CHAPTER 4

AN APPROXIMATION OF THE WAVE RUN-UP
FREQUENCY DISTRIBUTION

By

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The distribution of wave steepness (H/T^2) for fully
developed sea is obtained from Bretschneider's joint
distribution of wave height and wave period. This
steepness distribution is used with standard wave run-
up curves to develop a frequency curve of wave run-up.
Use of this run-up distribution curve will permit more
accurate estimation of the variability in wave run-up
for design cases, and particularly the percent of time
in which run-ups will exceed that predicted for the
significant wave. The distribution may also be used
with normal overtopping procedures to determine more
accurate estimates of overtopping quantities.

Wave run-up may be defined as the vertical height above mean water
level to which water from a breaking wave will rise on a structure face.
Accurate design data on the height of wave run-up is needed for determi-
nation of design crest elevations of protective structures subject to
wave action such as seawalls, beach fills, surge barriers, and dams.
Such structures are normally designed to prevent wave overtopping with
consequent flooding on the landward side and, if of an earth type,
possible failure by rearface erosion.

Because of the importance of wave run-up elevations in determining
structure heights and freeboards, a great deal of work has been done in
the past six years in an attempt to relate wave run-up to incident wave
characteristics, and slope or structure characteristics. Compilations
based largely on laboratory experimental work have been made and have re-
sulted in curves similar to those shown in Figure 1 which is reprinted[1]*
from the U. S. Beach Erosion Board Technical Report No. 4. Such curves
most frequently have related the dimensionless ratio of relative run-up
(R/H_o) to incident wave steepness in deep water (H_o/T^2), as a function
of structure type or slope. (H_o is the equivalent deep water wave height.)
The curves shown in Figure 1 are of this type, and pertain to structures
having a depth of water greater than three wave heights at the toe of
the structure; this depth limitation in effect means that the wave breaks
directly on the structure. The curves shown in Figure 1 are a portion
of a set of five separate figures, covering different structure depths
(d/H_o). All are published in Beach Erosion Board Technical Report
Number 4.[1]

These curves were derived primarily from small scale laboratory
tests. Further laboratory tests with much larger waves (heights two
to five feet) have shown that a scale effect exists for some conditions.

*Numbers in parentheses indicate references listed at end of report.

AN APPROXIMATION OF THE WAVE RUN-UP FREQUENCY DISTRIBUTION

<p style="text-align:center">TABLE I</p>

<p style="text-align:center">JOINT DISTRIBUTION OF H AND T FOR ZERO CORRELATION
Number of Waves Per 1,000 Consecutive Waves for Various Ranges in Height and Period</p>

Range in Relative Height $H/\bar{H}$	RANGE IN RELATIVE PERIOD $T/\bar{T}$											Cumulative
	0-0.2	0.2-0.4	0.4-0.6	0.6-0.8	0.8-1.0	1.0-1.2	1.2-1.4	1.4-1.6	1.6-1.8	1.8-2.0	0-2.0	
0-0.2	0.03	0.50	2.05	4.86	7.68	8.09	5.31	1.92	0.34	0.03	30.81	30.81
0.2-0.4	0.10	1.41	5.81	13.78	21.76	23.92	15.05	5.44	0.98	0.07	88.32	119.13
0.4-0.6	0.14	2.06	8.54	20.23	31.95	33.65	22.10	7.99	1.44	0.11	128.21	247.34
0.6-0.8	0.16	2.40	9.91	23.48	37.08	39.06	25.65	9.27	1.67	0.12	148.80	396.14
0.8-1.0	0.16	2.40	9.92	23.51	37.13	39.11	25.69	9.28	1.67	0.12	148.99	545.13
1.0-1.2	0.15	2.14	8.87	21.02	33.19	34.97	22.96	8.30	1.49	0.11	133.20	678.33
1.2-1.4	0.12	1.74	7.21	17.07	26.96	28.40	18.65	6.74	1.21	0.09	108.19	786.52
1.4-1.6	0.09	1.30	5.37	12.72	20.09	21.16	13.90	5.02	0.90	0.07	80.62	867.14
1.6-1.8	0.06	0.90	3.72	8.82	13.93	14.67	9.64	3.48	0.63	0.05	55.90	923.04
1.8-2.0	0.03	0.48	1.99	4.72	7.45	7.85	5.15	1.86	0.33	0.03	29.89	952.93
2.0-2.2	0.03	0.42	1.72	4.09	6.45	6.80	4.47	1.61	0.29	0.02	25.90	978.83
2.2-2.4	0.01	0.18	0.76	1.80	2.84	2.99	1.97	0.71	0.13	0.01	11.40	990.23
2.4-2.6	0.01	0.09	0.39	0.93	1.47	1.55	1.02	0.37	0.07		5.90	996.13
2.6-2.8		0.04	0.18	0.43	0.67	0.71	0.47	0.17	0.03		2.70	998.83
0-3.0	1.09	16.06	66.44	157.46	248.65	262.93	172.03	62.16	11.18	0.83		
Cumulative	1.09	17.15	83.59	241.05	489.70	752.63	924.66	986.82	998.00	998.83		

<p style="text-align:right">(Bretschneider —1959)</p>

FIGURE I WAVE RUN-UP FOR SPECIFIC VALUES OF $\dfrac{H'_o}{T^2}$ $\left(\dfrac{d}{H'_o} > 3\right)$

Methods of taking this scale effect into account are also discussed
in Technical Report Number 4,[1] and will not be further covered here.
Use of these curves with appropriate scale correction appears to
give quite accurate determinations of wave run-up for smooth struc-
tures, if the waves are of the same type as generated in the labora-
tory. These laboratory waves are simple repetitive waves, with each
successive wave being essentially identical to the one preceding and
following it.

Unfortunately, waves in nature are not generally of this type.
In nature, no two waves are exactly alike, each successive wave being
different from the preceding and following ones by a greater or lesser
amount. The difference in successive waves is apparent in both height
and period (or length). Consequently, there is some question as to
the method in which the laboratory-derived run-up curves should be
applied to actual wave conditions in nature. Practice in the past has
been to apply these curves using the so called significant wave, which
is a hypothetical wave having statistically described characteristics.
Its height is defined as the average height of the upper one-third of
the waves in the wave train, and its period is the average period of
these higher waves. If the waves in a wave train are grouped accord-
ing to their various heights, a statistical distribution is essentia-
lly the same regardless of the actual magnitude of the heights, or
the state of generation or decay of the wave train. The significant
wave, representing the average of the higher one-third of the waves,
is exceeded by only thirteen percent of the waves in the wave train.
If this wave height is used for design run-up considerations, the
run-up obtained will be exceeded by only a small percentage of the
waves, and damage is unlikely to occur. Experience has proved this
assumption generally valid.

In the few cases where the importance of completely preventing
wave overtopping of a structure is quite crucial, normal practice has
been to compute a spectrum of wave run-ups using these same run-up
curves and obtaining a range of wave steepness values based on the
varying height values and a constant significant period assumed appli-
cable to all of the higher waves. It is recognized that such computa-
tions are in error, but it has been assumed that the error would be
relatively slight and of little importance.

Several years ago, however, a joint distribution of wave height
and period was described by Bretschneider[2] for the particular case
of fully developed sea - that is the case where the wind has been
blowing long enough over a great enough distance to generate waves
which are in a steady state condition. The fully developed sea condi-
tion is that for the case when the correlation coefficient between
wave height and wave period (or length) is zero.

50

AN APPROXIMATION OF THE WAVE RUN-UP
FREQUENCY DISTRIBUTION

This correlation coefficient has been defined as

$$r(\eta,\lambda) = \frac{\overline{\eta\lambda} - 1}{\left[\left(\overline{\eta^2}-1\right)\left(\overline{\lambda^2}-1\right)\right]^{\frac{1}{2}}}$$

where $\eta = \dfrac{H}{\overline{H}}$

and $\lambda = \dfrac{L}{\overline{L}} = \dfrac{T^2}{\overline{T^2}}$

The bar indicates average values and H, L, and T, are, respectively, the wave height, length, and period. For the special case of zero correlation, the probability of both a particular value of height and length occurring simultaneously may be given non-dimensionally as

$$p(\eta,\lambda) = p(\eta) \cdot p(\lambda)$$

Utilizing this equation and the expressions for the individual height and length distribution functions, Bretschneider[2] derives a cumulative joint distribution for height and length as

$$P(\eta,\lambda) = \left[1 - e^{-\frac{\pi\eta^2}{4}}\right]\left[1 - e^{-\frac{\pi\lambda^2}{4}}\right]$$

which may be put in terms of period as

$$P(\eta,\tau) = \left[1 - e^{-\frac{\pi\eta^2}{4}}\right]\left[1 - e^{-.675\,\tau^4}\right]$$

where $\tau = \dfrac{T}{\overline{T}}$

This equation gives the percent of waves (P) having simultaneous values of relative height (η) and relative period (τ) equal to or less than stated values.

With this equation, Bretschneider has obtained a table showing the percentage of waves in a consecutive wave train that would be expected to occur having various values of relative height ($\eta = H/\overline{H}$) and period

$(\tau = T/\bar{T})$. (2) These values may be used to obtain an approximate frequency tabulation of relative wave steepness by taking values of η/τ^2 since

$$\frac{\eta}{\tau^2} = \frac{\dfrac{H}{\bar{H}}}{\left(\dfrac{T}{\bar{T}}\right)^2} = \frac{H}{T^2} \bigg/ \frac{\bar{H}}{(\bar{T})^2}$$

This has been done, assuming that the mid-value of η and τ for each η and τ range is approximately appropriate to the frequency tabulated. The resulting curve is shown in Figure 2.

Now the average wave steepness $\overline{\left(\dfrac{H}{T^2}\right)} = \dfrac{\pi}{2}\ \dfrac{\bar{H}}{\overline{T^2}}$ (3) for fully developed sea, and

$$\overline{T^2} = 1.079\ (\bar{T})^2$$

so that for fully developed sea

$$\overline{\left(\frac{H}{T^2}\right)} = \frac{\pi}{2}\ \frac{\bar{H}}{1.079\ (\bar{T})^2} = 1.456\ \frac{\bar{H}}{(\bar{T})^2}$$

As the frequency curve for steepness in terms of $\bar{H}/(\bar{T})^2$ has already been obtained, a frequency curve in terms of average steepness $\overline{(H/T^2)}$ can now also be obtained. This has been done, and the frequency curve of relative steepness (for fully developed sea) is shown also in Figure 2.

However, for design cases it is more frequently the significant height and period which is available, particularly if wave characteristics are obtained on a hindcasting basis. The steepness distribution in terms of significant wave parameters would therefore also be of use. They may be easily obtained since, as has been shown by Longuet-Higgins (4)

$$H_{1/3} = 1.60\ \bar{H}$$

and, according to Bretschneider, for fully developed sea (zero correlation)

$$T_{1/3} = \bar{T}$$

Then substituting above,

$$\frac{H_{1/3}}{T_{1/3}^2} = 1.6\ \frac{\bar{H}}{(\bar{T})^2} = 1.10\ \overline{\left(\frac{H}{T^2}\right)}$$

AN APPROXIMATION OF THE WAVE RUN-UP
FREQUENCY DISTRIBUTION

This curve is shown in Figure 2. It represents the distribution of wave steepness for a fully developed sea as a ratio of the steepness of the significant wave.

The limiting wave steepness in deep water as given by Reid and Bretschneider[5] from Michell's work[6] is $H/T^2 = 0.88$. However, in the frequency distribution shown, values of steepness as great as 250 times the average steepness were derived. Even with a relatively low value for the average wave steepness, these values would still considerably exceed the limiting value of 0.88. These waves must then be breaking and represent the proportion of waves that at any one instant are exceeding the critical steepness and breaking either as relatively small whitecaps or, less frequently, as relatively large breakers.

An estimate of the relative amount of these breaking waves would be interesting, and is readily available if the average steepness can be obtained. For the case of fully developed sea, the steepness of the significant wave $(H_{1/3}/T^2_{1/3})$ is about 0.06.[2] With this steepness value, and the steepness distribution as shown in Figure 2, a value of about 0.17% is derived as the approximate proportion of waves in a fully developed sea which are breaking at any one particular time.

However, the case of a truly fully developed sea seldom if ever occurs in nature, requiring as it does relatively unattainable durations and fetches. The case of more practical interest is that of the normal storm generating area, in which the steepness of the significant wave is more nearly equal to 0.22 (the value most frequently used for design purposes). (This value corresponds to an average steepness of 0.20). If one assumes that the Bretschneider relationship for joint distribution of height and period for fully developed sea is approximately valid also for this condition, then a value of about 2.2% is obtained as the approximate proportion of waves in a normal generating area which are in the process of breaking at any particular time. This estimate would seem of the right order of magnitude, although no specific observations of this measure appear to be available to check its validation.

Actually the Bretschneider joint distribution is for a zero correlation between wave height and period. For the more normal generating area zero correlation does not occur, but the actual correlation of approximately 0.2 is not far removed. The true distribution would be skewed somewhat toward the long period values, but the approximation of the Bretschneider distribution would still appear reasonable (see his Figure 7.2 for example[2]), particularly when the general accuracy of the distributions as a whole is considered. Consequently a rough estimate of the proportion of whitecaps and breaking waves in a generating area can reasonably be made as about 2% or a little more.

53

Using this steepness distribution and the run-up curve given in Figure 1, a distribution of individual relative run-ups may be derived. This distribution has been obtained for the more interesting case of a generating area where $H_{1/3}/T^2_{1/3} = 0.22$ and is shown in Figure 3. Again, the assumption is made that the Bretschneider joint distribution is also approximately valid for the normal generating case. Figure 3a shows the distribution of individual relative run-ups; that is, the run-ups as related to the height of the particular wave associated with that particular run-up. Figure 3b shows this distribution normalized by division by the relative run-up of the significant wave, also as a function of slope.

The distribution shown in Figure 3 is of interest, but of still greater interest would be the distribution of run-up alone, or of run-up as a ratio of the run-up of the significant wave. Figure 3b shows this relation, but as a ratio to the wave height distribution $(H/H_{1/3})$. Despite the fact that the height distribution[4] is known it appears impossible to accurately obtain the $R/R_{1/3}$ distribution directly from these two known distributions since all three are definitely interrelated.

However, the initial steepness distribution obtained can also be tabulated (from Bretschneider's table) to give percentage values of occurrence of particular steepnesses as associated with particular relative heights $(H/\bar{H})$ and periods $(T/\bar{T})$. Such a table is initially in normalized form, $(H/T^2)/(\overline{H/T^2})$, but may be put in terms of particular steepnesses if a particular value of $\overline{(H/T^2)}$ is assumed. (Note that assumption of a particular value of $\overline{(H/T^2)}$ is tantamount to assumption of a particular value of $H_{1/3}/T^2_{1/3}$). Once this value has been assumed, a similar table for relative run-up (R/H) may be derived from Figure 1 using these steepness values; this table will show the percentage values of occurrence of relative run-up as associated with particular relative heights $(H/\bar{H})$ and periods $(T/\bar{T})$. Each relative run-up (R/H) may then be multiplied by the value of relative height $(H/\bar{H})$ associated with it to obtain a table showing a percentage distribution of $R/\bar{H}$ – that is, run-up as a ratio to the average height of the wave train considered.

These values of $R/\bar{H}$ may then be accumulated to give a frequency diagram. Since $H_{1/3} = 1.6 \bar{H}$, a frequency diagram of $R/H_{1/3}$ may in turn be obtained from this diagram. The desired frequency distribution $R/R_{1/3}$ may be obtained by dividing this $R/H_{1/3}$ distribution by the known value of $R_{1/3}/H_{1/3}$ associated with the significant wave steepness $(H_{1/3}/T^2_{1/3})$ assumed above in obtaining the original non-normalized steepness distribution table.

This process has been carried out for two slopes (1 on 6 and 1 on 2-1/4) using the steepness distribution for fully developed sea (i.e., $H_{1/3}/T^2_{1/3} = 0.06$) and four slopes (1 on 6, 1 on 3, 1 on 2-1/4, and 1 on 1-1/2) assuming a significant wave steepness $H_{1/3}/T^2_{1/3} = 0.22$, as applicable to most normal generating areas. The latter case again

FIGURE 3a APPROXIMATE DISTRIBUTION OF ACTUAL RELATIVE RUN-UP FOR A GENERATING AREA ($N_{1/3}/\tau^2_{1/3} = 0.22$) AS A FUNCTION OF SLOPE (for $d/N_o > 3$)

FIGURE 3b NORMALIZED DISTRIBUTION OF ACTUAL RELATIVE RUN-UP FOR A GENERATING AREA ($H_{1/3}/\tau^2_{1/3} = 0.22$) AS A FUNCTION OF SLOPE (for $d/N_o > 3$)

FIGURE 2 FREQUENCY DIAGRAM OF WAVE STEEPNESS FOR FULLY DEVELOPED SEA

assumes that the Bretschneider distribution for fully developed sea
is also approximately valid for such generating areas.

These distribution curves are shown in Figure 4. They all lie
quite close to one another, particularly in the higher run-up ranges
which are of most interest. Of especial importance is the fact that
the distributions derived for the two wave steepnesses (0.06 and
0.22) are very nearly exactly identical for each of the two slopes
determined.

That portion of the distribution curves for lower values of
$R/R_{1/3}$ is somewhat less exactly determined, probably because of the
relatively large proportionate ranges of $H/\bar{H}$ and $T/\bar{T}$ used to obtain
these values. This use resulted in a noticeable stair-step plot of
the lower end of the distribution, and the curves are drawn as
dashed in this region. For the higher values however the plotted
points showed very little scatter, and a fairly exact curve may be
drawn. The determination of the distribution of the lower values
is primarily of academic interest since, of course, it is the dis-
tribution of values higher than $R_{1/3}$ which is of paramount practical
engineering design value.

The distribution curves for these four slopes are very nearly
the same, and within the limits of approximation implied by the method
of their obtention, may be considered to be the same. The outer
limits of these curves are shown in Figure 5 to describe the band of
frequencies determined. An average line has been drawn within this
band as a single approximate run-up distribution to be used for all
slopes – for waves still in the generating area, and for structure
depths, d/H, greater than 3. This curve is suprisingly close to
the distribution curve for wave heights, which is also shown in
Figure 5. In fact, within the approximations and assumptions used
in obtaining the run-up distribution curve, the run-up distribution
is probably equally well represented by the height distribution curve –
at least in the area of engineering interest which is generally for
the run-ups in excess of $R_{1/3}$. And usage of the height distribution
which is relatively widely known probably facilitates general appli-
cation. From either curve, it is seen that about 13% of the run-
ups exceed the run-up of the significant wave, that about 1% of the
run-ups will be 1.5 times $R_{1/3}$, and that about 1 in 1000 will be 2
times $R_{1/3}$.

Use of the distribution shown in Figure 5 results in appreciably
higher values of run-up (approximately twice the increase in run-up
above $R_{1/3}$) for any particular frequency value than use of the earlier
approximation involving run-ups computed on the basis of steepness
values obtained from the height distribution and a constant significant
period. It is felt, however, to give a more accurate estimate of
what actually occurs in nature.

56

AN APPROXIMATION OF THE WAVE RUN-UP
FREQUENCY DISTRIBUTION

FIGURE 4 FREQUENCY DISTRIBUTION OF WAVE RUN-UP FOR ANY PARTICULAR
WAVE TRAIN AS A FUNCTION OF STRUCTURE SLOPE
(for waves still in the generating area and structure depth $d/H_o > 3$)

FIGURE 5 FREQUENCY DISTRIBUTION OF WAVE RUN-UP
(for waves still in the generating area and structure depth $d/H_o > 3$)

COASTAL ENGINEERING

Derivation of the run-up distribution has been only for relative structure depths, d/H, greater than 3. However, the same process can be carried through for other relative structure depths, and this work is now underway.

Actual validation of such a run-up distribution must await considerably improved field observations, or test in a wave flume which can generate a complex wave train having the statistical properties of actual ocean waves. Such a generator is now under design for one of the Beach Erosion Board wave flumes, and it is hoped its later use will permit more exact empirical test of this run-up distribution. It is quite possible, for example, that the run-up of a particular wave in an actual wave train depends more on the wave height or run-up of the immediately preceding wave, than on its own steepness. But until observations permit accurate checking, the run-up distribution presented herein is felt to be a more realistic estimate than earlier methods.

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