

## Chapter 5

### BORE INCEPTION AND PROPAGATION BY THE NONLINEAR WAVE THEORY

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#### ABSTRACT

The integration of the equations of the nonlinear shallow-water theory by a finite difference scheme based on the method of characteristics is programmed for digital computers. In the program, the equations of the bore propagation are coupled to the equations of the nonlinear theory, and thus a procedure for predicting the motion of the entire wave, including the bore, is established. Waves of irregular shape and experimental data are treated by an iterative method. Laboratory experiments on the inception and propagation of bores also are presented.

#### INTRODUCTION

The theory of waves of small amplitude, which has been the basis for the classical studies of ocean waves, predicts an infinite wave amplitude at the shoreline. The shallow-water theory, on the other hand, presents no such difficulty, and is more appropriate for the study of waves near the coast. However, before a strong case for the shallow-water theory can be established, it is desirable that (a) a method of solution for its routine application be developed and (b) fairly good agreement between the theory and observation be shown. The purpose of this paper is to present the results of an effort made to achieve these objectives. The principles for obtaining the solutions of the equations of the nonlinear shallow-water theory on a digital computer are described. It is shown that by using an iterative method, it is feasible to feed experimental and field data into the computer program. Finally, experiments of an exploratory nature are described and the results of the computed and experimental values are compared.

#### THEORY

The equations of the nonlinear shallow-water theory in two-dimensional flow are

$$(\partial/\partial x)[u(h + \eta)] + (\partial\eta/\partial t) = 0 \quad (1)$$

$$(\partial u/\partial t) + (u\partial u/\partial x) = -(g\partial\eta/\partial x) + (1/\rho)(\partial\tau/\partial x) \quad (2)$$

In these equations, the x-axis is taken along the bottom,  $h$  is the depth of water,  $\eta$  is the water surface elevation above the steady-state position,  $\rho$  is the water density, and  $\tau$  is the shear stress along the bottom. The derivation of these equations is given by Stoker (1957). In (2), the last term on the right is added to the corresponding inviscid equation to take care of the friction effects. In the terms of the friction slope  $S_f$ ,  $(1/\rho)(\partial\tau/\partial x) = gS_f$ . (1) and (2) are the equations of the first-order shallow-water theory in which it is assumed that the pressure distribution is hydrostatic or that the vertical component of the acceleration has a negligible effect on the pressure. Friedrichs (1948) has shown that by using a perturbation scheme higher-order theories may be developed from the hydrodynamical equations. In a higher-order theory, the vertical acceleration may not be neglected and the pressure distribution need not be hydrostatic. This presentation, however, is only concerned with the first-order nonlinear theory.

Equations (1) and (2) constitute a system of first-order hyperbolic equations. In such systems, smooth solutions do not necessarily exist for all time; after a finite time a smooth solution may cease to be smooth and later tend to a discontinuity that may behave quite differently from the smooth solution (Jeffrey and Taniuti, 1964). Physically, it is implied that the wave may develop a bore and the propagation of the bore will not be given by the system of equations (1) and (2). The bore speed, however, can be determined by the application of the momentum principle.

#### METHODS OF SOLUTION

Analytical solutions of the nonlinear equations are seldom known. Finite difference methods are therefore used to obtain numerical solutions. A survey of these methods is given by Forsythe and Wasow (1960). The numerical methods for the solution of hyperbolic equations may be classified broadly into (a) finite difference schemes using regular networks and (b) the method of characteristics. The first method employs fixed space and time intervals. It has been applied by Stoker and his associates to the movement of floods in rivers (Stoker, 1957) and by Keller et al. (1960) for determining the change in bore height and speed as it advances into nonuniform flow. However, as yet, no way has been found to predict the development of a bore by this method.

The second method is based on the characteristics of the differential equations (1) and (2). Because these equations are hyperbolic, the characteristics are real and numerical methods utilizing the characteristics have been developed.

## METHOD OF CHARACTERISTICS

The solution of equations (1) and (2) may be given by the intersection of the two sets of positive and negative characteristic curves. The slope of the positive characteristic is given by

$$dx/dt = u + c \quad (3)$$

where  $c$  is the wave celerity given by  $c = g\sqrt{h + \eta}$ . The slope of the negative characteristic is given by

$$dx/dt = u - c \quad (4)$$

Furthermore,

$$d(u + 2c - mt) = 0 \quad (5)$$

on a positive characteristic, and

$$d(u - 2c - mt) = 0 \quad (6)$$

on a negative characteristic, where  $m = -gS - gS_f$  and  $S$  is the rate of change of depth with distance  $x$ . The derivation of equations (3) through (6) and a description of the underlying principles are given by Stoker (1957). A critical analysis of the method of characteristics is given by Le Mehaute (1963). Wave transformation in shoaling water has been investigated by an approximate analytical method based on characteristics by Kishi (1963). Equations (3) through (6) serve as the basis for numerical computations by the method of characteristics. These four ordinary differential equations replace the system of partial differential equations (1) and (2). Once the initial values are known, solutions may be obtained step by step.

The required initial values may be prescribed in several ways. But for free surface flows, it is recommended that the initial values be given as (a) the values of  $u$  and  $c$  as functions of  $x$  at time  $t = t_0$  and (b) the values of  $c$  as functions of time  $t$  at  $a$ , given  $x = X_0$ . The first set is known from the steady-state conditions prior to the arrival of the wave and the second set is known from a knowledge of the variation of the water surface elevation with time at  $x = X_0$ .

The values of  $u$ , the particle velocity, need not be prescribed at  $x = X_0$  as a function of time. This is a fortuitous circumstance so far as experimental data are concerned, because the particle velocity is difficult to measure. But the choice of initial values in the manner described above rests on a deeper principle, the choice being influenced by the role of the negative characteristics. The negative characteristics issued from the leading wave elements interact with and modify the original wave (Ho, 1962). Stoker (1957) has suggested, in this context, that the values of either  $u$  or  $c$ , but not both, should be prescribed as initial values. An analogous situation has been observed in flood waves in rivers where the stage discharge curve may have a loop. At a given location, for the same surface elevation, the water velocity at the wave front may be different from that at the rear of the wave. The water velocity at the rear is subjected to modification by the negative characteristic issued from the wave front. This may be called the "backwater effect." With the choice of the initial values in the recommended manner, the "backwater effect" is taken into consideration.

The method of characteristics has an inherent advantage in that it provides insight into the physical phenomenon under study. Thus, the inception of a bore is predicted by the intersection of adjacent positive characteristics. The bore speed is given by (Stoker, 1957)

$$V = \sqrt{g(h + \eta)(2h + \eta)/2h} \quad (7)$$

The bore propagation is determined by coupling the bore equation (7) to the equations of the shallow-water theory. In terms of the bore strength defined by

$$M = V/\sqrt{g(h + \eta)} \quad (8)$$

it is found that (Keller et al., 1960)

$$V/\sqrt{gh} = M(2M^2 - 1) \quad (9)$$

$$c/\sqrt{gh} = \sqrt{2M^2 - 1} \quad (10)$$

and

$$\frac{1}{h} \frac{dh}{dM} = \frac{-4(M + 1)(M - 0.5)^2(M^3 + M^2 - M - 0.5)}{(M - 1)(M^2 - 0.5)(M^4 + 3M^3 + M^2 - 1.5M - 1)} \quad (11)$$

The changes in bore height and speed as the bore advances into nonuniform flow are calculated from (11). If a positive characteristic should meet the boreline at a later time, then a new value for the bore strength  $M$  should be calculated. The new value of  $M$  should be taken from the characteristic that intersects the boreline.

An alternate method for the calculation of the bore propagation based on the continuity and momentum, as used in hydraulics, is given by Freeman and Le Mehaute (1964).

#### NUMERICAL PROCEDURE

The integration of the equations of the shallow-water theory by the method of characteristics may be performed very efficiently on a digital computer. In this method, the shallow-water theory is represented by the ordinary characteristic differential equations (3) through (6), rather than the original partial differential equations (1) and (2). The numerical solutions of equations (3) through (6) are obtained by replacing the differential equations by the corresponding difference equations. Only the procedure for the investigation of irregular waves that develop a bore at the wave front and whose profiles are given by discrete point functions of the time will be described. An iterative method, perhaps feasible only in machine calculations, constitutes a key element of the computational procedure. A study on regular waves and waves developing bores at intermediate points on their profiles and the wave runup is currently under way for the U.S. Naval Civil Engineering Laboratory and will be reported on at a later date.

The solution of the equations of the shallow-water theory may be presented in the form of sets of values of flow quantities from which a network of characteristics could be plotted on the  $x,t$  plane. The procedure for the determination of the network, in accordance with the discussion on the initial values, is patterned after de Prima (Stoker, 1948). Essentially, it consists of first establishing the initial characteristic depicting the motion of the wave front and then issuing the negative characteristics from it. The values of  $c$  on the  $t$ -axis are given at discrete points as input data. By introducing these values at appropriate points in the computations, the network is extended to cover the pertinent portion of the  $x,t$  plane. Brief descriptions of the various phases of the procedure are given as follows.

#### INITIAL CHARACTERISTIC

From  $(X_0, t_0)$  (Fig. 1), the initial characteristic  $C_0$  is drawn first. This is a positive characteristic, and the values of  $u$  and  $c$  for  $C_0$  are known from the steady-state flow

conditions. If an appropriate time interval,  $\delta t$ , is chosen on the characteristic segment  $P_1P_2$ , then

$$t(P_2) = t(P_1) + \delta t \quad (12)$$

The value of  $x$  at  $P_2$  is calculated from equation (13), which in turn is obtained from the finite difference forms of (3) and (5):

$$x(P_2) = x(P_1) + [u(P_1) + c(P_1) - 0.5m(P_1) \cdot \delta t] \cdot \delta t \quad (13)$$

#### INTERIOR CHARACTERISTICS

For purposes of illustration, let it be assumed that the calculation has progressed to the line  $P_1J_1$ , with all the characteristics to the left of this line being known. Therefore, at each of the node points  $P_1, Q_1, R_1, \dots, J_1$ , the slopes of the positive characteristics are given by (3). The values of  $x$ ,  $u$ , and  $c$  are also known at  $P_2$ , since it lies on the initial characteristic. Then a negative characteristic can be drawn from  $P_2$  to intersect a positive characteristic drawn from  $Q_1$ , the point of intersection being  $Q_2$ . The finite difference forms of (3) through (6) provide four simultaneous algebraic equations from which the four unknown quantities  $u$ ,  $c$ ,  $t$ , and  $x$  at  $Q_2$  can be calculated. The computation then proceeds to  $R_2$ ,  $S_2$ , and on up to  $K'$ . If  $K'$  coincides with  $K$ , one of the discrete points on the  $t$ -axis given as input data, then one chain of operation is completed. To advance the computations to the right, storage locations in the machine occupied by  $P_1, Q_1, R_1, \dots, J_1$  will be taken over by  $P_2, Q_2, R_2, \dots, J_2$ , and the storage locations occupied by  $P_2, Q_2, R_2, \dots, J_2$  will be vacated and reserved for the results of the next chain. The maximum number of storage locations required to handle this part of the program is that which can accommodate the computations for a single chain. By conducting the computations in a sequence of chains and recording the results after each chain, large outputs may be obtained from a relatively small computer.

If  $K'$  does not correspond to a discrete point  $K$  given by the input data, an iterative method will be used to pass a negative characteristic through  $K$ .

#### ITERATIVE METHOD

Let it be assumed that the difference between the values of  $t(K')$  and  $t(K)$ , where  $K$  is in the neighborhood of  $K'$  and

its position is fixed by the input data, is larger than a prescribed tolerance interval. Then, the value of  $\delta t$  in (12) will be adjusted, and the chain of operations on  $P_2, Q_2, R_2, \dots, K'$  will be resumed. The operation will be repeated as many times as necessary until the difference between the values of  $t(K')$  and  $t(K)$  lies within the prescribed tolerance interval.

#### BORE PROPAGATION

From a computational viewpoint, the inception of the bore is indicated when two adjacent points on a negative characteristic coincide with each other. The slope of the boreline on the  $x, t$  plane is given by

$$dx/dt = V + u_0 \quad (14)$$

where  $u_0$  is the water velocity in the water ahead of the bore. The bore path is then determined step by step, using a procedure similar to Freeman and Le Mehaute's (1964). The changes in the bore speed between the consecutive intersections of the positive characteristics with the bore path are taken from the numerical solution of (11).

#### COMPUTER PROGRAM

The integration of the equations of the nonlinear shallow-water theory by the method of characteristics, together with the determination of the inception and propagation of the bore, has been programmed in FORTRAN. The program includes the iterative method for the treatment of experimental data for irregular waves. A conceptual flow diagram is given in Fig. 2.

The following input data are required for each run:

- $h_1$ , the initial steady-state depth at  $X_0$
  - $N_1$ , the number of points taken on the wave on the  $t$ -axis
  - $n$ , the coefficient of friction in Manning's formula
  - $S$ , the rate of change of depth in the  $x$ -direction
  - $T$ , the final value of time on the  $t$ -axis
  - $t_0$ , the initial value of time on the  $t$ -axis
  - $u_1$ , the initial steady-state velocity at  $X_0$
  - $X_f$ , the value of  $x$  at the downstream station
  - $X_0$ , the value of  $x$  at the upstream station
  - $\Delta$ , the time interval between consecutive points on the  $t$ -axis
- Values of  $c$  at  $X_0$  as functions of time.

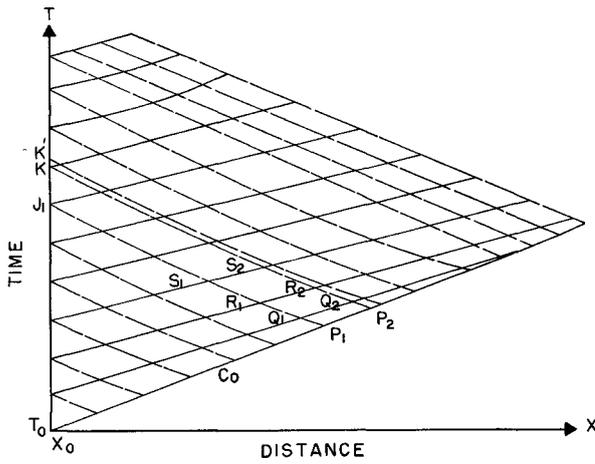


FIG. 1 --NETWORK OF CHARACTERISTICS

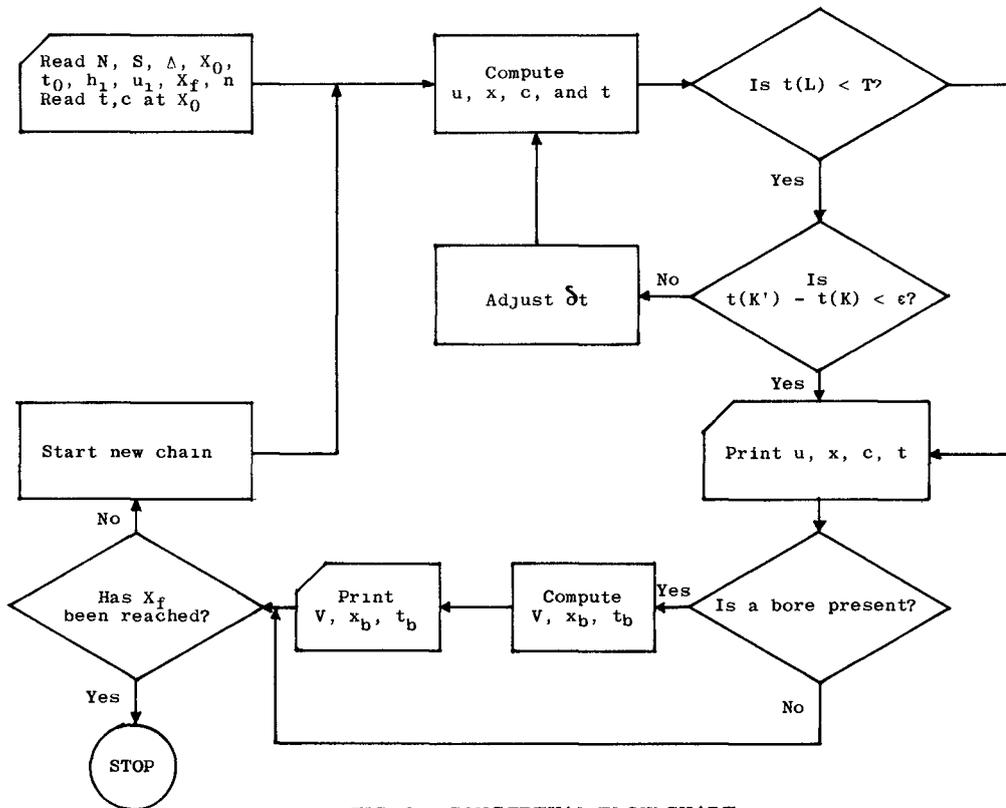


FIG. 2. --CONCEPTUAL FLOW CHART

The output information is printed out in the form of tables of values of depth, distance, time, water velocity, wave celerity, bore speed, and bore position.

### EXPERIMENTS

Experiments of an exploratory nature involving waves of irregular shape were performed in a glass-walled rectangular channel. The disturbances were generated by manipulating the valve controlling the flow into the channel. Two resistance probes, one near the upstream end and the other farther downstream, were installed in the channel. The probes operate on the principle that the resistance across two wires immersed in water is a function of the depth of immersion. The probe outputs, representing the variation of the water surface elevation with time, were amplified and recorded on oscillograph recorders. The principles and details for this type of instrumentation are described by Wiegel (1955).

To obtain a comparison of the experimental and computed values, the surface elevation data as recorded at the upstream probe were fed into the computer program, from which the theoretical data at the downstream probe could be obtained. Some of these results are shown in Figs. 3, 4, and 5. In Figs. 3 and 4, the water is initially at rest. In Fig. 5, the flow is initially steady but nonuniform and accelerating in the x-direction. An examination of these figures indicates that there is excellent agreement between the computed and experimental values for the time of arrival of the bore. Perhaps the conditions at the bore front caused by the pile-up of the water there may not be faithfully represented by the theory in its present form. The computed wave behind the bore is seen to conform in general outline to the recorded wave. Because of the difficulty of maintaining constant calibration curves for the probes, the experimental results should be considered to be providing qualitative evidence in support of the theory for the portion of the wave behind the bore. It is felt that more extensive experiments will favor the establishment of the nonlinear shallow-water theory as a powerful tool for the study of wave processes near the coast.

### CONCLUSION

A finite difference scheme for the solutions of the equations of the nonlinear shallow-water theory by the method of characteristics for digital computer applications is described. The significance of the proper method for choosing the initial values is discussed. The computer program includes the test for the inception of bores and methods for the calculation of bore speed and height. The use of an iterative method makes it possible to treat waves of irregular arbitrary shape. Exploratory experiments on the inception and propagation of bores are also presented.

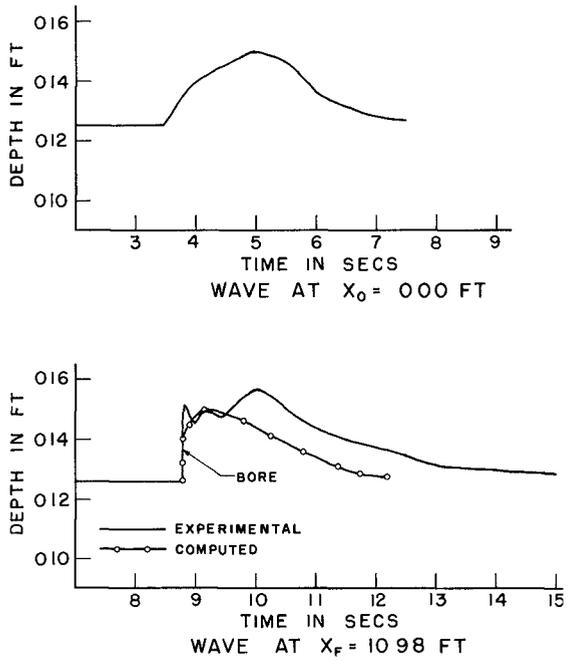
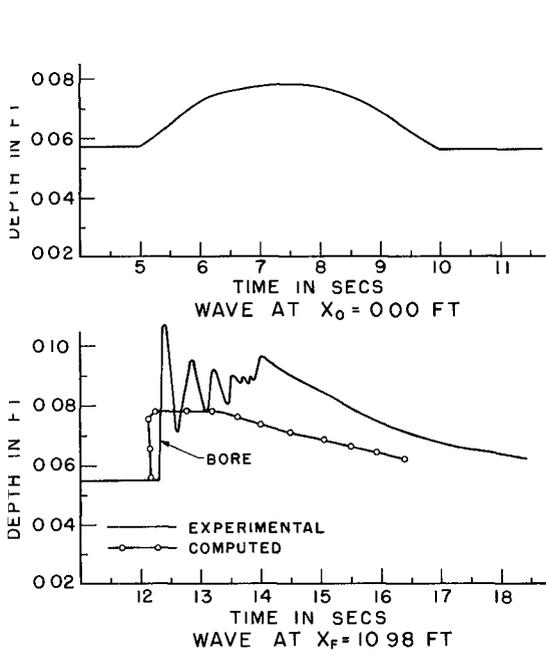


FIG 3 --BORE PROPAGATION IN WATER OF CONSTANT DEPTH TYPE A

FIG 4 --BORE PROPAGATION IN WATER OF CONSTANT DEPTH TYPE B

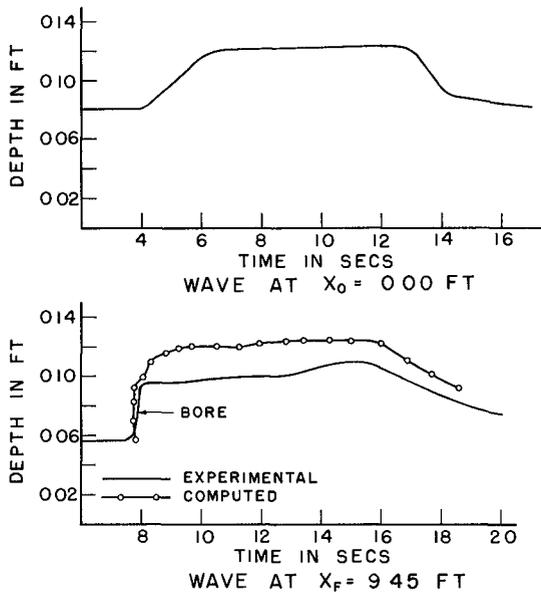


FIG 5 --BORE PROPAGATION IN NONUNIFORM FLOW

The accuracy, economy, and speed of the digital computer have eliminated the extensive calculations involved in integrating the equations of the nonlinear shallow-water theory. It is hoped that the feasibility of successful computer programming will encourage the greater use of the nonlinear theory by the coastal engineer. However, further research, with improved equipment and instrumentation, is needed to establish the range of validity of the theory so that it could be applied with confidence to predict the wave motion near the coast.

#### ACKNOWLEDGEMENTS

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