## CHAPTER 30

# VELOCITY AND SHEAR STRESS IN WAVE BOUNDARY LAYERS

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## SYNOPSIS

The critical force for the entrainment of sediment on the ocean floor is the maximum, instantaneous shear force. A numerical estimate for the stress is made from the third approximation of a second order boundary layer theory for oscillating laminar flow. The analytical derivation satisfies both the case of flow in an oscillating water tunnel and the case of a progressive (Airy) wave, where the shear distribution depends on the form of the velocity distribution in the boundary layer. Solution for the velocity is on the basis of iteration in an infinite series, where the convective terms are numerically evaluated from lower order solutions. For the boundary shear, the phase lead is found to be less than predicted by linear theory, and although the correction at the third approximation is small compared to lower approximations, the modified vertical distribution provides a basis for the correction of shear measurements, obtained by indirect means, to the boundary value.

## INTRODUCTION

In coastal engineering, one of the critical problems is the prediction of the motion of sediment or solid pollutants in the marine environment. Although it is realized that the critical parameter for the entrainment of sedimentary particles is the instantaneous shear force, an accurate assessment of its numerical value for various wave and boundary conditions has been difficult to obtain. In part this stems from the difficulty in analytical modeling of the velocity and shear stress distributions near a solid boundary, but it is also the result of the scarcity of experimental data on these parameters in the zone of shoaling waves. The importance of understanding boundary layer behavior under ocean waves can be well illustrated with the findings of Stone and Summers (1972) who have determined that 95% of the total load moves within 15 cm of the bottom in the nearshore zone and 80% of which is carried as bedload. These figures agree with Saville's (1950) results, where 40-100% of the sediment was found to be transported as bedload in a model basin.

In this paper we are concerned with the modeling of the velocity distribution in laminar flow of oscillating boundary layers of the form found in oscillating water tunnels and of a progressive (Airy) wave. Consequently, we evaluate the shear stress distribution analytically and appraise the contribution the third approximation makes on the phase and amplitude of the instantaneous friction in order to understand the force distribution at the boundary.

The boundary layer theory of Schlichting (1932) is the basis of developments reported herein, with higher approximations of Kestin and others (1961, 1967) and Shah (1970). Similar approaches can be found in the studies of Hill and Stenning (1960), Hori (1962), Hunt and Johns (1963), Dore (1968), Noda (1971) and others.

## THE BOUNDARY LAYER

### 1. The velocity distribution

The distribution of velocity in the potential flow region of harmonically varying flows without displacement is of the form:

$$U(x,t) = U_o(x)\cos\omega t = Re\left(U_o(x)e^{i\omega t}\right) \tag{1}$$

where $U_o$ is the velocity amplitude at the edge of the boundary layer, $\omega = 2\pi/T$ is the characteristics frequency of oscillation of period $T$, and $Re$ denotes the real part of the complex variable.

The periodic flow in the free stream region produces oscillations in the fluid near the solid boundary. If the amplitudes of fluctuation are small, such as $(ka,\ a/d) \ll 1$, the boundary layer behavior can be calculated by using Fourier series for arbitrary fluctuations of the free stream with time (Hill and Stenning, 1960). This method involves linearization of the stream function $\psi(x,y,t)$, which is expanded in an infinite series

$$\psi(x,y,t;\varepsilon) = \varepsilon\left(\psi_1(z) + \varepsilon\psi_2(z) + \varepsilon^2\psi_3(z) + \ldots\right)e^{i\omega t} \tag{2}$$

following Schlichting's (1932) technique. According to Kestin and others (1967)

$$\psi = \delta^{-1}U_o(x)\sum_{n=1}^{\infty}\chi_n(\eta,t) \tag{3}$$

given

$$\eta = y/\delta \tag{4}$$

and

$$\delta = (\omega/2\nu)^{1/2} \tag{5}$$

where $\delta$ is the thickness of the boundary layer. The parameter $\chi_n$ represents the dimensionless stream function.

For incompressible, irrotational flow the stream function and the velocity potential can be equated through the velocity terms, so that

$$-\frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = u$$
$$\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} = v \tag{6}$$

where $\psi(x,y,t)$ and $\phi(x,y,t)$ meet the requirement

$$\nabla^2\phi = \nabla^2\psi = 0 \tag{7}$$

Considering now, that motion is only periodic in the potential flow region, i.e. steady flow is externally not superimposed on the boundary layer, we can write the velocity distribution in the two-dimensional case in two Fourier series

$$u(x,y,t) = u_0(x,y) + \sum_{n=1}^{\infty} u_n(x,y)e^{in\pi t} + \sum_{n=1}^{\infty} u_n^*(x,y)e^{-in\pi t}$$
$$v(x,y,t) = v_0(x,y) + \sum_{n=1}^{\infty} v_n(x,y)e^{in\pi t} + \sum_{n=1}^{\infty} v_n^*(x,y)e^{-in\pi t} \tag{8}$$

where $u_0$, $v_0$ are components of the steady "streaming" arising from the second approximation and $u^*$, $v^*$ represent the complex conjugates of $u$, $v$. Complex variable theory is used to minimize computations in this expansion procedure.

We shall now establish the motion in the boundary layer by writing the appropriate momentum equation to $O(\varepsilon^2)$

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} +$$
$$\frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_2}{\partial x} + v_1 \frac{\partial u_2}{\partial y} + v_2 \frac{\partial u_1}{\partial y} + v_2 \frac{\partial u_2}{\partial y} +$$
$$\frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_3}{\partial x} + u_3 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_3}{\partial y} + v_3 \frac{\partial u_1}{\partial y} =$$
$$\frac{\partial U_0}{\partial t} + U_0 \frac{\partial U_0}{\partial x} + V_0 \frac{\partial U_0}{\partial y} + \ldots + \tag{9}$$
$$\nu \left( \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_2}{\partial y^2} + \frac{\partial^2 u_3}{\partial y^2} \right)$$

If initially we assume $u = O(1)$ and $v = O(\varepsilon)$ where $\varepsilon = U_0/\omega\delta$ is the perturbation parameter, we can neglect all convective acceleration terms involving $v_{1,2,3}$ and $V_0$. We also assume the pressure gradient across the layer to be small compared to its magnitude just outside the layer, and use this term to match the flows across the upper boundary. We now make use of the reduced equation:

$$\frac{\partial}{\partial t}(u_1 + u_2 + u_3) + \frac{\partial u_1}{\partial x}(u_1 + u_2 + u_3) +$$
$$\frac{\partial u_2}{\partial x}(u_1 + u_2) + u_1 \frac{\partial u_3}{\partial x} =$$
$$\frac{\partial U_0}{\partial t} + U_0 \frac{\partial U_0}{\partial x} + \ldots + \nu \frac{\partial^2}{\partial y^2}(u_1 + u_2 + u_3) \tag{10}$$

to the third approximation. Each of the velocity terms in Eq.10 must separately satisfy continuity, therefore

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0$$
$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \tag{11}$$
$$\frac{\partial u_3}{\partial x} + \frac{\partial v_3}{\partial y} = 0$$

As $u_2 = u_{21} + u_{22}$, where $u_{21}$ is the periodic and $u_{22}$ is the steady component, the boundary conditions appropriate to Equations 10, 11 are

$$u_1 = \begin{cases} 0 \text{ at } & y = 0 \\ U_0 \text{ at } & y = \delta \end{cases}$$

$$u_{21} = u_3 = \begin{cases} 0 \text{ at } & y = 0 \\ 0 \text{ at } & y = \infty \end{cases}$$

$$u_{22} = \begin{cases} 0 \text{ at } & y = 0 \\ O(\varepsilon) \text{ at } & y = \delta \end{cases}$$

$$v_1 = v_2 = v_3 = \begin{cases} 0 \text{ at } & y = 0 \\ < \varepsilon \text{ at } & y = \delta \end{cases}$$

According to Stuart (1963), the steady velocity at the outer edge of the boundary layer is of the order $\varepsilon U_0$, which defines a Reynolds number $R_s = U_0^2/\omega\nu = a^2/\delta^2$. This second perturbation parameter is of $O(1)$ when $\varepsilon \ll 1$, which condition is required for maintaining a non-decaying man transport velocity in the "outer boundary layer", defined by Stuart (1966), and also to validate linearizing the equations of motion. When $R_s$ is large, $u_{22}$ progressively diminishes in the outer layer, the interaction between flow within and outside the boundary layer can be shown to remain negligible, and the steady streaming will not be influenced significantly by the presence of potential flow.

Following Kestin et al (1967), the general expression for the stream function to the nth approximation is of the form:

$$\psi = \frac{1}{\delta}\Big[\frac{U_o^{n-1}}{\omega^{n-1}}\frac{d^{n-1}U_o}{dx^{n-1}}\Big\{\zeta_{na1}(\eta)e^{ni\omega t}+\zeta_{nb1}(\eta)e^{(n-2)i\omega t}+\ldots\Big\}+ \frac{U_o^{n-2}}{\omega^{n-1}}\frac{dU_o}{dx}\frac{d^{n-2}U_o}{dx^{n-2}}\Big\{\zeta_{na2}(\eta)e^{ni\omega t}+\zeta_{nb2}(\eta)e^{(n-2)i\omega t}+\ldots\Big\}+\ldots\Big] \tag{12}$$

Differentiating and retaining the real part, three approximations of the horizontal velocity components are obtained.

$$u_1 = Re\left[U_o\zeta'_{1a0}(\eta)e^{i\omega t}\right] \tag{13}$$

$$u_2 = Re\left[\frac{U_o}{\omega}\frac{dU_o}{dx}\left(\zeta'_{2a1}(\eta)e^{i2\omega t}+\zeta'_{2b1}(\eta)\right)\right] \tag{14}$$

$$u_3 = Re\left[\left(\frac{U_o}{\omega}\right)^2\frac{d^2U_o}{dx^2}\left(\zeta'_{3a1}(\eta)e^{i3\omega t}+\zeta'_{3b1}(\eta)e^{i\omega t}\right)+ \frac{U_o}{\omega^2}\left(\frac{dU_o}{dx}\right)^2\left(\zeta'_{3a2}(\eta)e^{i3\omega t}+\zeta_{3b2}(\eta)e^{i\omega t}\right)\right] \tag{15}$$

As $y\to\delta$ the steady component becomes

$$u_{22} = -3/4\left(U_o/\omega\ dU_o/dx\right) \tag{16}$$

The third approximation for the velocity component normal to the boundary, and the differential equations for $\zeta$ with their pertinent boundary conditions are omitted for the sake of brevity. These can be found in the article by Shah (1970). The first two terms have been determined by Schlichting (1932).

$$\zeta_1(\eta) = \zeta_{1a0}(\eta) = \frac{1-i}{2}+\eta+\frac{1-i}{2}e^{-(1+i)\eta} \tag{17}$$

$$\zeta_{2a}(\eta) = \zeta_{2a1}(\eta) = \frac{1+i}{4\sqrt{2}}e^{-\sqrt{2}(1+i)\eta}+\frac{i}{2}\eta e^{-(1+i)\eta}-\frac{1-i}{4\sqrt{2}} \tag{18}$$

$$\zeta_{2b}(\eta) = \zeta_{2b1}(\eta) = \frac{13}{8}-\frac{1}{8}e^{-2\eta}-\frac{3}{2}e^{-\eta}\cos\eta-e^{-\eta}\sin\eta- \frac{\eta}{2}e^{-\eta}\sin\eta-\frac{3}{4}\eta \tag{19}$$

For the third approximation Shah (1970) obtained

$$\zeta_{3a}(\eta) = \zeta_{3a1}(\eta) = (1-i)\Big\{\Big(\frac{5}{16\sqrt{3}} + \frac{1+2\sqrt{2}}{16(\sqrt{2}+1)} - \frac{1}{4}\Big) - \frac{5}{16\sqrt{3}} e^{-\sqrt{3}(1+i)\eta} - \frac{1}{4\sqrt{2}} e^{-\sqrt{2}(1+i)\eta} - \frac{1}{8}\Big[(1+i)\eta - \Big(1+\frac{1}{\sqrt{2}}\Big)\Big]e^{-(1+i)\eta} + \frac{1-\sqrt{2}}{16(\sqrt{2}+1)} e^{-(\sqrt{2}-1)(1+i)\eta} + \frac{1}{8} e^{-2(1+i)\eta}\Big\} \tag{20}$$

$$\zeta_{3b}(\eta) = \zeta_{3b1}(\eta) = \Big[\frac{1}{8}(2-i)\eta^2 - \frac{1}{8}(1+2i)\eta + \frac{1}{1200}(608-521i)\Big]e^{-(1+i)\eta} - \Big[\frac{1}{8}\eta + \frac{1}{16}(5+7i)\Big]e^{-(1-i)\eta} - \frac{i}{80}e^{-(3+i)\eta} - \frac{1}{48}(1-i)e^{-2(1+i)\eta} - \Big[\frac{1}{20}(2+i)\eta + \frac{1}{200}(47+96i)\Big]e^{-2\eta} - \frac{3}{4}i\eta + \frac{1}{600}(37+806i) \tag{21}$$

$$\zeta_{3c}(\eta) = \zeta_{3a2}(\eta) = -\frac{3}{16\sqrt{3}}(\sqrt{2}+1)(1-i)e^{-\sqrt{3}(1+i)\eta} - \Big[\frac{1}{2}\eta - \frac{1+\sqrt{2}}{4\sqrt{2}}(1-i)\Big]\cdot e^{-\sqrt{2}(1+i)\eta} + \Big[-\frac{1}{4}(1+i)\eta^2 + \frac{1}{4}\eta + \frac{1-3\sqrt{2}}{8\sqrt{2}}(1-i)\Big]e^{-(1+i)\eta} + \frac{3-2\sqrt{2}}{16(\sqrt{2}+1)}(1-i)e^{-(\sqrt{2}+1)(1+i)\eta} + \frac{1-i}{16\sqrt{6}(\sqrt{2}+1)}(12+9\sqrt{2}-7\sqrt{6}+2\sqrt{3}) \tag{22}$$

$$\zeta_{3d}(\eta) = \zeta_{3b2}(\eta) = \Big[\frac{i}{12}\eta^3 + \frac{1}{16}(11+3i)\eta^2 + \frac{1}{16}(16-13i)\eta + \frac{1}{800}(903-951i)\Big]e^{-(1+i)\eta} + \Big[-\frac{1}{20}(3+4i)\eta + \frac{1}{200}(12-159i)\Big]e^{-2\eta} - \Big[\frac{1}{8}(1-i)\eta^2 + \frac{1}{8}(9+2i)\eta + \frac{1}{16}(13-31i)\Big]e^{-(1-i)\eta} + \frac{1}{80}e^{-(3+i)\eta} - \frac{3}{2}i\eta + \frac{1}{800}(-311+3137i) \tag{23}$$

Distribution of the horizontal velocity for phase angles $\omega t = 0, \pi/3, \pi/2, 2\pi/3, \pi$ is shown in Figure 1 for $4/\omega(dU_o/dx) = 16/\omega^2(U_o d^2U_o/dx^2) = 1$. Comparison of the velocity profiles indicate that the third approximation is effective near

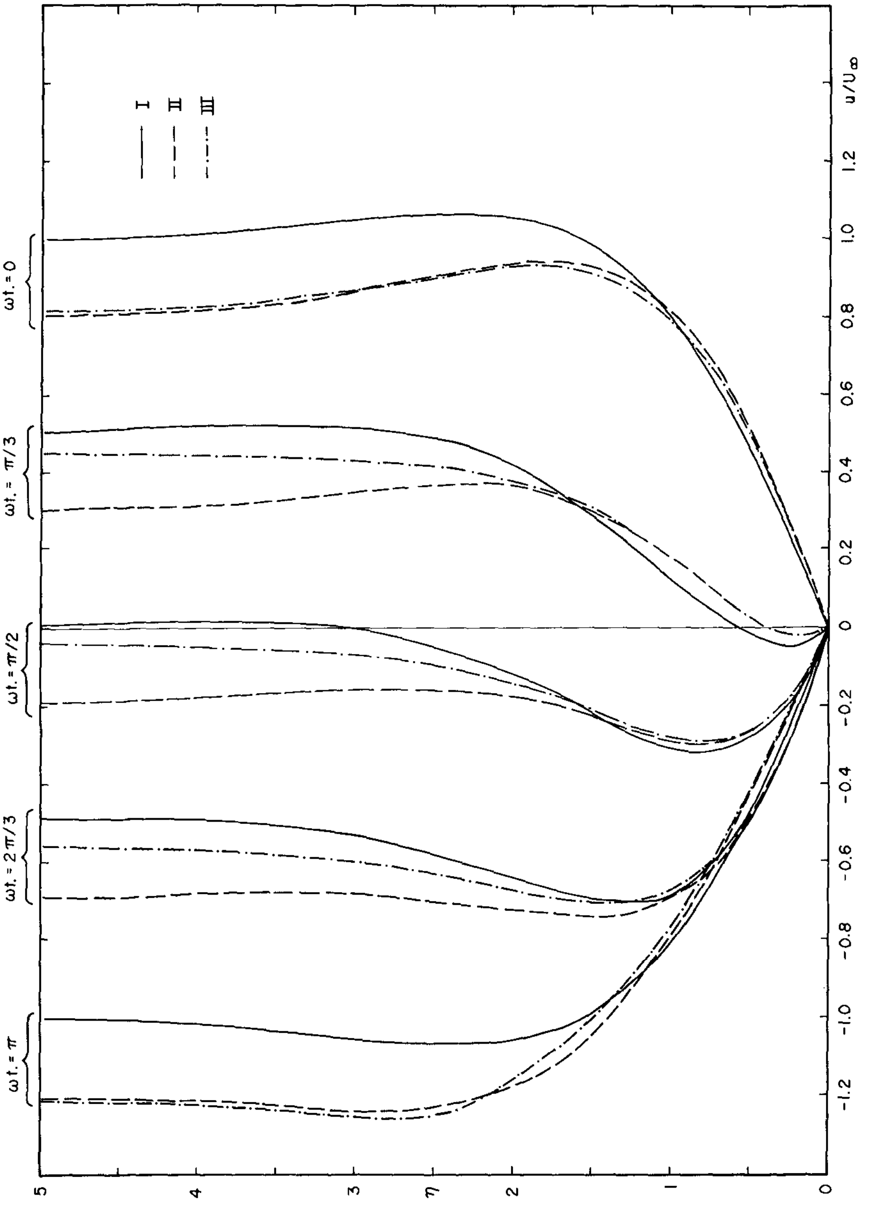


Figure 1

the outer edge of the boundary layer, and especially near flow reversal. It does not, however, contribute to the mass transport velocity owing to the fact that all velocity functions in the third approximation are periodic.

2. Case of a progressive wave

When the external flow is represented by a progressive gravity wave, the appropriate form of the stream function, according to Dore (1968) and Johns (1968), is

$$\psi(x,y,t;\varepsilon) = \frac{a\omega d}{\sinh k_0 d}\left[\psi_0(z)+\varepsilon\psi_1(z)+\dots\right]e^{i(k_0x-\omega t)} \tag{23}$$

where

$$z = y+d \tag{24}$$

and the wave number can be expanded as

$$k = k_0 + \varepsilon k_1 + \dots \tag{25}$$

The second term was given by Biesel (1949) as

$$k_1 = \frac{k_0^2 d\,(1+i)}{2k_0 d + \sinh 2k_0 d} \tag{26}$$

The boundary conditions are

$$\psi_0 = \psi_1 = 0 \quad \text{at} \quad z = d$$
$$\psi_0 = \psi_1 = \text{constant at} \quad z = 2d+\xi$$

where the free surface is specified by

$$\xi = ae^{i(k_0x-\omega t)}$$

and the depth $d$, has its origin at the still water surface.

With a change of variables for the zero order stream function we can write

$$\Psi_o(y)e^{i(k_o x-\omega t)} = A \sin ik_o z \ \sin(k_o x-\omega t) \tag{27}$$

where

$$A = -\frac{a\omega}{k_o \sinh k_o d} \tag{28}$$

Retaining the real part and displacing the coordinate system by $\pi/4$, we obtain the horizontal velocity by differentiation of Eq. 27

$$-\frac{\partial \Psi_o}{\partial y} = U_o = -Ak_o \cosh k_o z \ \cos(k_o x-\omega t) \tag{29}$$

at the outer edge of the boundary layer. Similarly,

$$\frac{\partial U_o}{\partial x} = Ak_o^2 \cosh k_o z \ \sin(k_o x-\omega t) \tag{30}$$

$$\frac{\partial^2 U_o}{\partial x^2} = -Ak_o^3 \cosh k_o z \ \cos(k_o x-\omega t) \tag{31}$$

We assume now, that as $\eta \to 1$, terms of $O(\varepsilon, \varepsilon^2)$ in $k_o d$ remain small for the fixed viscosity in order that the condition of viscous effects outside the boundary layer be avoided. Therefore it follows, that the zero order velocity is considered to adequately describe the free stream oscillation. This is justifiable, since the point of interest is not in the flow far from the boundary, where mass transport and interaction terms are significant; rather in establishing the magnitude of shear stress near $\eta = 0$; which is discussed in the following section. The argument is based on $R_s$ being large, whereby interaction between the mass transport velocity and the inviscid flow in the potential region are neglected, following Stuart's (1966) arguments. When $R_s$ is of $O(1)$, however, the periodic component of the horizontal velocity, obtained in the second approximation, must be matched asymptotically with the first order velocity $\varepsilon U_1$ at the interface, so that the boundary condition reflected in Equation 9 is satisfied. That is,

$$u_{21}(x,y) = Re\left[\frac{U_0}{\omega}\frac{dU_0}{dx}\zeta'_{2b}(\eta)\right]e^{i2\omega t} - \varepsilon U_1(x,y,t) \tag{32}$$

as shown by Longuet-Higgins (1953) and Noda (1971). Evaluation of interaction terms resulting from the simultaneous perturbation of inner and outer flows remains a continuing research interest of the author.

## THE SHEAR STRESS DISTRIBUTION

To assess the potential force available, say for the entrainment of sediment, it is important to know the maximum value of the instantaneous shear at the bottom, its distribution normal to the boundary and its phase advance in respect to the horizontal velocity.

In laminar flow, the shear stress distribution is derived from the velocity gradient

$$\tau = \nu\rho\left(\frac{\partial u}{\partial y}\right) \tag{33}$$

which expanded to the third approximation becomes, in general

$$\sum_{n=1}^{3} \tau_n = \nu\rho\left[\frac{\partial u_1}{\partial \eta} + \frac{\partial u_2}{\partial \eta} + \frac{\partial u_3}{\partial \eta}\right] \tag{34}$$

where $u_1$, $u_2$ and $u_3$ are given in Equations 13, 14, and 15 respectively.

The boundary conditions for the auxiliary functions are

$$\left.\begin{array}{l} \zeta_1''(\eta),\ \zeta_{2a}''(\eta),\ \zeta_{2b}''(\eta) \\ \zeta_{3a}''(\eta),\ \zeta_{3b}''(\eta),\ \zeta_{3c}''(\eta),\ \zeta_{3d}''(\eta) \end{array}\right\} = \left\{\begin{array}{ll} \text{finite at} & \eta = 0 \\ 0 \quad \text{at} & \eta = \infty \end{array}\right.$$

Solutions for all $\zeta''$ are:

$$\zeta_1''(\eta) = (1+i)\,e^{-(1+i)\eta} \tag{35}$$

$$\zeta_{2a}''(\eta) = -\frac{\sqrt{2}\,(1-i)}{2}\,e^{-\sqrt{2}(1+i)\eta} - e^{-(1+i)\eta}\left(\eta-(1-i)\right) \tag{36}$$

$$\zeta_{2b}''(\eta) = e^{-\eta}\left[\cos\eta - 2\sin\eta + \eta\cos\eta\right] - \frac{1}{2}e^{-2\eta} \tag{37}$$

$$\zeta_{3a}''(\eta) = -\frac{1}{2}i\eta e^{-(1+i)\eta} + \frac{1+i}{4}\left[\left(3+\frac{1}{\sqrt{2}}\right)e^{-(1+i)\eta} - \frac{5\sqrt{3}}{2}e^{-\sqrt{3}(1+i)\eta} - 2\sqrt{2}\,e^{-\sqrt{2}(1+i)\eta} - \frac{1}{2}e^{-(\sqrt{2}+1)(1+i)\eta} + 4e^{-2(1+i)\eta}\right] \tag{38}$$

$$\zeta_{3b}''(\eta) = \frac{1}{8}\left[\left\{(2+4i)\eta^2 - (6+2i)\eta + \frac{521+608i}{75}\right\}e^{-(1+i)\eta} + (2i\eta-(5-3i))e^{-(1-i)\eta} + \frac{3-4i}{5}e^{-(3+i)\eta} - \frac{4}{3}(1+i)\,e^{-2(1+i)\eta}\right] - \frac{1}{10}\left(2\eta(2+i) + \frac{27+86i}{5}\right)e^{-2\eta} \tag{39}$$

$$\zeta_{3c}''(\eta) = \frac{3\sqrt{3}\,(\sqrt{2}+1)(1+i)}{8}e^{-\sqrt{3}(1+i)\eta} + \frac{1}{4}\left[2(1-i)\eta^2 + 10i\eta - \left(4+\frac{1}{\sqrt{2}}\right)(1+i)\right]e^{-(1+i)\eta} + \left[-2i\eta + \left(\frac{1}{\sqrt{2}}+\sqrt{2}+1\right)\cdot(1+i)\right]e^{-\sqrt{2}(1+i)\eta} - \frac{2\sqrt{2}-3}{8}(\sqrt{2}+1)(1+i)e^{-(\sqrt{2}+1)(1+i)\eta} \tag{40}$$

$$\zeta_{3d}''(\eta) = \frac{1}{4}\left[-\frac{2}{3}\eta^3 + (1-i)\eta^2 + \frac{1}{2}(4+5i)\left\{(1+i)\eta^2 - 2\eta\right\} + \frac{7-3i}{4}\left\{(1+i)\eta - 1\right\} + \frac{113+414i}{50}\right]e^{-(1+i)\eta} + \frac{3+4i}{10}\left[-2\eta+1\right]e^{-2\eta} + \frac{27+139i}{50}e^{-2\eta} + \frac{4+3i}{40}e^{-(3+i)\eta} \tag{41}$$
$$\frac{1}{4}\left[(1+i)\eta^2 - 2i\eta - 2(6-5i)\right]e^{-(1-i)\eta}$$

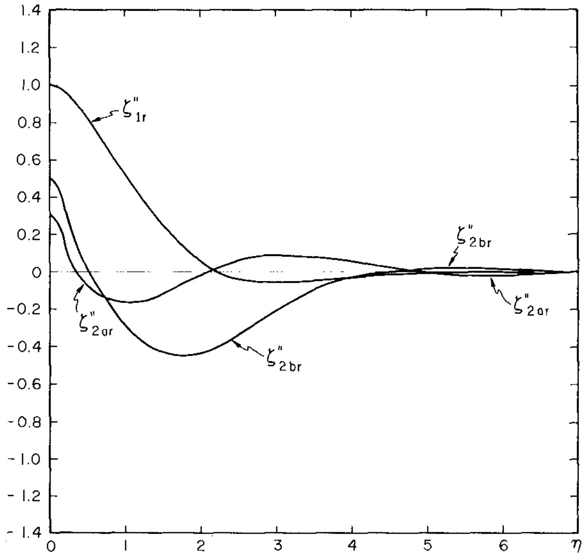


Figure 2

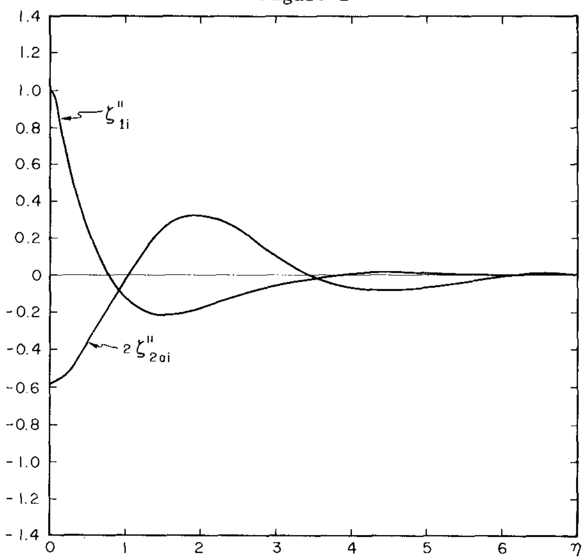


Figure 3

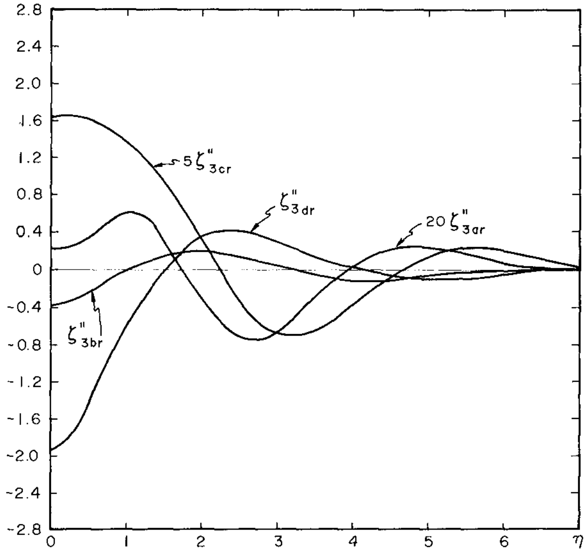


Figure 4

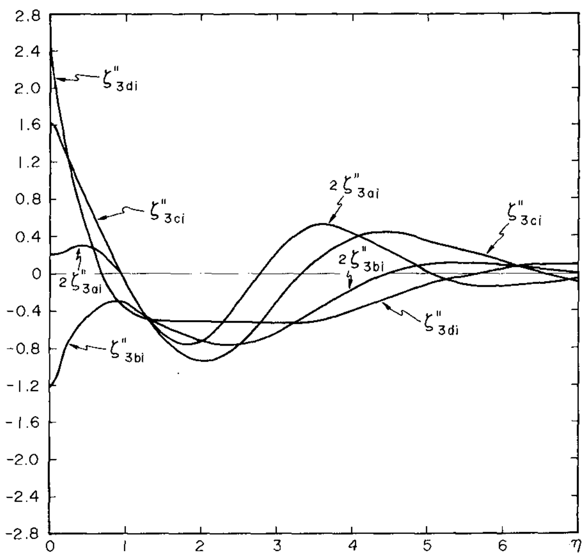


Figure 5

Equations 35-37 are the auxiliary functions of the first and second approximations; these are depicted in Figures 2 and 3 separated into real and imaginary components. Similarly, Figures 4 and 5 show components of the third approximation (Equations 38-41). It is evident, that $\zeta_1''(\eta)$ considerably outweighs the numerical contribution of all other components, except that of $\zeta_{3d}''(\eta)$. However, this function is only effective in correcting the first approximation.

We can now write the complete expression for the shear stress distribution correct to

$$\left(\frac{\tau}{\rho U_o^2}\right)\frac{U_o}{\delta} = \frac{dU_o}{dx}\zeta_{2b}''(\eta) + \left\{\omega\zeta_1''(\eta) + \frac{U_o}{\omega}\frac{d^2U_o}{dx^2}\zeta_{3b}''(\eta) + \frac{1}{\omega}\left(\frac{dU_o}{dx}\right)^2\zeta_{3d}''(\eta)\right\}e^{i(\omega t-\theta)} + \frac{dU_o}{dx}\cdot \zeta_{2a}''(\eta)\,e^{i2(\omega t-\theta)} + \left\{\frac{U_o}{\omega}\frac{d^2U_o}{dx^2}\zeta_{3a}''(\eta) + \left(\frac{dU_o}{dx}\right)^2\zeta_{3c}''(\eta)\right\}e^{i3(\omega t-\theta)} \tag{42}$$

The first term on the right side is the steady contribution equivalent to $\partial u_{2t}/\partial\eta$ . The phase advance is represented by $\theta$. It is seen from this equation, grouped to equivalence in the harmonic terms, that the contribution made by the third approximation is two-fold, one of which is correct terms of $O(\epsilon^0)$. In fact, this is numerically more significant, than all terms of $O(\epsilon^2)$ . In other words, the contribution of the third approximation to the distribution of $\tau(y)$ is small.

An example of the shear stress distribution of Equation 42 is given for an Airy wave of H = 2.0 ft, T = 10 sec and d = 30 ft in Figure 6, where the normalized shear is shown as a function of $\eta$ . Phase advance of the maximum shear over the forcing velocity is found to be near $\theta = \pi/6$ in absolute units, instead of $\pi/4$ predicted by linear theory. This is equivalent to 30°, and compares well with the experimental phase lead obtained by Jonsson (1966) in an oscillating water tunnel, although the flow in his boundary layer was turbulent. It appears, therefore, that the additional nonlinearity due to the Reynolds stresses, or the diffusion of vorticity into the inviscid region reduces the phase lead of the stress; this is supported in part by the analytical work of Srivastava (1966) who

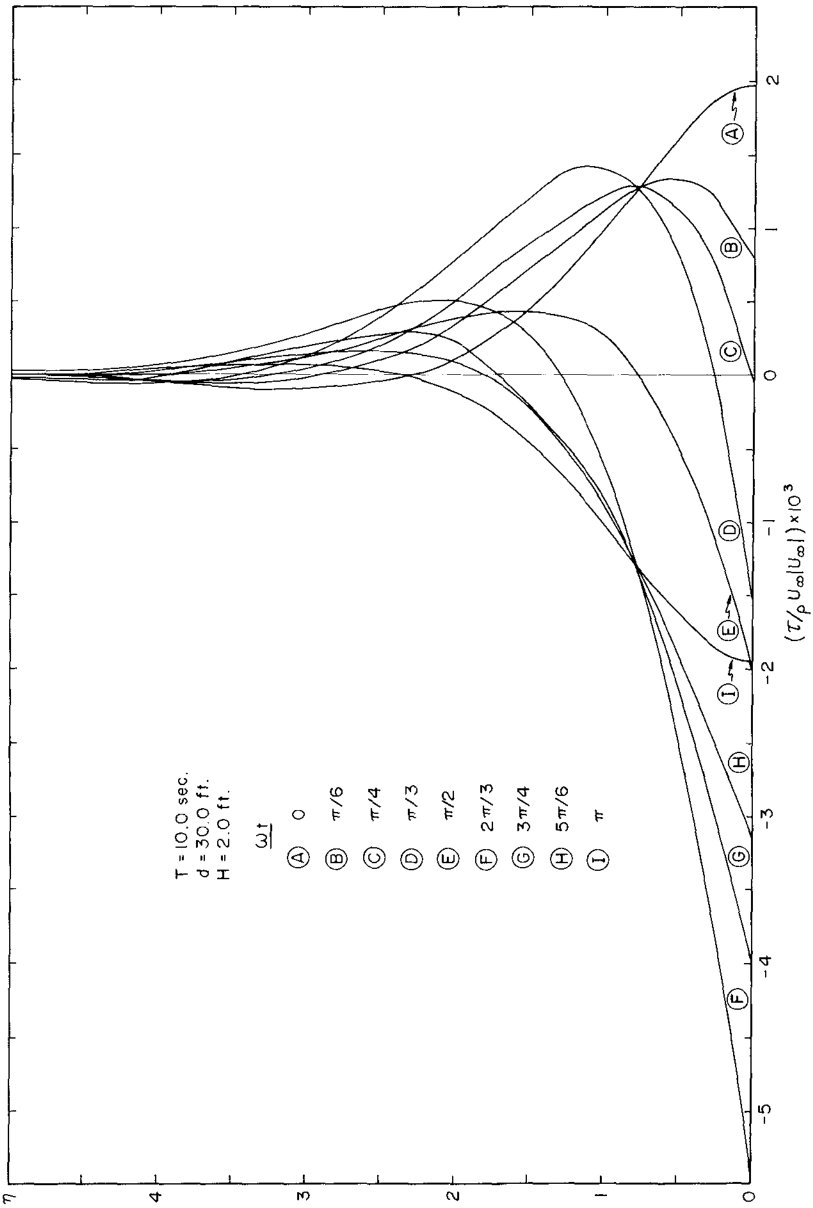
T = 10.0 sec.
d = 30.0 ft.
H = 2.0 ft.
ωt
A 0
B π/6
C π/4
D π/3
E π/2
F 2π/3
G 3π/4
H 5π/6
I π
η
$(\tau/\rho U_\infty|U_\infty|) \times 10^3$

Figure 6

obtained progressive reductions in the phase by increasing the order of the solution for the shear stress.

## CONCLUSIONS

An analytical solution for the shear stress distribution to the third approximation in the boundary layer of laminar flow established the following:

(1) The maximum, instantaneous shear stress has a phase difference $\theta = \pi/6$ over $u$, agreeing with Jonssons's (1966) experimental data;

(2) The third approximation modifies the numerical estimate of the second approximation boundary shear by no more than 6 percent, which is below the experimental error associated with shear measurements;

(3) When the shear stress is measured indirectly at some elevation above the solid boundary, the perturbation procedure described herein enables one to correct the experimental data for the true value of the bottom skin friction. This is particularly applicable to experiments where current meters or Preston probes are used to survey the velocity distribution near the boundary.

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## NOTATION

| | | | |
|---|---|---|---|
| $a$ | Wave amplitude | $z=y+d$ | Normalized distance or coordinate |
| $d$ | Local water depth | $\delta$ | Boundary layer thickness |
| $e$ | Exponential | $\varepsilon$ | Perturbation parameter |
| $g$ | Gravitational acceleration | $\zeta$ | Function of $\eta$ |
| $H$ | Wave height | $\eta=y/\delta$ | Dimensionless coordinate, normal to boundary |
| $i$ | $\sqrt{-1}$ | $\Theta$ | Phase advance of the shear str |
| $k=2\pi/L$ | Wave number | $\nu$ | Kinematic viscosity |
| $L$ | Wavelength | $\xi$ | Vertical displacement of water surface from still water level |
| $Re$ | Real part of complex number | $\rho$ | Density of fluid |
| $R_s=U_0/\omega\nu$ | Reynolds number | $\tau$ | Horizontal shear stress |
| $T$ | Wave period | $\omega=2\pi/T$ | Wave number |
| $t$ | Time | $\phi$ | Velocity potential |
| $U_0$ | Free stream velocity | $\chi$ | Dimensionless stream function |
| $u_{1,2,3}$ | Velocity in the boundary layer horizontal component | $\psi$ | Stream function |
| $v_{1,2,3}$ | Velocity in the boundary layer vertical component | $\nabla$ | Laplacian operator |
| $x$ | Horizontal distance or coordinate | | |
| $y$ | Vertical distance or coordinate | | |