\zeta^2 = \zeta_1^2 + \zeta_2^2 \tag{11}$$

If one of the component spectra (perhaps, the high frequency component spectrum) is not narrow-banded, then the significant wave height of the entire spectrum is analytically derived as outlined in Appendix A. In order to compare the difference between the two approaches, Figure 16 is prepared.

Table 1 Group of significant wave heights used for analysis

GROUP	SIGNIFICANT WAVE HEIGHT	
	NOMINAL	RANGE
I	4.0 ft (1.22 m)	Less than 5.5 ft (1.68 m)
II	6.5 (1.98)	5.5 ft (1.68 m) ---- 7.5 ft (2.29 m)
III	10.0 (3.05)	7.5 (2.29) ---- 12.5 (3.81)
IV	15.0 (4.57)	12.5 (3.81) ---- 17.5 (5.34)
V	20.0 (6.10)	17.5 (5.34) ---- 22.5 (6.86)
VI	25.0 (7.62)	22.5 (6.86) ---- 27.5 (8.38)
VII	30.0 (9.15)	27.5 (8.38) ---- 32.5 (9.91)
VIII	35.0 (10.67)	32.5 (9.91) ---- 37.5 (11.43)
IX	40.0 (12.20)	37.5 (11.43) ---- 42.5 (12.96)
X	45.0 (13.72)	Higher than 42.5 ft (12.96 m)

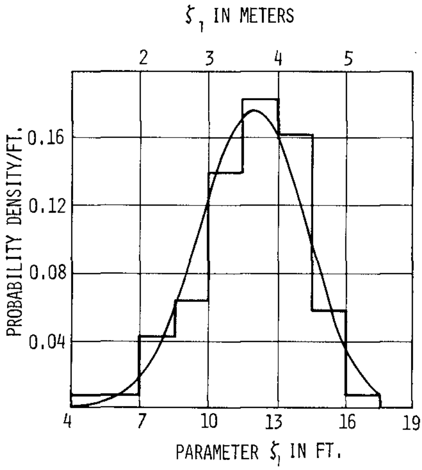

Figure 12 Histogram and probability density function of parameter ζ_1 for Group IV

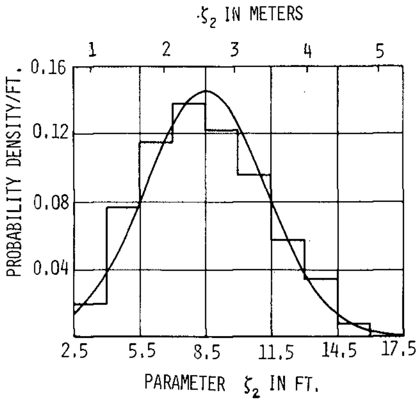

Figure 13 Histogram and probability density function of parameter ζ_2 for Group IV

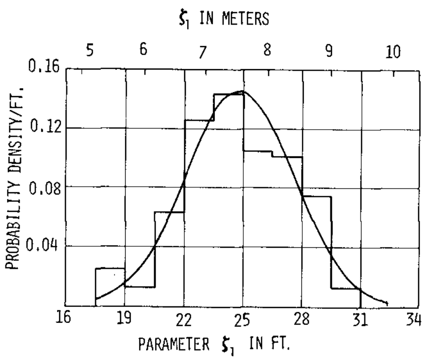

Figure 14 Histogram and probability density function of parameter ζ_1 for Group VII

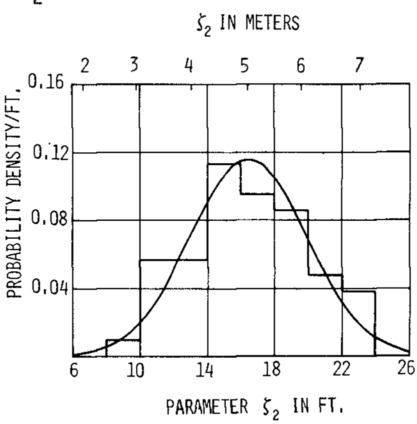

Figure 15 Histogram and probability density function of parameter ζ_2 for Group VII

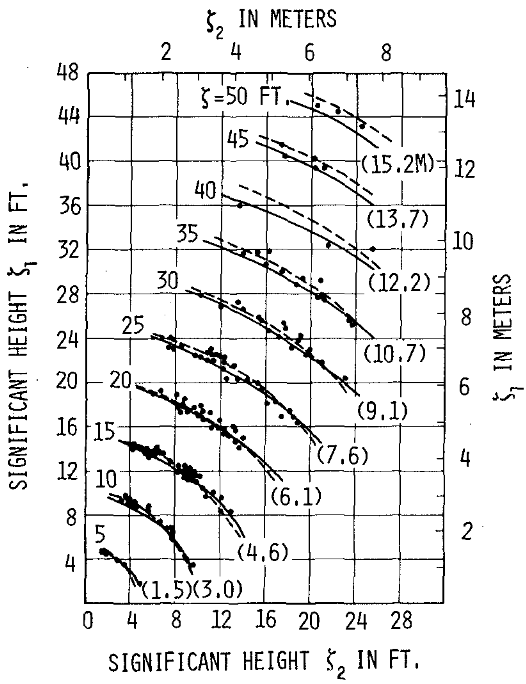

Figure 16 Relationship between significant wave heights ζ_1, ζ_2, and ζ

The solid lines in Figure 16 show the relationship between significant wave heights, ζ_1, ζ_2, and ζ, as obtained from Equation (A-10) in Appendix A, while the dashed lines show the relationships obtained from Equation (11). Included also in the figure are the results of the analysis of measured wave spectra whose significant wave heights are within ±2.5 percent of the specified height. For example, for significant wave height, ζ = 9.1 m (30 ft.), the results of the analysis of wave spectra with significant heights from 8.9 m (29.3 ft.) to 9.4 m (30.8 ft.) are plotted in the figure. As can be seen in the figure, the two lines obtained from Equations (A-10) and (11) do not differ appreciably, and that the results of analysis of the measured spectra fall on these lines. Hence, the simple formula given in Equation (11) may be used for the functional relationship between ζ_1 and ζ_2. Thus, for a given sea severity of significant wave height ζ, the six parameters involved in Equation (10) can be reduced to five parameters by taking the ratio of the two significant wave heights, ζ_1/ζ_2. For convenience, let $\zeta_1/\zeta_2 = \tan\theta$, and the statistical property of θ will be discussed hereafter.

The derivation of the probability density function of $\theta=\tan^{-1}(\zeta_1/\zeta_2)$(where, $0 < \theta < 90$) is outlined in Appendix B, and comparisons between histogram and the probability density function given in Equation (B-3) are shown in Figures 17 and 18 which pertain to Groups IV and VII, respectively. As can be seen in these examples, the derived probability density function agrees well with the histograms and hence the statistical analysis will be made using this probability function.

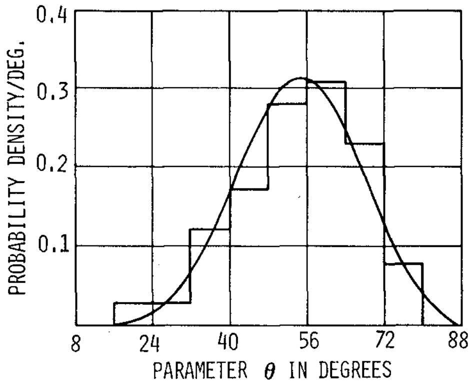

Figure 17 Histogram and probability density function of parameter Θ for Group IV

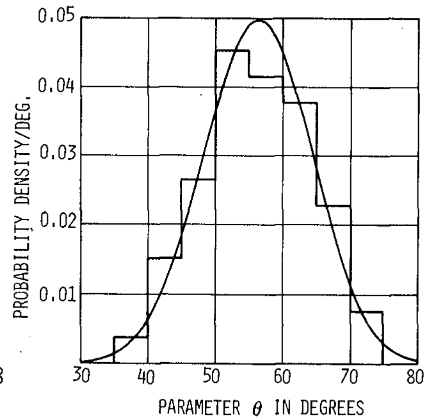

Figure 18 Histogram and probability density function of parameter Θ for Group VII

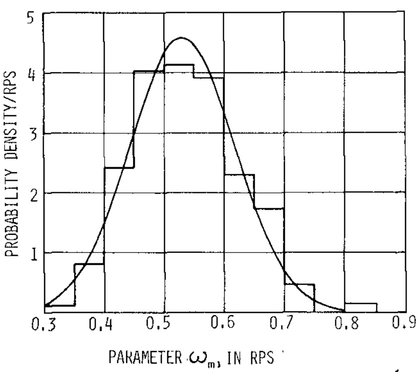

Figure 19 Histogram and probability density function of parameter ω_{m1} for Group IV

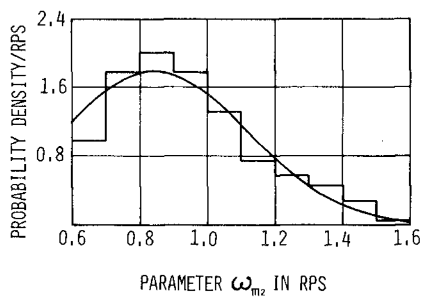

Figure 20 Histogram and probability density function of parameter ω_{m2} for Group IV

Next, the probability density function of the modal frequencies, ω_{m_1} and ω_{m_2}, will be obtained. In the determination of the modal frequencies for a given wave spectrum, no limitation is set for the value of ω_{m_1}, namely, $0 < \omega_{m_1} < \infty$. However, the modal frequency, ω_{m_2}, has to be greater than ω_{m_1}, and furthermore, the restriction is made such that ω_{m_2} should not be less than 0.6 in order to avoid the possibility of an unrealistic representation of the spectral shape. For example, suppose a spectrum has triple peaks, two of which are in the frequency range below 0.6, then the computer program may take these two peaks (even though the energy is not appreciable) and discard the third peak in the higher frequency range which is associated with the wind-generated seas. In this case, it may be more appropriate to represent the two peaks in the frequency range less than 0.6 by one peak, and consider the energy in the higher frequency range.

The results of analysis of the available data have indicated that ω_{m_1} and ω_{m_2} both obey the normal distribution law, although ω_{m_2} is truncated at 0.6. As an example, comparisons between histogram and the probability density function for Group IV are shown in Figures 19 and 20.

The value of the shape parameter, λ_1, is much higher than that of λ_2, in general; however, it appears that both follow the gamma probability law. Comparisons between the histogram and the gamma probability density function for λ_1 and λ_2 for Group IV are shown in Figure 21 and 22, respectively. Other examples of the comparisons for Group V (ζ = 6.10 m, 20 ft.) are shown in Figures 23 and 24.

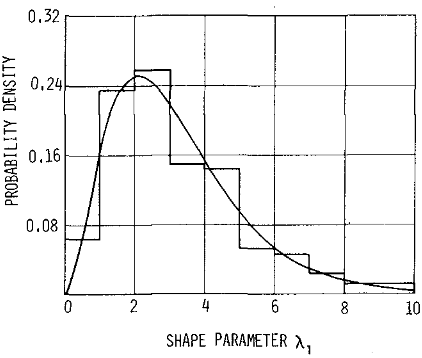

Figure 21 Histogram and probability density function of parameter λ_1 for Group IV

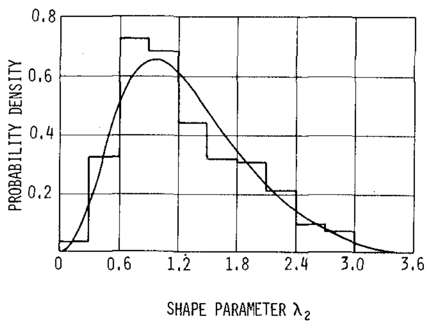

Figure 22 Histogram and probability density function of parameter λ_2 for Group IV

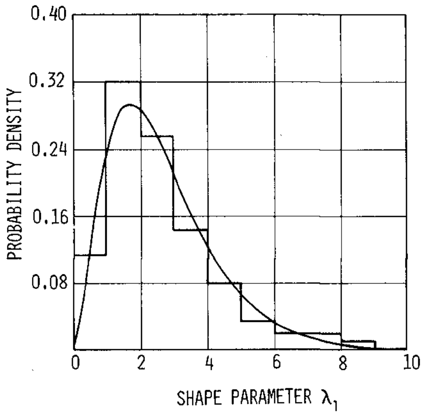

Figure 23 Histogram and probability density function of parameter λ_1 for Group V

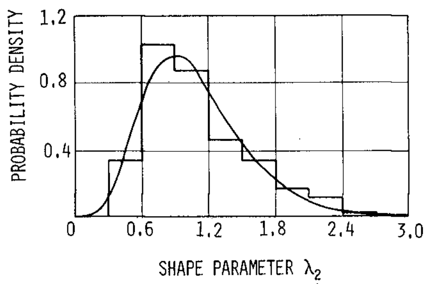

Figure 24 Histogram and probability density function of parameter λ_2 for Group V

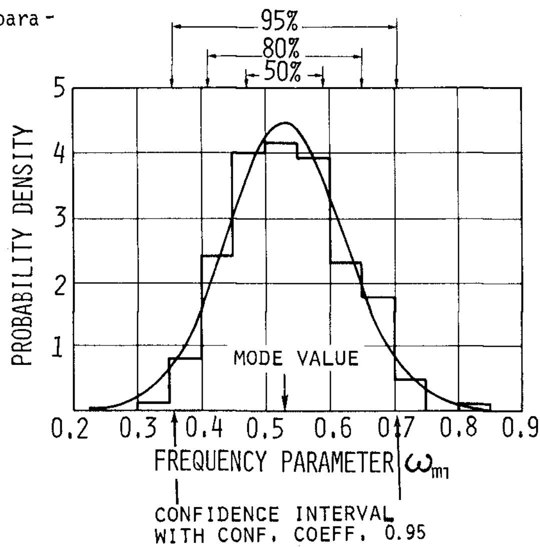

Figure 25 Mode value and upper and lower-ground values of ω_{m1} for a confidence coefficient 0.95 (Group IV)

FAMILY OF WAVE SPECTRA

In this section, a family of wave spectra will be developed for a desired sea severity using the probability function applicable for each parameter in an attempt to represent various shapes of spectra associated with a storm.

In the development of the family of spectra, an attempt was made first to divide each probability density function into seven divisions as shown at the top of Figure 25. The figure is an example for the modal frequency, ω_{m_1} for Group IV (nominal significant height 4.75 m, 15 ft.). Then six values were determined by which 50%, 80%, and 95% of the probability density function was covered as shown at the top of the figure. For each value of ω_{m_1}, the value of each of the other parameters was determined from the original data by taking their respective averages in the region of ±5% of ω_{m_1}. For example, the θ-value for $\omega_{m_1} = 0.65$ (the upper value for 80% coverage) was determined by taking the average value of θ from a sample which belonged to ±5% of $\omega_{m_1} = 0.65$ ($0.62 < \omega_{m_1} < 0.68$) in Group IV. Since there are five parameters, θ, ω_{m_1}, ω_{m_2}, λ_1, and λ_2, this procedure resulted in a family consisting of a total of 25 spectra for a given sea severity. Judging from the results of this procedure, however, it appears that 25 wave spectra are too many to be considered since many of the shapes look alike.

To reduce the number of spectra involved, another approach is to choose three values for each parameter; namely, the modal value, the upper and the lower values determined from a confidence band for a confidence coefficient of 0.95. For example, the most probable ω_{m_1} is obtained as 0.53 for Group IV as shown in Figure 25, and the upper and lower bound values are 0.71 and 0.36, respectively. Similarly, three ω_{m_1}-values are evaluated for all other groups (except Groups IX and X) listed in Table 1, and the results are plotted in Figure 26 as a function of significant wave height, ζ. Included also in this figure are the formulae representing the data points so that the ω_{m_1} values for a desired sea severity can be evaluated.

Next, the values of the other parameters for each ω_{m_1} are determined for all groups in the same fashion as mentioned earlier, the results of which are plotted in Figures 27 through 30 as a function of ζ. It is noted that the parameter values shown in these figures are all associated with the parameter ω_{m_1}. The formulae representing the data points are also included in these figures. Although the formulae are derived without consideration of data points for severe seas (ζ = 12.2 m, 40 ft. or above), the formulae may still be applicable for severe seas, since the data does not show serious scatter in relatively severe seas. Thus, a set of three spectra associated with the parameter ω_{m_1} can be drawn for an arbitrarily specified sea severity. As an example, Figure 31 shows a set for a significant wave height of 4.0 m (13.1 ft.). As can be seen in the figure, the dominant peaks of the spectra range from $\omega_{m_1} = 0.38$ to 0.74, and the spectral shape becomes more broad as the modal frequency ω_{m_1} increases.

The above discussion along with Figures 25 through 31 pertain to the parameter ω_{m_1}. The same procedure as that to derive a set of three spectra associated with ω_{m_1}, is carried out for the other four parameters, and thus a total of fifteen mathematical spectra can be established for a given sea severity. Of these fifteen spectra, five are associated with the mode value of the five parameters. The results of the analysis have

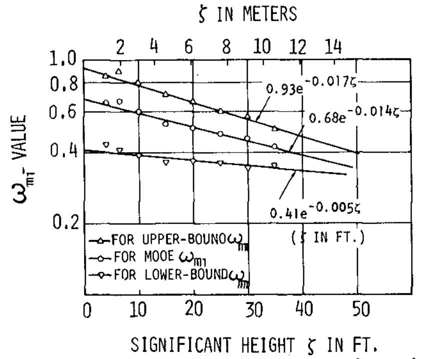

Figure 26 ω_{m1}-values as a function of significant height

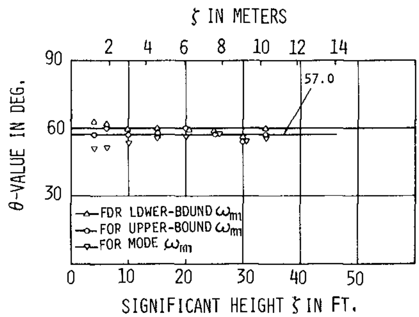

Figure 27 Θ-values for probable, upper and lower-bound values of ω_{m1} as a function of significant height

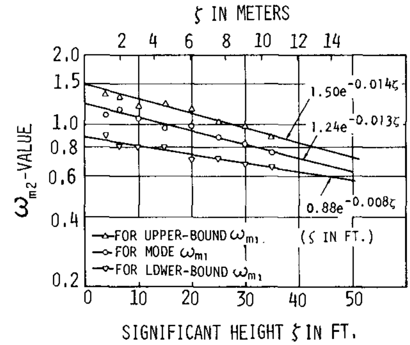

Figure 28 ω_{m2}-values for probable, upper and lower-bound values of ω_{m1} as a function of significant height

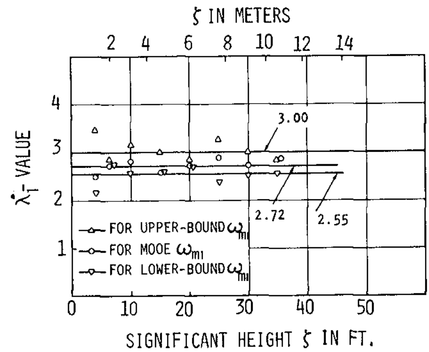

Figure 29 λ_1-values for probable, upper and lower-bound values of ω_{m1} as a function of significant height

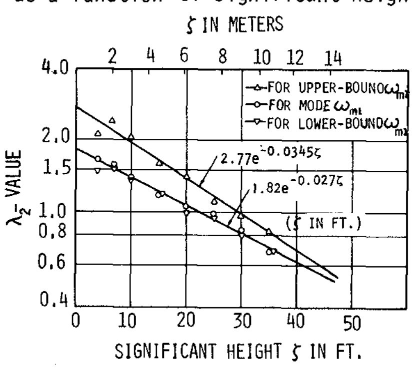

Figure 30 λ_2-values for probable, upper and lower-bound values of ω_{m1} of significant height indicated, however, that the shapes of these five spectra are nearly the same as shown in Figure 32. Hence, the spectrum associated with the mode value of the parameter θ may be chosen as representative of the five spectra determined from the mode values of the parameters.

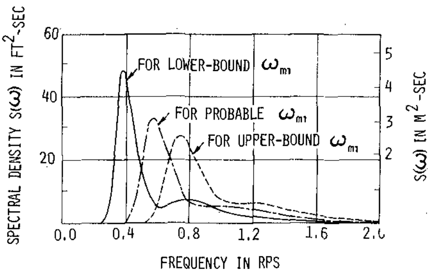

Figure 31 Spectral density function for probable, lower and upper-bound values of ω_{m1} (Sig. height 4.0 m, 12.1 ft)

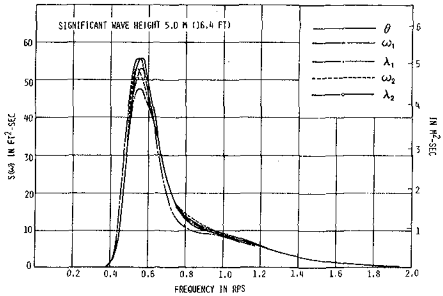

Figure 32 Comparison of five spectra associated with the mode value of five parameters (Sig. height 5.0 m, 16.4 ft)

Table 2a Values of six-parameters (ft-units)

	ζ_1	ζ_2	ω_{m1}	ω_{m2}	λ_1	λ_2
Most Probable Spectrum	$0.84\ \zeta$	$0.54\ \zeta$	$0.70\ e^{-0.014\ \zeta}$	$1.15\ e^{-0.012\ \zeta}$	3.00	$1.54\ e^{-0.019\ \zeta}$
95% Confidence Spectra	$0.95\ \zeta$	$0.31\ \zeta$	$0.70\ e^{-0.014\ \zeta}$	$1.50\ e^{-0.014\ \zeta}$	1.35	$2.48\ e^{-0.031\ \zeta}$
	$0.65\ \zeta$	$0.76\ \zeta$	$0.61\ e^{-0.012\ \zeta}$	$0.94\ e^{-0.011\ \zeta}$	4.95	$2.48\ e^{-0.031\ \zeta}$
	$0.84\ \zeta$	$0.54\ \zeta$	$0.93\ e^{-0.017\ \zeta}$	$1.50\ e^{-0.014\ \zeta}$	3.00	$2.77\ e^{-0.034\ \zeta}$
	$0.84\ \zeta$	$0.54\ \zeta$	$0.41\ e^{-0.005\ \zeta}$	$0.88\ e^{-0.008\ \zeta}$	2.55	$1.82\ e^{-0.027\ \zeta}$
	$0.90\ \zeta$	$0.44\ \zeta$	$0.81\ e^{-0.016\ \zeta}$	$1.60\ e^{-0.010\ \zeta}$	1.80	$2.95\ e^{-0.032\ \zeta}$
	$0.77\ \zeta$	$0.64\ \zeta$	$0.54\ e^{-0.012\ \zeta}$	0.61	4.50	$1.95\ e^{-0.025\ \zeta}$
	$0.73\ \zeta$	$0.68\ \zeta$	$0.70\ e^{-0.014\ \zeta}$	$0.99\ e^{-0.012\ \zeta}$	6.40	$1.78\ e^{-0.021\ \zeta}$
	$0.92\ \zeta$	$0.39\ \zeta$	$0.70\ e^{-0.014\ \zeta}$	$1.37\ e^{-0.012\ \zeta}$	0.70	$1.78\ e^{-0.021\ \zeta}$
	$0.84\ \zeta$	$0.54\ \zeta$	$0.74\ e^{-0.016\ \zeta}$	$1.30\ e^{-0.012\ \zeta}$	2.65	$3.90\ e^{-0.026\ \zeta}$
	$0.84\ \zeta$	$0.54\ \zeta$	$0.62\ e^{-0.012\ \zeta}$	$1.03\ e^{-0.009\ \zeta}$	2.60	$0.53\ e^{-0.021\ \zeta}$

ζ = significant wave height in feet

Table 2b Values of six-parameters (m-units)

	ζ_1	ζ_2	ω_{m1}	ω_{m2}	λ_1	λ_2
Most Probable Spectrum	$0.84\ \zeta$	$0.54\ \zeta$	$0.70\ e^{-0.046\ \zeta}$	$1.15\ e^{-0.039\ \zeta}$	3.00	$1.54\ e^{-0.062\ \zeta}$
95% Confidence Spectra	$0.95\ \zeta$	$0.31\ \zeta$	$0.70\ e^{-0.046\ \zeta}$	$1.50\ e^{-0.046\ \zeta}$	1.35	$2.48\ e^{-0.102\ \zeta}$
	$0.65\ \zeta$	$0.76\ \zeta$	$0.61\ e^{-0.039\ \zeta}$	$0.94\ e^{-0.036\ \zeta}$	4.95	$2.48\ e^{-0.102\ \zeta}$
	$0.84\ \zeta$	$0.54\ \zeta$	$0.93\ e^{-0.056\ \zeta}$	$1.50\ e^{-0.046\ \zeta}$	3.00	$2.77\ e^{-0.112\ \zeta}$
	$0.84\ \zeta$	$0.54\ \zeta$	$0.41\ e^{-0.016\ \zeta}$	$0.88\ e^{-0.026\ \zeta}$	2.55	$1.82\ e^{-0.089\ \zeta}$
	$0.90\ \zeta$	$0.44\ \zeta$	$0.81\ e^{-0.052\ \zeta}$	$1.60\ e^{-0.033\ \zeta}$	1.80	$2.95\ e^{-0.105\ \zeta}$
	$0.77\ \zeta$	$0.64\ \zeta$	$0.54\ e^{-0.039\ \zeta}$	0.61	4.50	$1.95\ e^{-0.082\ \zeta}$
	$0.73\ \zeta$	$0.68\ \zeta$	$0.70\ e^{-0.046\ \zeta}$	$0.99\ e^{-0.039\ \zeta}$	6.40	$1.78\ e^{-0.069\ \zeta}$
	$0.92\ \zeta$	$0.39\ \zeta$	$0.70\ e^{-0.046\ \zeta}$	$1.37\ e^{-0.039\ \zeta}$	0.70	$1.78\ e^{-0.069\ \zeta}$
	$0.84\ \zeta$	$0.54\ \zeta$	$0.74\ e^{-0.052\ \zeta}$	$1.30\ e^{-0.039\ \zeta}$	2.65	$3.90\ e^{-0.085\ \zeta}$
	$0.84\ \zeta$	$0.54\ \zeta$	$0.62\ e^{-0.039\ \zeta}$	$1.03\ e^{-0.030\ \zeta}$	2.60	$0.53\ e^{-0.069\ \zeta}$

ζ = significant wave height in meters

Thus, a total of eleven spectra derived in the above are considered as a family of wave spectra for a specified sea severity. The values of six parameters for these eleven spectra are expressed in terms of significant-wave height, ζ, and are tabulated in Table 2 so that a family of spectra for a desired sea can be generated from Equation (10). In these eleven spectra, one is considered as the "most probable spectrum" representing a specified sea, and the remaining ten spectra are those expected to occur with 95 percent confidence.

Examples of the family of spectra thus derived are shown in Figures 33 through 36. Figures 33 and 34 are for significant wave heights of 1.25 m (4.1 ft.) and 3.0 m(9.8 ft.), respectively, and the shapes of wave spectra vary considerably in these seas of relatively mild severity. The frequency domain where the predominant wave energy exists varies to a great extent (from 0.4 to 1.2) depending on the shape of spectra. On the other hand, Figure 35 is for a significant wave height of 9.0 m (29.5 ft.), and the frequency domain where the predominant wave energy exists varies to a much less extent (from 0.3 to 0.7) in this severe sea although a variety of spectral shapes may still be observed.

Figure 36 is for a significant wave height of 13.0 m (42.6 ft.), an example of very severe seas. A family of spectra shown in this figure is of particular interest since the values of all parameters for this family of spectra are determined by extending the lines of the analysis of data, examples of which are shown earlier in Figures 26 through 30. In order to examine whether or not the family of spectra thus generated represents the measured spectra, Figure 37 is prepared. The figure shows an observed spectrum which has the same significant wave height as that of the family shown in Figure 36, 13.0 m (42.6 ft.). Included also in the figure is the computer-generated six-parameter spectrum. From the comparison of these two figures, it can be seen that the example given in Figure 37 agrees well with one of the members of the family (the most probable spectrum) given in Figure 36. Thus, it appears that the families of spectra generated using the parameter values obtained from the extension of the lines in the analysis of the data can be considered to represent realistic sea spectra.

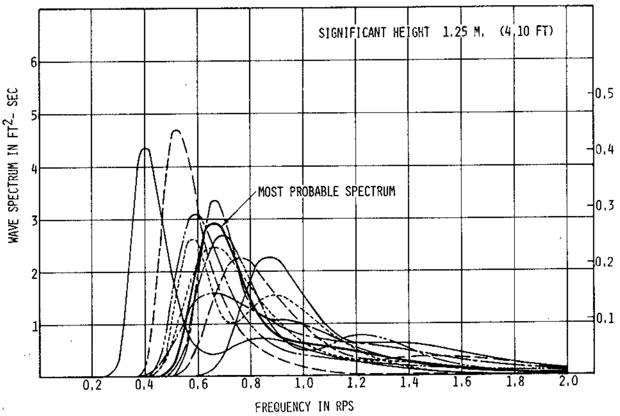

Figure 33 Family of spectra for significant wave height of 1.25 m (4.1 ft.)

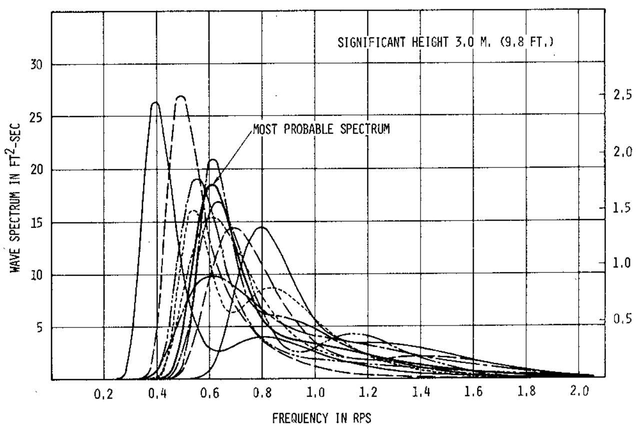

Figure 34 Family of spectra for significant wave height of 3.0 m (9.8 ft.)

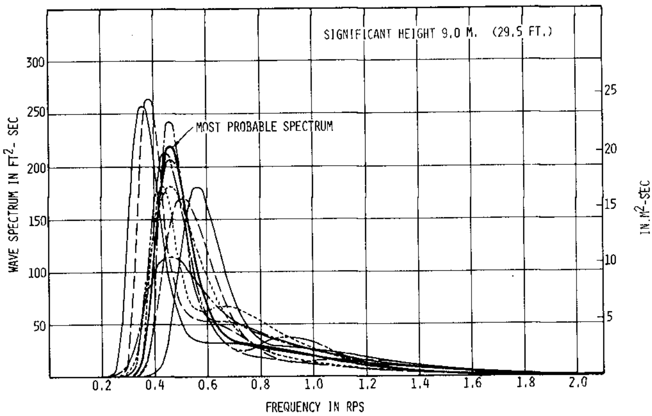

Figure 35 Family of spectra for significant wave height of 9.0 m (29.5 ft.)

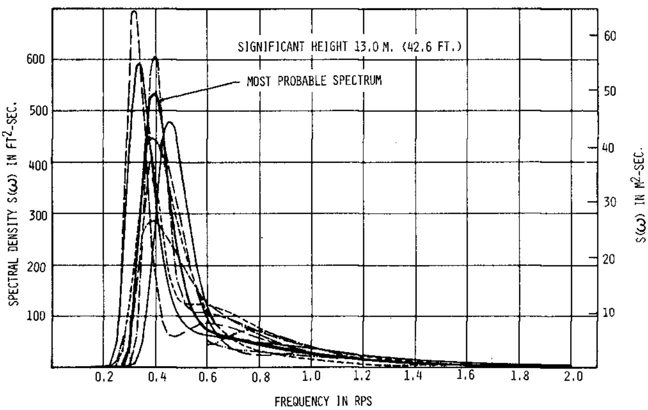

Figure 36 Family of spectra for significant wave height of 13.0 m (42.6 ft.)

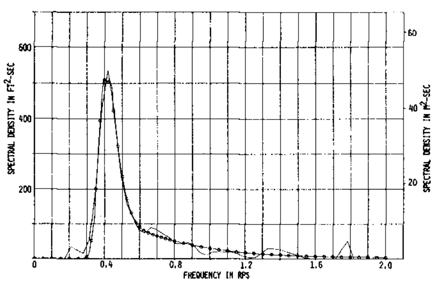

Figure 37 Comparison of measured and six-parameter spectrum (significant height 13.0 m, 42.6 ft.)

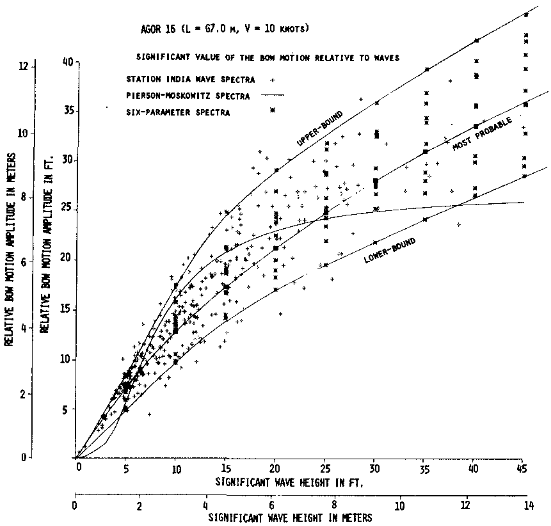

Figure 38 Significant amplitudes of the vertical relative bow motion of a catamaran in various seas (10-knot speed)

APPLICATION OF SIX-PARAMETER SPECTRA

The six-parameter wave spectra can be applied to evaluate responses such as motions and wave-induced forces, etc., of marine vehicles and structures in a seaway for design consideration. Figure 38 shows, as an example of the application, the significant values of the vertical bow motion relative to waves of a catamaran (length 67 m, 220 ft.) for 10-knot speed in various sea severities. Included also in the figure are the responses evaluated by using the Pierson-Moskowitz fully-developed sea spectra, and those due to individual wave spectra (about 300) observed at Station India in the North Atlantic Ocean [9].

As can be seen in the figure, for a specified sea severity, there are 11 responses (one for each member of the family of the six-parameter spectra), and one of the family members yields the largest response, and another yields the smallest response. The most probable spectrum given in Table 2 yields the most probable response. By connecting the points obtained in each sea severity, we may establish the most probable response, the upper-bound and the lower-bound responses which will provide useful information for design. The upper and lower-bounds cover the majority of the responses obtained from using the measured wave spectra, and it may safely be said for this case that the values of the six parameters obtained from a statistical analysis of the data yield a family of spectra representing realistic seas reasonably well with 95 percent confidence.

The results of a similar application made for predicting extreme wave-induced loads on an ocean structure may be found in Reference [10].

The six-parameter wave representation may be used for the analysis of wave spectra associated with particular situations such as breaking waves, wave run-up, etc. The measured spectra may be expressed in terms of six parameters and a statistical analysis may be carried out on the parameters to find significant factors which influence the spectra.

Another application of the six-parameter representation is to store or file a massive amount of measured data for statistical analysis. In the case where consecutive measurements are made within a certain time interval, the data may be stored in the form of six variables, the values of which change gradually with time, and hence the sea condition associated with the growth and decay of a storm may be discussed from a statistical analysis of the parameters.

CONCLUSIONS

Consideration of various shapes of wave spectra for a given sea severity is of particular significance for evaluation of response of ocean and coastal structures to waves, since the magnitude of responses is greatly influenced by the relative location of the modal frequencies of waves and response of individual structures. In an attempt to develop a systematic series of wave spectra covering a variety of spectral shapes, a series of wave spectra which involves six parameters are newly developed.

In the development of the six-parameter wave spectra, the spectra are decomposed into two parts, and each part is expressed by a mathematical formula with three parameters; i.e., significant wave height, modal frequency, and shape parameter, and the entire spectrum is expressed by the combination of two sets of three-parameter spectra. The parameters are determined numerically such that the difference between theoretical and observed spectra is minimal. The six-parameter spectra thus obtained appear to represent almost all stages of the sea condition associated with a storm.

Then, a total of 800 spectra observed in the North Atlantic Ocean are classified into 10 groups depending on severity, and for each group a statistical analysis is carried out on the parameters taking into account the correlation between them. From the results of analysis, a family consisting of eleven sets of mathematical spectra is established for a given sea severity. In these eleven spectra, one is considered as the "most probable spectrum" representing a specified sea, and the remaining ten spectra are those expected to occur with 95 percent confidence. The values of six parameters for these eleven spectra are expressed in terms of significant wave height, and are tabulated in Table 2 of the paper so that a family of spectra for a desired sea severity can be generated from the formula (Equation 10) given in the paper.

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APPENDIX A: SIGNIFICANT WAVE HEIGHT FOR A SPECTRUM CONSISTING OF HIGH AND LOW FREQUENCY COMPONENTS

Let x(t) be the wave profile of an irregular sea which is a normal random process with zero mean and a narrow-band spectrum, and let x(t) be comprised of two components; $x_1(t)$ and $x_2(t)$, which correspond to the wave profile of the low frequency and the high frequency components, respectively. Here, $x_1(t)$ and $x_2(t)$ are both normal random processes; however, in the following it is assumed that $x_1(t)$ is a narrow-banded but $x_2(t)$ is not necessarily narrow-banded. The amplitude, frequency, and variance are denoted by A, ω, and σ^2, respectively, and the subscripts 1 and 2 refer to the low and high frequency components, respectively.

Then, from the assumption described above, the wave profile x(t) may be written in the following form:

$$x(t) = A(t) \operatorname{Cos}(\omega_2 t - \Theta(t)) \tag{A-1}$$

The low-frequency component wave profile becomes,

$$x_1(t) = A_1(t) \operatorname{Cos}(\omega_1 t - \Theta_1(t)) \tag{A-2}$$

On the other hand, the high frequency component wave may be expressed in the following form:

$$x_2(t) = x_{2c}(t) \operatorname{Cos}(\omega_2 t) + x_{2s}(t) \operatorname{Sin}(\omega_2 t) \tag{A-3}$$

where,

$$X_{2c}(t) = \sum \left[a_n \operatorname{Cos}(n\omega - \omega_2)t + b_n \operatorname{Sin}(n\omega - \omega_2)t \right]$$

$$x_{2s}(t) = \sum \left[a_n \operatorname{Sin}(n\omega - \omega_2)t - b_n \operatorname{Cos}(n\omega - \omega_2)t \right]$$

and

$$a_n = \frac{2}{T} \int_0^T x_2(t) \operatorname{Cos}(n\omega t)\, dt$$

$$b_n = \frac{2}{T} \int_0^T x_2(t) \operatorname{Sin}(n\omega t)\, dt$$

Here, $x_{2c}(t)$ and $x_{2s}(t)$ are independent normal processes, and their joint probability density function can be written as,

$$f(x_{2c}, x_{2s}) = \frac{1}{2\pi\sigma_2^2}\, e^{-\frac{x_{2c}^2 + x_{2s}^2}{2\sigma_2^2}} \tag{A-4}$$

From the condition that $x(t) = x_1(t) + x_2(t)$, Equations (A-1), (A-2), and (A-3) yield:

$$x(t) = \left[x_{2c}(t)+A_1(t)\mathrm{Cos}\psi\right]\mathrm{Cos}\ \omega_2 t + \left[x_{2s}(t)-A_1(t)\mathrm{Sin}\psi\right]\mathrm{Sin}\omega_2 t \tag{A-5}$$

where,

$$\psi = (\omega_1-\omega_2)t-\Theta_1$$

Thus, from Equations (A-1) and (A-5), we can derive the following relationship:

$$\begin{aligned} x_{2c} &= A\ \mathrm{Cos}\ \Theta - A_1\ \mathrm{Cos}\ \psi \\ x_{2s} &= A\ \mathrm{Sin}\ \Theta + A_1\ \mathrm{Sin}\ \psi \end{aligned} \tag{A-6}$$

Since the joint probability density function of x_{2c} and x_{2s} is given in Equation (A-4), the joint probability density function of A and θ can be obtained from Equations (A-4) and (A-6) by transformation of two random variables (x_{2c}, x_{2c}) to (A, θ), and therefrom the marginal probability density function of A can be obtained. This part of the work was done by Middleton [A-1], and he derived the following formula:

$$f(A) = \int_0^{2\pi} f(A,\Theta)\, d\Theta = \frac{A}{\sigma_2^2}\, e^{-\frac{A^2+A_1^2}{2\sigma_2^2}}\, I_0\left(\frac{AA_1}{\sigma_2^2}\right), \quad 0 \le A < \infty \tag{A-7}$$

The probability density function of the amplitude, A, given in the above, however, is expressed in terms of the amplitude of the low frequency component waves, A_1. Hence, Equation (A-7) can be considered as a conditional probability density function $f(A|A_1)$. In order to express the probability density function of A in terms of the variance of the two component waves σ_1^2 and σ_2^2, the condition is used that the low frequency wave component has a narrow-band spectrum and hence its amplitude follows the Rayleigh probability law. Thus, the probability density function function of A can be written as follows:

$$\begin{aligned} f(A) &= \int_0^{\infty} f(A|A_1)\, f(A_1)\, dA_1 \\ &= \frac{A}{\sigma_2^2}\, e^{-\frac{A^2}{2\sigma_2^2}} \int_0^{\infty} I_0\left(\frac{AA_1}{\sigma_2^2}\right) \frac{A_1}{\sigma_1^2}\, e^{-\frac{1}{2}\left(\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right)A_1^2}\, dA_1 \end{aligned} \tag{A-8}$$

By carrying through the integration involved in the above equation, the following formula can be derived:

$$f(A) = \left(\frac{1}{1+\mu}\right) \frac{A}{\sigma_2^2} e^{-\frac{A^2}{2\sigma_2^2}}$$

$$\times \left(1 + \sqrt{\frac{\pi}{2}} \sqrt{\frac{\mu}{1+\mu}} \frac{A}{\sigma_2^2} e^{\left(\frac{\mu}{1+\mu}\right)\left(\frac{A}{2\sigma_2}\right)^2} I_{\frac{1}{2}}\left(\frac{\mu}{1+\mu}\left(\frac{A}{2\sigma_2}\right)^2\right)\right) \quad \text{(A-9)}$$

where,

$$\mu = \left(\frac{\sigma_1}{\sigma_2}\right)^2 , \qquad I_{\frac{1}{2}}(Z) = \sqrt{\frac{2}{\pi Z}} \text{ Sinh } Z$$

Equation (A-9) is the probability density function of the amplitude, A, in terms of two variances, σ_1^2 and σ_2^2, which are equal to the area under the low and high frequency components, respectively, of the wave spectrum. The significant wave height of the spectrum, ζ, is then obtained numerically from Equation (A-9) as follows:

$$\zeta = 3 \int_{A_*}^{\infty} A\, f(A)\, dA \quad \text{(A-10)}$$

where, A_* can be found from

$$\int_0^{A_*} f(A)\, dA = 2/3$$

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APPENDIX B: DERIVATION OF PROBABILITY DENSITY FUNCTION OF $\theta = \tan^{-1}(\zeta_1/\zeta_2)$

In the text it is discussed that the significant wave heights ζ_1 and ζ_2 both obey the normal probability law, and that the concept of truncation is necessary for the significant height for the low frequency component, ζ_1.

Let μ_1 and σ_1^2 be the mean and variance, respectively, of the significant height ζ_1 belonging to the i-th group in Table 1 given in the text, and let μ_2, σ_2^2 be the mean and variance of the significant height ζ_2. Since ζ_1 and ζ_2 may not necessarily be statistically independent, let ρ be the correlation coefficient which can be evaluated from the sample belonging to the i-th group. Then, by taking into account the truncation of ζ_1 at $(\zeta_i)_{max}$, the joint probability density function of ζ_1 and ζ_2 belonging to the i-th group can be written as.

$$f(\zeta_1,\zeta_2) = \frac{1}{1-\int_{(\zeta_i)_{max}}^{\infty} f(\zeta_1)\, d\zeta_1} \; \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

$$\times\, e^{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{\zeta_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{\zeta_1-\mu_1}{\sigma_1}\right)\left(\frac{\zeta_2-\mu_2}{\sigma_2}\right) + \left(\frac{\zeta_2-\mu_2}{\sigma_2}\right)^2\right\}} \qquad \text{(B-1)}$$

$$0 < \zeta_1 < (\zeta_i)_{max} \;, \quad 0 < \zeta_2 < \infty$$

where, $f(\zeta_1)$ is the probability density function of the significant height ζ_1 which is a normal distribution with mean μ_1 and variance σ_1.

Let $\eta = \zeta_1/\zeta_2$. Then, from Equation (B-1), the probability density function of η becomes,

$$f(\eta) = \frac{1}{1-\int_{(\zeta_i)_{max}}^{\infty} f(\zeta_1) d\zeta_1} \; \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_0^{\infty} \zeta_2$$

$$\times\, e^{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{\zeta_2\eta-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{\zeta_2\eta-\mu_1}{\sigma_1}\right)\left(\frac{\zeta_2-\mu_2}{\sigma_2}\right) + \left(\frac{\zeta_2-\mu_2}{\sigma_2}\right)^2\right\}} d\zeta_2 \qquad \text{(B-2)}$$

$$0 < \eta < \infty$$

Next, let $\theta = \tan^{-1}\eta = \tan^{-1}(\zeta_1/\zeta_2)$. Then, the probability density function of θ can be obtained numerically using Equation (B-2) as follows:

$$f(\Theta) = \Big[f(\eta)\Big]_{\eta = \tan\Theta} \mathrm{Sec}^2\,\Theta \;, \quad 0 < \Theta < 90 \qquad \text{(B-3)}$$