## CHAPTER 139

### MATHEMATICAL AND HYDRAULIC MODELS OF TIDAL WAVES

by

Jürgen Sündermann<sup>1)</sup> and Hans Vollmers<sup>2)</sup>

### ABSTRACT

A systematical comparison between experiments in hydraulic models and numerical computations for the case of the propagation of tidal waves in schematic estuaries is carried out. It is the aim of the investigations to present the advantages and disadvantages, the possibilities and limitations of the two models. The results show, generally, a fair agreement between measured and computed quantities even in the case of the occurence of horizontal eddies within the fluid.

### INTRODUCTION

Whereas the hydraulic model has acted as an indispensable tool of coastal engineers in years past, purely mathematical methods gained in importance only in recent times. This development can be attributed primarily to the advances made in computer technology. In principle both methods the hydraulic model and the hydrodynamical-numerical (HN) method, are well suited in simulating the dynamics of natural water bodies. However, before one can ascertain how the two methods can be optimally combined so that the model simulates natural processes to a high degree of accuracy, it is first necessary to investigate the advantages and disadvantages of each, their possibilities and limitations. This paper presents further results of a systematic

 Prof.Dr., Technische Universität Hannover Germany
Dr.-Ing., Bundesanstalt für Wasserbau Hamburg comparison of the two methods for tidal waves in schematic estuaries. Preliminary results were given by H. Vollmers at the 12th Coastal Engineering Conference [1].



Fig. 1 Schematic estuaries

In order to gain experience and to determine some basic guiding principles, work commenced on simple, yet representative models. These are to be gradually adapted to natural configurations. Fig. 1 shows the forms of the schematic estuaries used. They are similar in shape to those found along the German North Sea coast. The special layout and dimension of the models was determined by the space available in the laboratory, to location and setup of the measuring apparatus and, also, the characteristics of the HN-models and the available computer capacity.

In each case, we assumed a constant depth of 15 m. The width at the entrance of the region was 4 km. Forms A to D, that's those more distinctly one-dimensional, had a length of 55 km. Forms E and F, more two-dimensional, had a length of 16 km. The diameter of the circular extensions was 10 km in case E (Jade) and 5 km in case F (Dollart). The coastal boundaries were approximated by vertical walls.

For the estuaries A to D, we studied the temporal and spatial behaviour of a  $M_2$  tide of amplitude 1.5 m. For the cases E and F, the amplitude was increased to 3 m in order to obtain distinct measurable effects. TIDAL WAVE MODELS

In the mathematical models used for comparison, the coriolis force was neglected as the hydraulic model could not simulate this effect. In order to obtain some first hand information on the effect of the coriolis force, in particular for case E and F, corresponding HN computations were carried out.

Due to core limitations, the mathematical model utilizes vertically meaned horizontal velocities. Comparison with measured velocities had therefore to be limited to mean values. A HN-model which includes the third dimension and thus permits a realization of the full 3-dimensional velocity structure, has now also been completed [2].

#### THE HYDRAULIC MODEL

As the current processes examined here are, to a large extent, influenced by inertial and gravity forces, we can assign to the transformation ratio nature/model the wellknown Froude Number. In the hydraulic model the length and width scale was set at 1:1000, the depth scale was set at 1:100. With these scales a mean tidal period lasted 7.45 minutes. The model floor is composed of sand immobile under existing velocities. This floor, preformed by appropriate waves (approximately 1 cm amplitude) allowed us to simulate friction to a good degree. In Fig. 2 next to the just analyzed form E, we may also see the contours of form D, and the dimension of the 55 km long rectangular channel, form A.

The tidal waves were produced by a mechanical control system. During the experiments water levels and current speed were measured with vibrating points System Delft, self registering floating gauges and micro propellers. Directions were determined either with the help of small paper strips or photographically with a floating body free to rotate in the horizontal plane. As boundary conditions in the mathematical model we used the values at the entrance as measured in the hydraulic model. While 2425

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carrying out the speed measurements not only the time variations over a tidal cycle were determined but also the vertical distribution. Measuring locations in the hydraulic model were equivalent to corresponding grid points in the mathematical model.



Fig. 2 Model installation

### THE MATHEMATICAL MODEL

The quasilinear, vertically integrated hydrodynamic equations were taken as the basis for the model. In order to solve the corresponding initial value problem numerically, an explicit method developed by Hansen, the HN method, was used [3]. The mathematical model based on this principle can be adapted to natural regions to a good degree. As empirical input data it is necessary to prescribe merely the tidal amplitude at the entrance and the geometry of the area in question.

The particular layout of the computational grid in the finite difference method is determined by the type of problem to be solved, the necessary accuracy one wishes to attain and the allowable computation expenditures. Model forms A to D were computed one-dimensionally, that is, not considering cross-streams but considering a variable cross-section. The estuary forms E and F were treated 2-dimension-

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ally. The following differential equations are used

(1.1) 
$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{r}{h+\delta} \sqrt{u^2 + v^2} u - fv - A_H \Delta u + g \frac{\partial \xi}{\partial x} = 0$$

(1) (1.2) 
$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{r}{h+\xi} \sqrt{u^2 + v^2} v + fu - A_H \Delta v + g \frac{\partial \xi}{\partial y} = 0$$

(1.3) 
$$\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial x} ((h + \xi)u) + \frac{\partial}{\partial y} ((h + \xi)v) = 0$$

with u,v	= components of the vertical meaned velocity vector in the x-, y-direction, respectively
Ś	<pre>= deviation of the sea surface from the undi- sturbed level</pre>
h	= mean water depth
t	= time
А <sub>Н</sub>	= horizontal eddy coefficient
f	= coriolis parameter
g	= acceleration of gravity
r	= friction coefficient

This system of hyperbolic partial differential equations is nonlinear and hence particularly suited to shallow water dynamics which are known to be distinctly nonlinear.

In addition, boundary conditions must be added as follows

(2.1)  $\underline{v}_n = 0$  along the coastline (2.2)  $\underline{\xi}(t) = A\cos(6t - 2t)$  at the entrance of the estuary where A = amplitude  $\underline{6} = frequency$  2t = phase of the incoming tidal wave.As initial conditions we assume a state of rest: (3)  $u = v = 0; \quad \underline{\xi} = 0$  In a one-dimensional calculation the equations are corresponding simplified.

Now a brief description of the HN models for the estuary forms A to F will be given. The appropriate grids and natural dimensions may be seen by Fig. 3. The layout of the grid points is well suited to the structure of the differential equations. The grid locations where tidal oscillation are generated are marked.





Fig. 3 Numerical grids for the estuaries

Sündermann

All models have the following scales:

∆x	= Δ y	≈ 500	) m	spatial grid distance be- tween two similar compu- tation points
h	= 15	m		water depth
s A t	ο D: Δ	t = 30	sec	time stop

Models E, F :  $\triangle t = 25$  sec

Mode1

In each case, the computation were carried out until stationary conditions were achieved. The maximum time to reach this state was 5 tidal periods.

### COMPARISON OF THE RESULTS

The numerous measurements and numerical results under consideration permit a very detailed comparison between the two methods. Within the limitations of this paper, only a representative selection can be given. Results will be given only to the extent where they extend those already reported at the Washington conference.





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Fig. 4 shows a comparison between the computed tidal amplitudes and the corresponding measured ones. The comparison is quite good. The maximum differences are of the order of a few centimeters; ever 50 % of the differences lie within measuring error. The convective terms were neglected in the equations of motion. Including these terms reduced the above differences. Fig. 5 shows a comparison between computed and measured tidal curves at cross-section 56 of estuary B (at the point of transition from rectangle to triangle) over several tidal periods. The agreement in both curves as regards form, amplitude and tidal duration is good.



Thw = tide high water Tnw = tide low water

Fig. 5 Computed and measured tidal curve for the water elevation in cross-section 56 of estuary B

With estuary forms E and F we essentially wanted to compare the current structure in the whole model rather to make a detailed comparison of water level and current velocity. Fig. 6 shows a comparison between computed and measured current velocities over a tidal period. Flood and ebb curves are very similar; the agreement is adequate. Shown also are curves for profile 5 in model A. One can see that



the characteristic distribution of flood and ebb agree.

Fig. 6 Computed and measured current velocities during one tidal period in cross section 5 of estuaries A and E

Of particular interest is the comparison of the velocity structures for form E. One could expect that separation regions would form at the circular expansion areas especially at flood times. The development of these separation zones under nonstationary conditions is particularly instructive. Initial computations with the mathematical model however showed that, except during a short time interval at slack water, no separation occurred. It seems that sufficient conservation of vorticity was not achieved in this model. This failure could be eliminated by the inclusion of the convective terms. With help of synchronous output of the measured velocities, we were able to simulate and compare a classical vortex. The comparison was good.

Figures 7 (a) to (d) show the computed velocity fields (marked with arrows) with the corresponding fields in the hydraulic model for various sequential tidal phases. In Fig. 7 (a) the vortex has just started to be formed near

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the expansion region. Fig. 7 (d) shows the vortex at its largest extension at the start of the ebb currents. It is just beginning to disintegrate. A boundary layer flow diminishes it in the central region of the estuary.



(ъ)

DRECTION OF THE VELOCITY AFTER T + 9500 SEC (DEGREE)

Fig. 7 a,b Computed and measured velocity fields in case of estuary E for two tidal phases separated by time interval of 1/12 tidal period

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DIRECTION OF THE VELOCITY AFTER T - 99325 SEC (DEGREE)



(d)



Fig. 7 c,d Computed and measured velocity fields in case of estuary E for two tidal phases seperated by time interval of 1/12 tidal period

The possible effect of the coriolis force in the more 2-dimensional formes E and F as also in nature is of particular interest. Computations showed that, although the water level distribution did not change significantly, the horizontal velocity structure at particular phases of the tide, were altered significantly. As an example, Fig. 8 shows the current distribution at the same time as in Fig. 7(d) but including the coriolis effect. One may see the large change - in particular the unsymmetric distribution of the direction field.

DIRECTION OF THE VELOCITY AFTER T = 103050 SEC (DEGREE)



Fig. 8 Velocity fields as in Fig. 7 (d) including the coriolis force

Figures 9 (a) and (b) show two current situations for estuary F. Two eddies have formed in the expansion area. Just before slack water the whole circular basin consists one large vortex. To be noted is the good comparison as regards the area of deformation. Just at this time the eddy also extends somewhat into the rectangular canal and causes a deflection of the streamlines. (a)





Fig. 9 a,b Computed and measured velocity fields in case of estuary F for two tidal phases separated by a time interval of 1/4 tidal period

#### CONCLUSIONS

Seen overall, the results of the hydraulic and mathematical model compare well for the tidal waves considered. The essential characteristics of the currents have been qualitatively reproduced. The quantitative comparison shows, nearly without exception, only small differences between measured and computed quantities. This is especially true for the simple forms A and B. For these forms the hydraulic model can be substituted by an equivalent mathematical one.

No detailed quantitative comparison have yet been made for the estuaries E and F. The development of the current system at particular tidal phases is however so similar in the two systems that, in general, the two methods can be considered equivalent.

On the basis of measurement we have some to the general important conclusion that for the HN models the convective terms could not be neglected when analyzing tidal phenomena. On the other hand, the insertion of a simple quadratic friction term with a constant friction parameter seemed adequate. A comparison of the computations, with and without the coriolis force, showed that for two-dimensional regions large differences can occur in the current patterns although the surface elevations are not appreciably changed.

The results presented here, indicate that it would be worthwhile to improve the mathematical model. Thus, we intend to include the vertical structure in future and, by using a variable spaced difference net, to be able to simulate the real geometry to a higher degree of accuracy. It would then seem possible to replace the hydraulic experiment, for cases similar to the ones examined by a HN model. The hydraulic model can, on the other hand, be utilized in those complex instances where satisfactory mathematical formulations do not as yet exist. REFERENCES

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