CHAPTER 9

A THEORY FOR WAVES OF FINITE HEIGHT

Charles L. Bretschneider *

ABSTRACT

A theory for waves of finite height, presented in this paper, is an exact theory, to any order for which it is extended. The theory is represented by a summation harmonic series, each term of which is in an unexpanded form. The terms of the series when expanded result in an approximation of the exact theory, and this approximation is identical to Stokes' wave theory extended to the same order. The theory represents irrotational - divergenceless flow. The procedure is to select the form of equations for the coordinates of the particles in anticipation of later operations to be performed in the evaluation of the coefficients of the series. The horizontal and vertical components of these coordinates are given respectively by the following:

$$kx = k(x-\xi) + \sum_{1}^{M} (kA_0)^N \frac{\cosh Nk(\ell + z - \eta)}{\sinh Nk\ell} \sin Nk(x - Ct - \xi)$$

and

$$kz = k(z-\eta) + \sum_{1}^{M} (kA_0)^N \frac{\sinh Nk(\ell + z - \eta)}{\sinh Nk\ell} \cos Nk(x - Ct - \xi)$$

where

x and z are the coordinates; $k = 2\pi/L$, wave number; $A_0 = H/2$, half wave height; $C = L/T$, wave celerity and t is time.** ℓ is a parameter related to the undisturbed mean water depth, d. The constant term $kz_0 = k(\ell - d)$; ξ and η are the horizontal and vertical displacements of the water particles from their respective position of no motion. $a_N = a_1$, a_2, a_3, etc., coefficients of the series as N = 1, 2, 3, etc. to N = M. From the above equations it is possible to deduce the expressions for velocity potential and stream function. The horizontal and vertical components of particle velocity are obtained by differentiating ξ and η with respect to time. Along the free surface $z - \eta = 0$ and $z = \eta_s$ and all expressions reduce to simple forms, which in turn saves considerable work in the evaluation of the coefficients. The coefficients are evaluated by use of Bernoulli's equation. The final form of the solution is given by two sets of equations. One set of equations (same as above) is used to compute the particle position and the second set (the first time derivatives of the above) is used to compute the components of particle velocity at the particle position. That is, the particles and velocities are referenced to the lines of the stream function and the velocity potential. Expanding the two sets of equations, by approximation methods, results in one set of equation for computing particle velocity and no equations are required for the particle position.The unexpanded form requiring two sets of equations,

*Hydraulic Engineer, Beach Erosion Board, U. S. Army Corps of Engineers

** See Appendix for Symbols, p 182

being an exact solution, is more accurate theoretically, than the Stokes or the expanded form to the same order. Coefficients have been formulated for all terms of the order one to five for both the unexpanded and the expanded form of the theory, and are presented in tabular form as functions of d/L, as consecutive equations.

INTRODUCTION

Since 1847 when Stokes first presented his classical work on the theory of oscillatory waves, a number of authors have contributed to this fascinating subject. The reference list, which may not necessarily be complete, is given at the end of this paper. Except for the fact that the various developments given in these references entail certain difficulties and in some cases minor errors, no further discussions will be made thereof.

There is much to be discussed in the present paper and the usual formality of elementaries will be minimized. The inertia of the air and the atmospheric pressure along the wave surface can be neglected; i.e. these quantities are zero with respect to themselves, and the pressure within the fluid is assumed equal in all directions. There is to be no flow across the boundaries, the sea bed being rigid, flat, and impermeable, and the fluid is inviscid. The waves are long crests and x , z , t represent the two dimensional coordinates with respect to time. x is the horizontal direction measured from the crest, positive in the direction of wave propogation. z is the vertical coordinate measured negative below and positive above the undisturbed water elevation. The undisturbed water elevation is the mean water depth, and is that level the water seeks when all wave motion is absent. Finally the flow is irrotational, and since divergenceless, is Laplacian.

The equations by which the motion is described are as follows:

$$p = g\rho z - \rho\frac{\partial\phi}{\partial t} - \frac{1}{2}\rho\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right\}$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

$$\frac{\partial\phi}{\partial z} = 0 \quad \text{at} \quad z = -d$$

$$\frac{\partial p}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial p}{\partial x} + \frac{\partial\phi}{\partial z}\frac{\partial p}{\partial z} = 0\ , \quad \text{when } p = 0$$

Where ϕ is the velocity potential, g acceleration of gravity; ρ density of fluid; and p the pressure.

The first equation is the usual equation of hydrodynamics, the second specifies irrotational flow, the third specifies no flow across the sea bed, and the fourth specifies no flow across the free wave surface.

Coordinates

The coordinates of the particles of water can be represented by a set of equations around which a theory can be developed. If the equations are selected in anticipation of later operations to be performed, then one might be able to minimize the work envolved. In the presence of wave motion the horizontal and vertical displacements (ξ, η) of the water particles from the position of rest or the position of no motion can be represented respectively as follows:

$$k\xi = \sum_{1}^{M} a_N (kA_o)^N \frac{\cosh Nk(\ell + z - \eta)}{\sinh Nk\ell} \sin Nk(x - Ct - \xi) \qquad (1)$$

$$k\eta = \sum_{1}^{M} a_N (kA_o)^N \frac{\sinh Nk(\ell + z - \eta)}{\sinh Nk\ell} \cos Nk(x - Ct - \xi) \qquad (2)$$

(See Figure 1)

It then follows that the x, z coordinates of the particles are obtained from:

$$kx = k(x - \xi) + \sum_{1}^{M} a_N (kA_o)^N \frac{\cosh Nk(\ell + z - \eta)}{\sinh Nk\ell} \sin Nk(x - \xi) \qquad (3)$$

and

$$kz = k(z - \eta) + \sum_{1}^{M} a_N (kA_o)^N \frac{\sinh Nk(\ell + z - \eta)}{\sinh Nk\ell} \cos Nk(x - \xi) \qquad ($$

In the above equations

$2A_o = H$, the wave height, vertical distance between crest and trough

$k = 2\pi/L$, the wave number

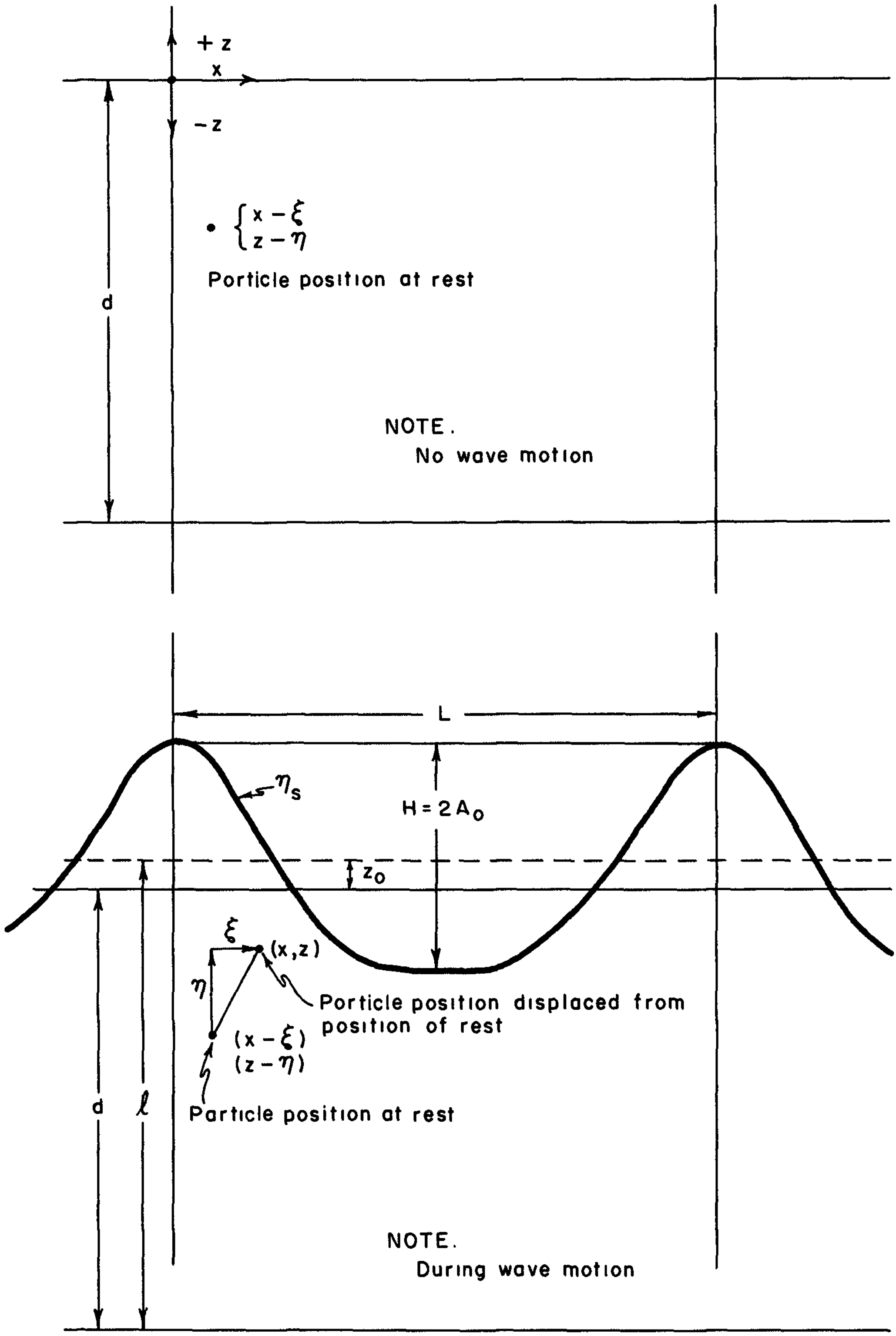

FIGURE I. SYSTEM OF COORDINATES

ℓ is a parameter related to the undisturbed water depth.

$z_o = \ell - d$ is a constant to be determined

x,z, are horizontal and vertical coordinates of the particle

ξ , η are the horizontal and vertical displacements of the particle from its initial undisturbed position of rest.

C = L/T is the wave celerity
L is the wave length
T is the wave period

$a_N = a_1, a_2, a_3, \ldots\ldots\ldots a_M$ (Mth order), consecutive coefficients of the series
N = 1, 2, 3, ... M (Mth order) corresponding to each of the above coefficients.

The parameter ℓ is related to the depth d according to

$$d = \frac{1}{L}\int_0^L \left(d + \eta_s\right) dx = \ell - z_o \tag{5}$$

where η_s is the surface elevation with respect to the undisturbed water elevation, and $kz_o = k(\ell - d)$. (See Figure 1)

The coefficients a_N, with the corresponding subscripts represent a convenient means for keeping track of the various terms of each order; i.e., a_1, is the first order term, a_2 and a_1^2 are the second order, a_3, a_1^3 and a_1 (a_2) are the third order terms, etc.

One of the conveniences of the system of coordinates used in the above equations is that the free surface conditions are obtained by setting $z - \eta = 0$, whence

$$k\eta_s = \sum_1^M a_N \left(kA_o\right)^N \cos Nk\,(x - \xi_s) - kz_o \tag{6}$$

where the constant kz_o is required as shown later

$$kx_s = k(x_s - \xi_s) + \sum_1^M a_N \,(kA_o)^N \frac{1}{\tanh Nk\ell} \sin Nk\,(x_s - \xi_s) \tag{7}$$

Horizontal and Vertical Components of Particle Velocity

The horizontal and vertical components of particle velocity may be obtained respectively from:

$$\frac{u}{C} = -\frac{1}{C}\frac{\partial \xi}{\partial t} = \frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial z} \tag{8}$$

and

$$\frac{w}{C} = \frac{1}{C}\frac{\partial \eta}{\partial t} = \frac{\partial \xi}{\partial z} = -\frac{\partial \eta}{\partial x} \tag{9}$$

whence

$$\begin{aligned}\frac{u}{C} &= (1-\frac{u}{C})\sum Na_N(kA_o)^N \frac{\cosh Nk(\ell+z-\eta)}{\sinh Nk\ell}\cos Nk(x-Ct-\xi)\\ &+ \frac{w}{C}\sum Na_N(kA_o)^N \frac{\sinh Nk(\ell+z-\eta)}{\sinh Nk\ell}\sin Nk(x-Ct-\xi)\end{aligned} \tag{10}$$

and

$$\begin{aligned}\frac{w}{C} &= (1-\frac{u}{C})\sum Na_N(kA_o)^N \frac{\sinh Nk(\ell+z-\eta)}{\sinh Nk\ell}\sin Nk(x-Ct-\xi)\\ &- \frac{w}{C}\sum Na_N(kA_o)^N \frac{\cosh Nk(\ell+z-\eta)}{\sinh Nk\ell}\cos Nk(x-Ct-\xi)\end{aligned} \tag{11}$$

The horizontal and vertical components of particle velocity are also given by:

$$\frac{u}{C} = -\frac{1}{C}\frac{\partial \phi}{\partial x} = -\frac{1}{C}\frac{\partial \psi}{\partial z} \tag{12}$$

and

$$\frac{w}{C} = -\frac{1}{C}\frac{\partial \phi}{\partial z} = +\frac{1}{C}\frac{\partial \psi}{\partial x} \tag{13}$$

Where ϕ and ψ are the velocity potential and the stream function respectively. It is seen from Equations 10 and 11 together with Equations 12 and 13 that the velocity potential and stream function except for arbitrary constants will have the following forms:

$$-\frac{k\phi}{C} = \sum_{1}^{M} a_N(kA_o)^N \frac{\cosh Nk(\ell+z+\psi/C)}{\sinh Nk\ell}\sin Nk(x-Ct+\phi/C) \tag{14}$$

$$-\frac{k\psi}{C} = \sum_{1}^{M} a_N(kA_o)^N \frac{\sinh Nk(\ell+z+\psi/C)}{\sinh Nk\ell}\cos Nk(x-Ct+\phi/C) \tag{15}$$

<u>Proof of Irrotational Flow</u>

Equations 10 and 11 represent irrotational flow irrespective of the actual values of the coefficients. That is:

$$\nabla^2 \phi = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{16}$$

Performing the above operation, it is found that

$$\begin{aligned}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = &-\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) \sum N a_N (kA_0)^N \frac{\cosh Nk(\ell + z - \eta)}{\sinh Nk\ell} \cos Nk(x - Ct - \xi) \\ &+\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) \sum N a_N (kA_0)^N \frac{\sinh Nk(\ell + z - \eta)}{\sinh Nk\ell} \sin Nk(x - Ct - \xi)\end{aligned} \tag{17}$$

The above does not yet prove that $\nabla^2 \phi = 0$ until the following is evaluated

$$\begin{aligned}\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = &-\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) \sum N a_N (kA_0)^N \frac{\cosh Nk(\ell + z - \eta)}{\sinh Nk\ell} \cos Nk(x - Ct - \xi) \\ &-\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) \sum N a_N (kA_0)^N \frac{\sinh Nk(\ell + z - \eta)}{\sinh Nk\ell} \sin Nk(x - Ct - \xi)\end{aligned} \tag{18}$$

For the summation terms of equations 17 and 18 to exist, the only possible solution of the simultaneous equations 17 and 18 is

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad \text{and} \quad \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = 0$$

therefore $\nabla^2 \phi = 0$

The above proof is more easily verified by performing the above operation on the equations given in the next section (Table I, for example).

Power Series Equations for Particle Velocity

In the development following it will be convenient to use

$$k(x - Ct - \xi) = \theta' \tag{19}$$

and

$$U = \sum N a_N (kA_0)^N \frac{\cosh Nk(\ell + z - \eta)}{\sinh Nk\ell} \cos N\theta' \tag{20}$$

and

$$W = \sum N a_N (kA_0)^N \frac{\sinh Nk(\ell + z - \eta)}{\sinh Nk\ell} \sin N\theta' \tag{21}$$

whence

$$\frac{u}{C} = (1 - \frac{u}{C})\, U + \frac{w}{C}\, W \tag{22}$$

and

$$\frac{w}{C} = (1 - \frac{u}{C})\, W - \frac{w}{C}\, U \tag{23}$$

The simultaneous solution of Equations 22 and 23 can be made by the process of resubstitution to as high an order as required where the Mth order will include all terms of U^p, W^q, and $U^r W^s$, where p = 0 to M, q = 0 to M, and r + s = 1 to M.

The process of resubstitutions leads to the following terms:

TABLE 1

M	u/C	w/C
1	U	W
2	$-U^2 + W^2$	$-2WU$
3	$U^3 - 3UW^2$	$-W^3 + 3WU^2$
4	$-U^4 + 6U^2W^2 - W^4$	$4W^3U - 4WU^3$
5	$U^5 - 10U^3W^2 + 5UW^4$	$W^5 - 10W^3U^2 + 5WU^4$
6	$-6U^6 + 15U^4W^2 - 15U^2W^4 + W^6$	$-6UW^5 + 20W^3U^3 - 6WU^5$
7	$U^7 - 21U^5W^2 + 35U^3W^4 - 7UW^6$	$-W^7 + 21W^5U^2 - 35W^3U^4 + 7WU^6$

It will be seen that a general expression can be written for u/C, having the following power series equation:

$$\frac{u}{C} = \left[K_{r,s}\right]_u U^r W^s \tag{24}$$

where

$$\left[K_{r,s}\right]_u = -1\,(-1)^r\,(-1)^{\frac{s}{2}}\;\frac{(r+s)!}{r!\;s!} \tag{25}$$

for $\left\{\begin{matrix} r = 0,1,2,3,4 \\ s = 0,2,4,6,8 \end{matrix}\right\}$ except for $\left\{\begin{matrix} r = 0 \\ s = 0 \end{matrix} \quad K_{0,0} = 0\right\}$

and M = r + s

For example, consider the 8th order (M = 8), in addition to those terms for M = 1 through M = 7, there will be the 8th order term for the following combinations of r,s = (8,0);
(6,2); (4,4); (2,6); and (0,8), whence from equation 24 the 8th order terms for u/C are

$$-U^8 + 28U^6W^2 - 70U^4W^4 + 28U^2W^6 - W^8$$

Similarily for the term w/C the power series equation:

$$\frac{w}{C} = \left[K_{r,s}\right]_w U^r \; W^s \tag{26}$$

where

$$\left[K_{r,s}\right]_w = (-1)^r \; (-1)^{\frac{s-1}{2}} \; \frac{(r+s)!}{r! \; s!} \tag{27}$$

for

$$\left\{\begin{matrix} r = 0,1,2,3,4 \\ s = 1,3,5,7 \end{matrix}\right\} \quad M = r + s$$

For example, the 8th order term will have the following combinations of r,s = (7,1); (5,3); (3,5); and (1,7), whence from equation 27 the 8th order terms for w/C are

$$-8U^7W + 56U^5W^3 - 56U^3W^5 + 8UW^7$$

Thus equations 25 and 27 can be used to obtain all terms from the first order to the Mth order respectively for u/C and w/C

Bernoulli's Equation

The problem of wave motion can be reduced to one of steady state by superimposing a steady current on the wave motion equal to the wave celerity but of opposite direction. This operation, known as the Rayleigh principle, leads to Bernoulli's equation applicable along the free surface, where it is assumed that everywhere along the free surface the pressure is constant or is zero with respect to atmospheric pressure, whence

$$(u_s - C)^2 + w_s^2 + 2g\eta_s \qquad = \text{constant}, \tag{28}$$

where the subscript s refers to the conditions at the free surface, Equation 28 can be written as follows:

$$\left(\frac{u_s}{C} - 1\right)^2 + \left(\frac{w_s}{C}\right)^2 + \frac{2g\eta_s}{C^2} = K ; \text{ constant} \tag{29}$$

or solving for $k\eta_s$

$$k\eta_s = \frac{kC^2}{g}\left\{\frac{u_s}{C} - \frac{1}{2}\left[\left(\frac{u_s}{C}\right)^2 + \left(\frac{w_s}{C}\right)^2\right] + \frac{k-1}{2}\right\} \tag{30}$$

It will be convenient to define the Bernoulli term as

$$B_s = \frac{u_s}{C} - \frac{1}{2}\left[\left(\frac{u_s}{C}\right)^2 + \left(\frac{w_s}{C}\right)^2\right] \tag{31}$$

Along the free surface equations 20 and 21, $z - \eta = 0$, whence

$$U_s = \sum Na_N (kA_0)^N X_N \cos N\theta' \tag{32}$$

and

$$W_s = \sum Na_N (kA_0)^N \sin N\theta' \tag{33}$$

where $X_N = \frac{1}{\tanh Nkl}$

From Table 1, one may obtain the Bernoulli term B_s which leads to the following terms:

TABLE II

Order (M)	$B_s = \frac{u_s}{C} - \frac{1}{2}\left[\left(\frac{u_s}{C}\right)^2 + \left(\frac{w_s}{C}\right)^2\right]$
1	U_s
2	$-3/2\ U_s^2 + \frac{1}{2} W_s^2$
3	$2U_s^3 - 2U_s W_s^2$
4	$-5/2\ U_s^4 + 5U_s^2 W_s^2 - \frac{1}{2} W_s^4$
5	$3U_s^5 - 10U_s^3 W_s^2 + 3U_s W_s^4$
6	$-7/2\ U_s^6 + 35/2\ U_s^4 W_s^2 - 21/2\, U_s^2 W_s^4 + \frac{1}{2} W_s^6$
7	$4U_s^7 - 28U_s^5 W_s^2 + 28U_s^2 W_s^5 - 4U_s W_s^6$

It will be seen that a general expression can be written for B_s, having the following power series equation:

$$B_s = \left[K_{r,s}\right]_{B_s} U_s^{\,r}\, W_s^{\,s} \tag{34}$$

where

$$\left[K_{r,s}\right]_{B_s} = -1(-1)^r\,(-1)^{\frac{s}{2}}\,\frac{(r+s+1)!}{2(r+1)!\,s!} \tag{35}$$

for $\begin{Bmatrix} r = 0,\ 1,\ 2,\ 3,\ 4,\ 5\ \ldots\ldots \\ s = 0,\ 2,\ 4,\ 6,\ 8\ldots\ldots\ldots \end{Bmatrix}$ except for $\begin{Bmatrix} r = 0 \\ s = 0 \end{Bmatrix}$; $K_{0,0} = 0\}$

For example, the 8th order terms will have the following combinationsof (r,s) = (8,0); (6,2); (4,4); (2,6); and (0,8); whence from Equation 35 the 8th order terms for B_s are:

$$-9/2\ U_s^{8} + 42\,U_s^{6}\,W_s^{2} - 63\,U_s^{4}W_s^{4} + 18\,U_s^{2}\,W_s^{6} - \tfrac{1}{2}\,W_s^{8}$$

Thus Equation 35 can be used to obtain the Bernoulli term B_s to as high an order as required. The term B_s will have an expanded form as follows:

$$\begin{aligned} B_s = & \left[B_{11} + B_{13}(kA_0)^2 + B_{15}(kA_0)^4 + B_{17}(kA_0)^6 + \cdot\ \cdot\right] kA_0 \cos\theta' \\ & + \left[B_{22} + B_{24}(kA_0)^2 + B_{26}(kA_0)^4 + \cdots\right](kA_0)^2 \cos 2\theta' \\ & + \left[B_{33} + B_{35}(kA_0)^2 + B_{37}(kA_0)^4 + \cdot\right](kA_0)^3 \cos 3\theta' \\ & + \left[B_{44} + B_{46}(kA_0)^2 + \cdots\right](kA_0)^4 \cos 4\theta' \\ & + \left[B_{55} + B_{57}(kA_0)^2\right](kA_0)^5 \cos 5\theta' \\ & \\ & + \left[B_{JJ} + B_{JJ+2}(kA_0)^2 + \cdot\ \cdot\right](kA_0)^J \cos J\theta' \\ & + B_M (kA_0)^M \cos M\theta' + R \end{aligned} \tag{36}$$

In the above, the first subscript refers to the terms corresponding with identical $(kA_0)^J$ Cos $J\theta'$, J being the general term. The second subscript refers to the order. For example, B_{15} is the fifth order term for Cos θ' , and B_{55} is the fifth order term for Cos 5 θ' R is a constant and represents the sum of the remainder terms for which no Cos N θ' exists.

<u>Procedure for the Evaluation of Coefficients</u>

The coefficients a_1, a_2, a_3...... a_M must be evaluated such that the surface boundary conditions are satisfied. The surface profile elevation with respect to the undisturbed water level is given by Equation 6.

The surface boundary conditions are satisfied when Equation 6 is made identical to Equation 30. To whatever order is required Equation 30 is a means by which the solution is obtained. Incidentally, such a solution is similar to a least squares solution in statistical theory.

It will be convenient to use an expanded form of Equation 30 as follows:

$k\eta_s = \sum a_N \, (kA_0)^N \cos N\theta'$, where

$$
\begin{aligned}
a_1 &= \left[A_{11} + A_{13}(kA_0)^2 + A_{15}(kA_0)^4 + A_{17}(kA_0)^6 + \cdot \right] \\
a_2 &= \left[A_{22} + A_{24}(kA_0)^2 + A_{26}(kA_0)^4 + \right] \\
a_3 &= \left[A_{33} + A_{35}(kA_0)^2 + A_{37}(kA_0)^4 + \cdot\cdot \right] \\
a_4 &= \left[A_{44} + A_{46}(kA_0)^2 + \right] \\
a_5 &= \left[A_{55} + A_{57}(kA_0)^2 + \cdot \cdot\cdot \right] + \cdots - kz_0
\end{aligned}
\qquad (37)
$$

The wave height $H = 2A_0$ is obtained from the difference between η_s at $\theta = 0$ and η_s at $\theta = \pi$, and since A_{11} will always be equal to unity as long as $H = 2A_0$, whence from equation 37,

$$
\begin{aligned}
0 = {} & (A_{13} + A_{33})(kA_0)^2 + (A_{15} + A_{35} + A_{55})(kA_0)^4 \\
& + (A_{17} + A_{37} + A_{57} + A_{77})(kA_0)^6
\end{aligned}
\qquad (38)
$$

Equating to zero terms of $(kA_0)^N$, one obtains the following:

$$
\begin{aligned}
A_{13} &= -A_{33} \\
A_{15} &= -(A_{35} + A_{55}) \\
A_{17} &= -(A_{37} + A_{57} + A_{77})
\end{aligned}
\qquad (39)
$$

etc.

The wave celerity can be expressed as follows:

$$
\frac{kC^2}{g} = F_1 + F_3(kA_0)^2 + F_5(kA_0)^4 + F_7(kA_0)^6 + \cdots\cdots \qquad (40)
$$

Using Bernoulli's Equation 30, together with Equations 37, 39, and 40 and equating like terms of $\cos N\theta$ one obtains the following set of equations:

$$\left[A_{11} + A_{13}(kA_0)^2 + A_{15}(kA_0)^4 + \quad\right] -$$
$$\left[F_1 + F_3(kA_0)^2 + F_5(kA_0)^4 + \cdot\right]\left[B_{11} + B_{13}(kA_0)^2 + B_{15}(kA_0)^4 + \cdots\right] = 0$$
$$\left[A_{22} + A_{24}(kA_0)^2 + \quad\right] - \left[F_1 + F_3(kA_0)^2 + \quad\right]\left[B_{22} + B_{24}(kA_0)^2 + \cdots\right] = 0$$
$$\left[A_{33} + A_{35}(kA_0)^2 + \quad\right] - \left[F_1 + F_3(kA_0)^2 + \cdots\right]\left[B_{33} + B_{35}(kA_0)^2 + \quad\cdot\right] = 0 \qquad (41)$$
$$\left[A_{44} + \quad\right] - \left[F_1 + \cdots\right]\left[B_{44} + \cdot\quad\right] = 0$$
$$\text{etc and } -kz_0 = \left[F_1 + F_3(kA_0)^2 + F_5(kA_0)^4 + \cdot\quad\right]\left[\frac{K-1}{2} + R\right]$$

The procedure is to expand each of the individual equations and then equate to zero like terms of $(KA_o)^N$. It will be convenient to present the higher order terms of the A's and the F 's in terms including the B's terms and the lower order term of A's and F's. Using Equations 41 (and also those of Equation 39) the results are summarized in Table III.

TABLE III

Term	Source	Order
$A_{11} = 1$	$H = 2A_0$	1
$F_1 = 1/B_{11}$	Eq 41	1 ond 2
$A_{22} = F_1\ B_{22}$	Eq 41	2
$A_{33} = F_1\ B_{33}$	Eq 41	3
$A_{13} = -A_{33}$	Eq 39	3
$F_3 = A_{13}F_1 - B_{13}F_1^2$	Eq 41	3 and 4
$A_{44} = F_1\ B_{44}$	Eq 41	4
$A_{24} = F_1\ B_{24} + F_3\ B_{22}$	Eq 41	4
$A_{55} = F_1\ B_{55}$	Eq 41	5
$A_{35} = F_1\ B_{35} + F_3\ B_{33}$	Eq 41	5
$A_{15} = -A_{35} - A_{55}$	Eq 39	5
$F_5 = A_{15}F_1 - F_1^2 B_{15} - F_1 F_3 B_{13}$	Eq 41	5 ond 6
$A_{66} = F_1\ B_{66}$	Eq 41	6
$A_{46} = F_1\ B_{46} + F_3\ B_{44}$	Eq 41	6
$A_{26} = F_1\ B_{26} + F_3\ B_{24} + F_5\ B_{22}$	Eq 41	6

The above scheme can be carried to as high an order as required, merely by writing down the additional terms. For example, the seventh order terms are obtained from Equation 41 as follows:

$$A_{77} = F_1 B_{77}$$

$$A_{57} = F_1 B_{57} + F_3 B_{55}$$

$$A_{37} = F_1 B_{37} + F_3 B_{35} + F_5 B_{33}$$

$$A_{17} = F_1 B_{17} + F_3 B_{15} + F_5 B_{13} + F_7 B_{11}$$

or F_7 from the last equation using $B_{11} = 1/F_1$ is as follows:

$$F_7 = F_1 A_{17} - F_1^2 B_{17} - F_1 F_3 B_{15} - F_1 F_5 B_{13}$$

Similarily the eighthorder terms can be written down directly as follows:

$$A_{88} = F_1 B_{88}$$

$$A_{68} = F_1 B_{68} + F_3 B_{66}$$

$$A_{48} = F_1 B_{48} + F_3 B_{46} + F_5 B_{44}$$

$$A_{28} = F_1 B_{28} + F_3 B_{26} + F_5 B_{24} + F_7 + F_7 B_{22}$$

Thus all expressions presented (Tables I, II, and III) can be carried to as high an order as required, with no difficulty whatsoever. These relations are convenient working parameters for the actual solution to a particular order.

<u>Example: Fifth Order Solution</u>

In order to continue the solution to any particular order, it is necessary to express the B_s - terms in terms of a_N, using Equations 32 and 33 and equations 34 and 35 (Table II). It will be seen from Table II that there will be cross product terms involving $\cos N\theta'$ and $\sin N\theta'$, and it will be necessary to replace these cross product terms using trigonometric identities. For example, the fifth order solution will require the terms of U_s, U_s^2, U_s^3, $U_s W_s^2$, etc. be determined. Using trigonometric identities these terms including all orders from one to five are as follows:

$$U_s = a_1 X_1 (kA_a) \cos\theta' + 2a_2 X_2 (kA_a)^2 \cos 2\theta' + 3a_3 X_3 (kA_a)^3 \cos 3\theta'$$

$$+ 4a_4 X_4 (kA_a)^4 \cos 4\theta' + 5a_5 X_5 (kA_0)^5 \cos 5\theta'$$

$$U_s^2 = \tfrac{1}{2} a_1^2 X_1^2 (kA_a)^2 + 2a_2^2 X_2^2 (kA_a)^4$$

$$+ \left[2a_1 X_1 a_2 X_2 (kA_a)^2 + 6a_2 X_2 a_3 X_3 (kA_a)^4\right] (kA_a) \cos\theta'$$

$$+ \left[\tfrac{1}{2} a_1^2 X_1^2 \quad + \quad 3a_1 X_1 a_3 X_3 (kA_a)^2\right] (kA_0)^2 \cos 2\theta'$$

$$+ \left[2a_1 X_1 a_2 X_2 + 4a_1 X_1 a_4 X_4 (kA_0)^2\right] (kA_a)^3 \cos 3\theta'$$

$$+ \left[2a_2^2 X_2^2 + 3a_1 X_1 a_3 X_3\right] (kA_a)^4 \cos 4\theta'$$

$$+ \left[4a_1 X_1 a_4 X_4 + 6a_2 X_2 a_3 X_3\right] (kA_a)^5 \cos 5\theta'$$

$$U_s^3 = \tfrac{3}{2} a_1^2 X_1^2 a_2 X_2 (kA_a)^4$$

$$+ \left[\tfrac{3}{4} a_1^3 X_1^3 (kA_0)^2 + (6a_1 X_1 a_2^2 X_2^2 + \tfrac{9}{4} a_1^2 X_1^2 a_3 X_3)(kA_0)^4\right] (kA_a) \cos$$

$$+ \left[3a_1^2 X_1^2 a_2 X_2 (kA_a)^2\right] (kA_a)^2 \cos 2\theta'$$

$$+ \left[\tfrac{1}{4} a_1^3 X_1^3 + (\tfrac{9}{2} a_1^2 X_1^2 a_3 X_3 + 3a_1 X_1 a_2^2 X_2^2)(kA_a)^2\right] (kA_a)^3 \cos 3\theta'$$

$$+ \left[\tfrac{3}{2} a_1^2 X_1^2 a_2 X_2\right] (kA_a)^4 \cos 4\theta'$$

$$+ \left[3a_1 X_1 a_2^2 X_2^2 + \tfrac{9}{4} a_1^2 X_1^2 a_3 X_3\right] (kA_a)^5 \cos 5\theta'$$

$$U_s^4 = \tfrac{3}{8} a_1^4 X_1^4 (kA_a)^4 + \left[4a_1^3 X_1^3 a_2 X_2 (kA_a)^4\right] kA_a \cos\theta'$$

$$+ \tfrac{1}{2} a_1^4 X_1^4 (kA_a)^4 \cos 2\theta'$$

$$+ 3a_1^3 X_1^3 a_2 X_2 (kA_a)^5 \cos 3\theta'$$

$$+ \tfrac{1}{8} a_1^4 X_1^4 (kA_a)^4 \cos 4\theta' + a_1^3 X_1^3 a_2 X_2 (kA_a)^5 \cos 5\theta'$$

$$U_s^5 = \tfrac{5}{8} a_1^5 X_1^5 (kA_a)^5 \cos\theta' + \tfrac{5}{16} a_1^5 X_1^5 (kA_0)^5 \cos 3\theta' + \tfrac{1}{16} a_1^5 X_1^5 (kA_0)^5 \cos 5\theta$$

$$W_s^2 = \frac{1}{2} a_1^2 (kA_0)^2 + 2 a_2^2 (kA_0)^4$$
$$+\left[2a_1 a_2 (kA_0)^2 + 6 a_2 a_3 (kA_0)^4 \right] (kA_0) \cos\theta'$$
$$+\left[3a_1 a_3 (kA_0)^2 - \frac{1}{2} a_1^2 \right] (kA_0)^2 \cos 2\theta'$$
$$+\left[4a_1 a_4 (kA_0)^2 - 2a_1 a_2 \right] (kA_0)^3 \cos 3\theta'$$
$$-\left[2a_2^2 + 3a_1 a_3 \right] (kA_0)^4 \cos 4\theta'$$
$$-\left[4a_1 a_4 + 6a_2 a_3 \right] (kA_0)^5 \cos 5\theta'$$

$$W_s^4 = \frac{3}{8} a_1^4 (kA_0)^4 + 2a_1^3 a_2 (kA_0)^5 \cos\theta'$$
$$- \frac{1}{2} a_1^4 (kA_0)^4 \cos 2\theta' - 3a_1^3 a_2 (kA_0)^5 \cos 3\theta'$$
$$+ \frac{1}{8} a_1^4 (kA_0)^4 \cos 4\theta' + a_1^3 a_2 (kA_0)^5 \cos 5\theta'$$

$$U_s W_s^2 = a_1^2 X_1 a_2 (kA_0)^4 - \frac{1}{2} a_1^2 a_2 X_2 (kA_0)^4$$
$$\left[\frac{1}{4} a_1^3 X_1 (kA_0)^2 + \left(2a_1 X_1 a_2^2 + \frac{3}{2} a_1^2 X_1 a_3 - \frac{3}{4} a_1^2 a_3 X_3 \right)(kA_0)^4 \right] (kA_0) \cos\theta'$$
$$+\left[a_1^2 a_2 X_2 (kA_0)^2 \right] (kA_0)^2 \cos 2\theta'$$
$$+\left[\left(\frac{3}{2} a_1^2 a_3 X_3 + 2a_1 a_2^2 X_2 - a_1 X_1 a_2^2 \right)(kA_0)^2 - \frac{1}{4} a_1^3 X_1 \right] (kA_0)^3 \cos 3\theta'$$
$$-\left[a_1^2 X_1 a_2 + \frac{1}{2} a_1^2 a_2 X_2 \right] (kA_0)^4 \cos 4\theta'$$
$$-\left[a_1 X_1 a_2^2 + \frac{3}{2} a_1^2 X_1 a_3 + 2a_1 a_2^2 X_2 + \frac{3}{4} a_1^2 a_3 X_3 \right] (kA_0)^5 \cos 5\theta'$$

$$U_s^2 W_s^2 = \frac{1}{8} a_1^4 X_1^2 (kA_0)^4$$
$$+\left[a_1^3 X_1^2 a_2 (kA_0)^4 \right] (kA_0) \cos\theta'$$
$$-\left[\left(\frac{1}{2} a_1^3 X_1^2 a_2 - \frac{1}{2} a_1^3 X_1 a_2 X_2 \right)(kA_0)^2 \right] (kA_0)^3 \cos 3\theta'$$
$$-\left[\frac{1}{8} a_1^4 X_1^2 \right] (kA_0)^4 \cos 4\theta'$$

$$-\left[\frac{1}{2}\, a_1^3 X_1^2 a_2 + \frac{1}{2}\, a_1^3 X_1\, a_2\, X_2\right] (kA_a)^5 \cos 5\,\theta'$$

$$U_s^3 W_s^2 = \left[\frac{1}{8}\, a_1^5\, X_1^3\, (kA_a)^4\right] kA_a \quad \cos\theta'$$

$$-\left[\frac{1}{16}\, a_1^5\, X_1^5\, (kA_o)^2\right] (kA_a)^3 \cos 3\,\theta'$$

$$-\left[\frac{1}{16}\, a_1^5\, X_1^5\right] (kA_o)^5 \cos 5\,\theta'$$

$$U_s\, W_s^4 = \left[\frac{1}{8}\, a_1^5\, X_1\; (kA_o)^4\right] (kA_a) \cos\theta'$$

$$-\left[\frac{3}{16}\, a_1^5 X_1\, (kA_a)^2\right] (kA_a)^3 \cos 3\,\theta'$$

$$+\left[\frac{1}{16}\, a_1^5 X_1\right] (kA_a)^5 \cos 5\,\theta'$$

Using the above expressions, together with Table II, it will be convenient to summarize the results in Table IV

TABLE IV — Terms of B_s to Fifth Order

R − Terms					
$(kA_0)^2 \cos 0$	$(kA_0) \cos\theta'$	$(kA_0)^2 \cos 2\theta'$	$(kA_0)^3 \cos 3\theta'$	$(kA_0)^4 \cos 4\theta'$	$(kA_0)^5 \cos 5\theta'$
$-3/4\ a_1^2 X_1^2$	$a_1 X_1$	$2 a_2 X_2$	$3 a_3 X_3$	$4 a_4 X_4$	$5 a_5 X_5$
$1/4\ a_1^2$		$-3/4\ a_1^2 X_1^2$	$-3 a_1 X_1 a_2 X_2$	$3 a_2^2 X_2^2$	$6 a_1 X_1 a_4 X_4$
	$(kA_0)^3 \cos\theta'$	$-1/4\ a_1^2$	$-a_1 a_2$	$-9/2\ a_1 X_1 a_3 X_3$	$-9 a_2 X_2 a_3 X_3$
	$-3 a_1 X_1 a_2 X_2$		$1/2\ a_1^3 X_1^3$	$-a_2^2$	$-2 a_1 a_4$
	$a_1 a_2$		$1/2\ a_1^3 X_1$	$-3/2\ a_1 a_3$	$-3 a_2 a_3$
	$3/2\ a_1^3 X_1^3$			$3 a_1^2 X_1^2 a_2 X_2$	$6 a_1 X_1 a_2^2 X_2^2$
	$-1/2\ a_1^3 X_1$			$2 a_1^2 X_1 a_2$	$9/2\ a_1^2 X_1^2 a_3 X_3$
				$a_1^2 a_2 X_2$	$2 a_1 X_1 a_2^2$
$(kA_0)^4$	$(kA_0)^5 \cos\theta'$	$(kA_0)^4 \cos 2\theta'$	$(kA_0)^5 \cos 3\theta'$	$-5/16\ a_1^4 X_1^4$	$3 a_1^2 X_1 a_3$
$-3 a_2^2 X_2^2$	$-9 a_2 X_2 a_3 X_3$	$-9/2\ a_1 X_1 a_3 X_3$	$-6 a_1 X_1 a_4 X_4$	$-5/8\ a_1^4 X_1^2$	$4 a_1 a_2^2 X_2$
a_2^2	$3 a_2 a_3$	$3/2\ a_1 a_3$	$2 a_1 a_4$	$-1/16\ a_1^4$	$3/2\ a_1^2 a_3 X_3$
$3 a_1^2 X_1^2 a_2 X_2$	$12 a_1 X_1 a_2^2 X_2^2$	$6 a_1^2 X_1^2 a_2 X_2$	$9 a_1^2 X_1^2 a_3 X_3$		$-5/2\ a_1^3 X_1^3 a_2 X$
$-2 a_1^2 X_1 a_2$	$9/2\ a_1^2 X_1^2 a_3 X_3$	$-2 a_1^2 a_2 X_2$	$6 a_1 X_1 a_2^2 X_2^2$		$-5/2\ a_1^3 X_1^2 a_2$
$a_1^2 a_2 X_2$	$-4 a_1 X_1 a_2^2$	$-5/4\ a_1^4 X_1^4$	$-3 a_1^2 a_3 X_3$		$-5/2\ a_1^3 X_1 a_2 X$
$-15/16\ a_1^4 X_1^4$	$-3 a_1^2 X_1 a_3$	$1/4\ a_1^4$	$-4 a_1 a_2^2 X_2$		$-1/2\ a_1^3 a_2$
$5/8\ a_1^4 X_1^2$	$3/2\ a_1^2 a_3 X_3$		$2 a_1 X_1 a_2^2$		$3/16\ a_1^5 X_1^5$
$-3/16\ a_1^4$	$-10 a_1^3 X_1^3 a_2 X_2$		$-15/2\ a_1^3 X_1^3 a_2 X_2$		$5/8\ a_1^5 X_1^3$
	$5 a_1^3 X_1^2 a_2$		$-5/2\ a_1^3 X_1^2 a_2$		$3/16\ a_1^5 X_1$
	$-a_1^3 a_2$		$5/2\ a_1^3 X_1 a_2 X_2$		
	$15/8\ a_1^5 X_1^5$		$3/2\ a_1^3 a_2$		
	$-5/4\ a_1^5 X_1^3$		$15/16\ a_1^5 X_1^5$		
	$3/8\ a_1^5 X_1$		$5/8\ a_1^5 X_1^3$		
			$-9/16\ a_1^5 X_1$		

Remembering the forms for a_N (Equation 34) it will be seen that certain a terms upon substitution will be transferred down the table from $(kA_0)^N$ to $(kA_0)^{N+2}$, $(kA_0)^{N+4}$, etc. The substitution and the proper tranfers result in the B_s terms and are conveniently summarized in Table V.

TABLE V — B_s — Terms to Fifth Order

$\left(\frac{-kz_0}{F_1+F_3(kA_0)^2}\right)=$	$B_{11}=$	$B_{22}=$	$B_{33}=$	$B_{44}=$	$B_{55}=$
$\frac{K-1}{2}$	X_1	$2A_{22}X_2$	$3A_{33}X_3$	$4A_{44}X_4$	$5A_{55}X_5$
+		$-3/4\ X_1^2$	$-3X_1A_{22}X_2$	$-3A_{22}^2X_2^2$	$-6X_1A_{44}X$
	$B_{13}=$	$-1/4$	$-A_{22}$	$-9/2\ X_1A_{33}X_3$	$-9A_{22}X_2A_3$
$(kA_0)^2$	$X_1\ A_{13}$		$1/2\ X_1^3$	$-A_{22}^2$	$-2A_{44}$
$-3/4\ X_1^2$	$-3X_1A_{22}X_2$		$1/2\ X_1$	$-3/2\ A_{33}$	$-3A_{22}A_3$
	A_{22}			$3X_1^2A_{22}X_2$	$6X_1A_{22}^2X_2$
$1/4$	$3/2\ X_1^3$			$2X_1A_{22}$	$9/2\ X_1^2A_{33}$
	$-1/2\ X_1$			$A_{22}X_2$	$2X_1A_{22}^2$
				$-5/16\ X_1^4$	$3X_1A_{33}$
+	$B_{15}=$	$B_{24}=$	$B_{35}=$	$-5/8\ X_1^2$	$4A_{22}^2X_2$
	$X_1\ A_{15}$	$2A_{24}X_2$	$3A_{35}X_3$	$-1/16$	$3/2\ A_{33}X_3$
$(kA_0)^4$	$-3X_1X_2A_{13}A_{22}$	$-3/2\ A_{13}X_1^2$	$-3X_1X_2A_{13}A_{22}$		$-5/2\ X_1^3A_{22}$
$-3/2\ X_1^2A_{13}$	$-3X_1X_2A_{24}$	$-1/2\ A_{13}$	$-3X_1X_2A_{24}$		$-5/2\ X_1^2A_2$
$1/2\ A_{13}$	$A_{13}A_{22}$	$-9/2\ X_1A_{33}X_3$	$-A_{13}A_{22}$		$-5/2\ X_1A_{22}$
$-3A_{22}^2X_2^2$	A_{24}	$3/2\ A_{33}$	$-A_{24}$		$-1/2\ A_{22}$
A_{22}^2	$9/2\ X_1^3A_{13}$	$6X_1^2A_{22}X_2$	$3/2\ A_{13}X_1^3$		$3/16\ X_1^5$
$3X_1^2A_{22}X_2$	$-3/2\ X_1\ A_{13}$	$-2A_{22}X_2$	$3/2\ A_{13}X_1$		$5/8\ X_1^3$
$-2X_1A_{22}$	$-9A_{22}X_2A_{33}X_3$	$-5/4\ X_1^4$	$-6X_1A_{44}X_4$		$3/16\ X_1$
$A_{22}X_2$	$3A_{22}A_{33}$	$1/4$	$2A_{44}$		
$-15/16\ X_1^4$	$12X_1A_{22}^2X_2^2$		$9X_1^2A_{33}X_3$		
$5/8\ X_1^2$	$9/2\ X_1^2A_{33}X_3$		$6X_1A_{22}^2X_2^2$		
$-3/16$	$-4X_1A_{22}^2$		$-3A_{33}X_3$		
	$-3X_1A_{33}$		$-4A_{22}^2X_2$		
	$3/2\ A_{33}X_3$		$2X_1A_{22}^2$		
	$-10X_1^3A_{22}X_2$		$-15/2\ X_1^3A_{22}X_2$		
	$5X_1^2A_{22}$		$-5/2\ X_1^2A_{22}$		
	$-A_{22}$		$5/2\ X_1A_{22}X_2$		
	$15/8\ X_1^5$		$3/2\ A_{22}$		
	$-5/4\ X_1^3$		$15/16\ X_1^5$		
	$3/8\ X_1$		$5/8\ X_1^3$		
			$-9/16\ X_1$		

Now from Table V, using the relations of Tables III, one obtains immediately the following fifth order solution.

TABLE VI

$$A_{11} = 1$$

$$F_1 = 1/X_1 = \tanh k\ell$$

$$A_{22} = \frac{F_1}{2F_1 X_2 - 1}\left\{\frac{3X_1^2 + 1}{4}\right\}$$

$$A_{33} = \frac{F_1}{3F_1 X_3 - 1}\left\{A_{22}(3X_1 X_2 + 1) - \frac{X_1}{2}(1 + X_1^2)\right\}$$

$$A_{13} = -A_{33}$$

$$F_3 = F_1 A_{13} - F_1^2\left[A_{13}X_1 - A_{22}(3X_1 X_2 - 1) + \frac{3X_1^3 - X_1}{2}\right]$$

$$A_{44} = \frac{F_1}{4F_1 X_4 - 1}\left\{A_{22}^2(3X_2^2 + 1) + \frac{3}{2}A_{33}(3X_1 X_3 + 1) - A_{22}(3X_1^2 X_2 + 2X_1 + X_2) + \frac{5X_1^4 + 10X_1^2 + 1}{16}\right\}$$

$$A_{24} = \frac{F_1}{2F_1 X_2 - 1}\left\{\frac{A_{13}}{2}(3X_1^2 + 1) + \frac{3}{2}A_{33}(3X_1 X_3 - 1) - 2A_{22}X_2(3X_1^2 - 1) + \frac{1}{4}(5X_1^4 - 1) - \frac{F_3}{4F_1}(8A_{22}X_3 - 3X_1^2 - 1)\right\}$$

$$\begin{aligned} A_{55} = \frac{F_1}{5F_1 X_5 - 1}\Big\{ & 2A_{44}(3X_1 X_4 + 1) + 3A_{22}A_{33}(1 + 3X_2 X_3) \\ & - 2A_{22}^2(3X_1 X_2^2 + X_1 + 2X_2) - \frac{3}{2}A_{33}(3X_1^2 X_3 + 2X_1 + X_3) \\ & + \frac{1}{2}A_{22}(5X_1^3 X_2 + 5X_1^2 + 5X_1 X_2 + 1) - \frac{X_1}{16}(3X_1^4 + 10X_1^2 + 3)\Big\} \end{aligned}$$

$$\begin{aligned} A_{35} = \frac{F_1}{3F_1 X_3 - 1}\Big\{ & A_{13}A_{22}(3X_1 X_2 + 1) + A_{24}(3X_1 X_2 + 1) - \frac{3}{2}A_{13}X_1(1 + X_1^2) \\ & + 2A_{44}(3X_1 X_4 - 1) - 3A_{33}X_3(3X_1^2 - 1) - 2A_{22}^2(3X_1 X_2^2 - 2X_2 + X_1) \\ & + \frac{A_{22}}{2}(15X_1^3 X_2 + 5X_1^2 - 5X_1 X_2 - 3) - \frac{1}{16}(15X_1^5 + 10X_1^3 - 9X_1) \\ & - \frac{F_3}{2F_1}(6A_{33}X_3 - 6A_{22}X_1 X_2 - 2A_{22} + X_1 + X_1^3)\Big\} \end{aligned}$$

TABLE VI con't

$$A_{15} = - A_{35} - A_{55}$$

$$F_5 = \Big\{ F_1 A_{15} - F_1^2 \Big[X_1 A_{15} - A_{13} A_{22}(3 X_1 X_2 - 1) - A_{24}(3 X_1 X_2 - 1) + \frac{3}{2} A_{13} X_1 (3 X_1^2 - 1) - 3 A_{22} A_{33}(3 X_2 X_3 - 1) + 4 A_{22}^2 X_1 (3 X_2^2 - 1) + \frac{3}{2} A_{33}(3 X_1^2 X_3 - 2 X_1 + X_3) - A_{22}(10 X_1^3 X_2 - 5 X_1^2 + 1) + \frac{1}{8}(15 X_1^5 - 10 X_1^3 + 3) - F_1 F_3 \Big[A_{13} X_1 - A_{22}(3 X_1 X_2 - 1) + \frac{X_1}{2}(3 X_1^2 - 1) \Big] \Big\}$$

The constant in Bernoullis' equation is obtained from the first column of Table V, as follows:

$$K = \Big\{ 1 + (kA_0)^2 \frac{3X_1^2 - 1}{2} + (kA_0)^4 \Big[A_{13}(3X_1^2 - 1) + 2A_{22}^2(3X_2^2 - 1) - 2A_{22}(3X_1^2 X_2 - 2X_1 + X_2) + \frac{1}{8}(15X_1^4 - 10X_1^2 + 3) \Big] - \frac{2kz_0}{F_1} \Big[1 - \frac{F_3}{F_1}(kA_0)^2 \Big] \Big\}$$

The above presentation of consecutive equations are in a convenient form for computing the A- terms and the F- terms for any selected value of $k\ell$, either by the long hand method or by use of a high speed computer. For example, consider $k\ell = 2\pi$ (deep water), then one obtains $\tanh k\ell = 1$; in fact, for $k\ell = 2\pi$, $\tanh Nk\ell = 1$, whence $X_1 = X_2 = X_3 = X_4 = X_5 = 1$. It will follow in turn:

$$A_{11} = 1,\ F_1 = 1,\ A_{22} = 1,\ A_{33} = \frac{3}{2},\ A_{13} = -\frac{3}{2},\ F_3 = 1,\ A_{44} = \frac{8}{3}$$

$$A_{24} = -\frac{5}{2},\ A_{55} = \frac{125}{24},\ A_{35} = -\frac{31}{6},\ A_{15} = -\frac{1}{24} \text{ and } F_5 = \frac{1}{2},$$ and

the constant in Bernoullis' equation becomes

$$K = 1 + (kA_0)^2 - 6(kA_0)^4 - 2kz_0\Big[1 - (kA_0)^2\Big]$$

The Undisturbed Mean Water Depth

The undisturbed mean water depth is obtained by use of Equations 5, 6, and 7, in which $\cos Nk(X - \xi_s)$ and $\sin Nk(X - \xi_s)$ are represented by sums of two products each respectively as follows:

$$\cos Nk(X - \xi_s) = \cos Nk\xi_s \cos Nkx + \sin Nk\xi_s \sin Nkx \qquad (42)$$

and

$$\sin Nk(X - \xi_s) = \cos Nk\xi_s \sin Nkx - \sin Nk\xi_s \cos Nkx \qquad (43)$$

Now cos Nkξ_s and sin Nkξ_s can be expanded by series to as high an order as required. For example, the fifth order expansion for equations 6 and 7 are as follows:

$$
\begin{aligned}
k\eta_s = {} & a_1\left[1-\frac{1}{2}(k\xi_s)^2+\frac{1}{24}(k\xi_s)^4\right] kA_0 \cos kx \\
& + a_1\left[k\xi_s-\frac{1}{6}(k\xi_s)^3\right] kA_0 \sin kx \\
& + a_2\left[1-2(k\xi_s)^2\right](kA_0)^2 \cos 2kx \\
& + a_2\left[2k\xi_s-\frac{4}{3}(k\xi_s)^3\right](kA_0)^2 \sin 2kx \\
& + a_3\left[1-\frac{9}{2}(k\xi_s)^2\right](kA_0)^3 \cos 3kx \\
& + a_3\left[3k\xi_s\right](kA_0)^3 \sin 3kx \\
& + a_4 (kA_0)^4 \cos 4kx \\
& + a_4\left[4k\xi_s\right](kA_0)^4 \sin 4kx \\
& + a_5 (kA_0)^5 \cos 5kx - kz_0
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
k\xi_s = {} & a_1 X_1\left[1-\frac{1}{2}(k\xi_s)^2+\frac{1}{24}(k\xi_s)^4\right](kA_0) \sin kx \\
& - a_1 X_1\left[k\xi_s-\frac{1}{6}(k\xi_s)^3\right](kA_0) \cos kx \\
& + a_2 X_2\left[1-2(k\xi_s)^2\right](kA_0)^2 \sin 2kx \\
& - a_2 X_2\left[2k\xi_s-\frac{4}{3}(k\xi_s)^3\right](kA_0)^2 \cos 2kx \\
& + a_3 X_3\left[1-\frac{9}{2}(k\xi_s)^2\right](kA_0)^3 \sin 3kx \\
& - a_3 X_3\left[3k\xi_s\right](kA_0)^3 \cos 3kx \\
& + a_4 X_4 (kA_0)^4 \sin 4kx \\
& - a_4 X_4\left[4k\xi_s\right](kA_0)^4 \cos 4kx \\
& + a_5 X_5 (kA_0)^5 \sin 5kx
\end{aligned}
\tag{45}
$$

In the above equations $X_N = \dfrac{1}{\tanh Nk\ell}$

The procedure for solution is first to eliminate $k\xi_s$ from the right hand side of Equation 45. This is done by the process of re-substitution: the first order is obtained as $k\xi_s = a_1 X_1 \; kA_s \; \sin kx$ and is substituted into equation 45 to obtain the second order, which in turn is again substituted into equation 45 to obtain the third order, etc. until the desired order is obtained. The resulting expression is then substituted into equation 44 to eliminate $k\xi_s$ from the right hand side, obtaining an expression for $k\eta_s$ independent of $k\xi_s$. Finally, this equation for $k\eta_s$ is substituted into equation 5 and the integration results is an expression for d/L as a function of ℓ/L. It will be convenient to write equation 5 as:

$$kz_0 = k(\ell - d) = \frac{1}{L}\int_0^L k\eta_s \; dx \tag{46}$$

It was found to the fourth order (also fifth order) that:

$$kz_0 = \frac{1}{2}\, a_1^2 X_1 \,(kA_0)^2 + a_2^2 X_2 \,(kA_0)^4 \tag{47}$$

Where all other terms vanished by integration. Based on equation 47, the sixth order term was predicted to be $3/2 \; a_3^2 \; X_3 \; (kA_0)^4$, and was then verified by the detailed process of resubstitution and integration. Based on the above findings one can suppose the following power series equation:

$$kz_0 = \frac{1}{2}\sum_{1}^{M} N a_N^2 X_N \,(kA_0)^{2N} \tag{48}$$

where N = 1, 2, 3, ...M, order M = 2N

For example, the eighth order term is found by setting N = 4, which results in

$$2\, a_4^2 X_4 \,(kA_0)^8$$

Since the depth is the known parameter it is desirable to obtain ℓ as a function of d, whence

$$k\ell = k(d + z_0) \tag{49}$$

Where $X_N = \dfrac{1}{\tanh Nk\ell}$, and letting $Y_N = \dfrac{1}{\tanh Nkd}$ by substituting $k(d + z_0)$ for $k\ell$ and using hyperbolic identities (sum of two products) one obtains

$$X_N = \frac{Y_N + \tanh Nkz_0}{1 + Y_N \tanh Nkz_0} \tag{50}$$

Equation 50 can be expanded to as high an order as required according to the following:

$$X_N = \left[Y_N + \tanh Nkz_0 \right]\left[1 - (Y_N \tanh Nkz_0 + (Y_N \tanh Nkz_0)^2 - \cdots \right] \quad (51$$

$$\tanh Nkz_0 = Nkz_0 - \frac{1}{3}(Nkz_0)^3 + \frac{2}{15}(Nkz_0)^5 - \frac{17}{630}(Nkz_0)^7 + \cdot \quad \cdot \Big] \tag{52}$$

Equations 51 and 52 are then used together with equation 47, and by the process of resubstitution kz_o is eliminated from the right hand side, and one obtains a relation of kz_o as a function of kd. For example to the sixth (also seventh order):

$$kz_0 = \frac{1}{2} a_1^2 Y_1 (kA_0)^2 + \left[a_2^2 Y_2 - \frac{1}{4} a_1^4 Y_1 \ (Y_1^2 - 1)\right] (kA_0)^4 \tag{53}$$

$$+ \left[\frac{3}{2} a_3^2 Y_3 - a_1^2 a_2^2 \ \frac{Y_2 (Y_1^2 - 1) + 2Y_1 (Y_2^2 - 1)}{2} + a_1^6 Y_1 (Y_1^2 - 1) \frac{Y_1^2 - 2}{8}\right] (kA_0)^6$$

Returning now to the fifth order solution, and from Table IV

$$a_1 = 1 + A_{13} (kA_0)^2 + A_{15} (kA_0)^4$$

$$a_2 = A_{22} + A_{24} (kA_0)^2 , \quad \text{whence}$$

$$kz_0 = \frac{1}{2} Y_1 \ (kA_0)^2 + \left[A_{22}^2 Y_2 - \frac{1}{4} Y_1 \ (Y_1^2 - 1) + A_{13} Y_1 \right] (kA_0)^4 \tag{54}$$

For the terms A_{22} and A_{13} above for the fifth order tanh $k\ell$ = tanh kd, and using A_{22} and A_{13} as obtained before one obtains for equation 54

$$kz_0 = K_2 \ (kA_0)^2 + K_4 (kA_0)^4 \quad \text{where} \tag{55}$$

$$K_2 = \frac{1}{2} Y_1$$

$$K_4 = \frac{Y_1}{64} \ (17 - 19 Y_1^2 - 21 Y_1^4 - 9 Y_1^6)$$

Accelerations

The horizontal and vertical components for the accelerations of the fluid particles are obtained respectively from the following expressions:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{1}{2}\left[\frac{\partial u^2}{\partial x} + \frac{\partial w^2}{\partial x}\right] \tag{56}$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{1}{2}\left[\frac{\partial u^2}{\partial z} + \frac{\partial w^2}{\partial z}\right] \tag{57}$$

The differential quantities on the right side of the above equations can be obtained by use of equations 24 and 26 together with equations 20 and 21, whence

$$\frac{\partial u}{\partial t} = C \left[K_{r,s}\right]_u \left[r\, U^{r-1} W^s \frac{\partial U}{\partial t} + s\, U^r W^{s-1} \frac{\partial W}{\partial t}\right] \tag{58}$$

$$\frac{\partial w}{\partial t} = C \left[K_{r,s}\right]_w \left[r\, U^{r-1} W^s \frac{\partial U}{\partial t} + s\, U^r W^{s-1} \frac{\partial W}{\partial t}\right] \tag{59}$$

$$\frac{1}{2}\frac{\partial u^2}{\partial x} = C^2 \left[K_{r,s}\right]_u U^r W^s \left[r\, U^{r-1} W^s \frac{\partial U}{\partial x} + s\, U^r W^{s-1} \frac{\partial W}{\partial x}\right] \tag{60}$$

$$\frac{1}{2}\frac{\partial w^2}{\partial x} = C^2 \left[K_{r,s}\right]_w U^r W^s \left[r\, U^{r-1} W^s \frac{\partial U}{\partial x} + s\, U^r W^{s-1} \frac{\partial W}{\partial x}\right] \tag{61}$$

$$\frac{1}{2}\frac{\partial u^2}{\partial z} = C^2 \left[K_{r,s}\right]_u U^r W^s \left[r\, U^{r-1} W^s \frac{\partial U}{\partial z} + s\, U^r W^{s-1} \frac{\partial W}{\partial z}\right] \tag{62}$$

$$\frac{1}{2}\frac{\partial w^2}{\partial z} = C^2 \left[K_{r,s}\right]_w U^r W^s \left[r\, U^{r-1} W^s \frac{\partial U}{\partial z} + s\, U^r W^{s-1} \frac{\partial W}{\partial z}\right] \tag{63}$$

In the above $\left[K_{r,s}\right]_u$ and $\left[K_{r,s}\right]_w$ are given respectively by equations 25 and 27.

Now $\frac{\partial U}{\partial t}$ and $\frac{\partial W}{\partial t}$ can be obtained from equations 20 and 21 respectively as follows:

$$\frac{\partial U}{\partial t} = \left(1 - \frac{u}{C}\right) kC \sum N^2 a_N (kA_0)^N \frac{\cosh Nk(\ell + z - \eta)}{\sinh Nk\ell} \sin N\theta' \tag{64}$$

$$- \left(\frac{w}{C}\right) kC \sum N^2 a_N (kA_0)^N \frac{\sinh Nk(\ell + z - \eta)}{\sinh Nk\ell} \cos N\theta'$$

$$\frac{\partial W}{\partial t} = \left(1 - \frac{u}{C}\right) kC \sum N^2 a_N (kA_0)^N \frac{\sinh Nk(\ell + z - \eta)}{\sinh Nk\ell} \cos N\theta' \tag{65}$$

$$- \left(\frac{w}{C}\right) kC \sum N^2 a_N (kA_0)^N \frac{\cosh Nk(\ell + z - \eta)}{\sinh Nk\ell} \sin N\theta'$$

In addition, one obtains the following:

$$\frac{\partial U}{\partial x} = -\frac{1}{C}\frac{\partial U}{\partial t} \tag{66}$$

$$\frac{\partial W}{\partial x} = -\frac{1}{C}\frac{\partial W}{\partial t} \tag{67}$$

$$\frac{\partial U}{\partial z} = \frac{\partial W}{\partial x} = -\frac{1}{C}\frac{\partial W}{\partial t} \tag{68}$$

$$\frac{\partial W}{\partial z} = -\frac{\partial U}{\partial x} = \frac{1}{C}\frac{\partial U}{\partial t} \tag{69}$$

Procedure for Computation

The first operation to be performed is the evaluation of the coefficients a_N, for example, the fifth order solution as outlined in Table V. This is done by selecting H, ℓ, and L, and perform computations to obtain the required a_N coefficients, water depth d and wave period T. These evaluations are then used to obtain expressions for the surface profile and the velocity potential.

The next step is to select $k(x - \xi)$ and $k(z - \eta)$, coordinates of the undisturbed particle positions, and from equations 3 and 4 compute kx and kz the coordinates of the particles. The surface profile is given for $z - \eta = 0$.

The next step is to compute U, W, $\frac{\partial U}{\partial t}$, $\frac{\partial W}{\partial t}$, $\frac{\partial U}{\partial x}$, $\frac{\partial W}{\partial x}$, $\partial U/\partial z$, and $\partial W/\partial z$, respectively by use of equations 20, 21, 64, 65, 66, 67, 68, and 69.

The horizontal and vertical components of $u, w, \frac{du}{dt}$ and $\frac{dw}{dt}$ are then obtained respectively using equations (24, 25), (26, 27), (56, 58, 60, 61) and (57, 59, 62, 63).

Transformation of equations to the form of Stokes'

The previous development resulted in equations in an unexpanded form. These equations can be expanded, using suitable approximations, and it will be shown that the expanded forms are identical to those obtained as outlined in Stokes' solution. The procedure is to expand the following identities.

$\cosh Nk\,(\ell + z - \eta) = \cosh Nk\,(d+z)\,\cosh Nk\,(z_0 - \eta) + \sinh Nk\,(d+z)\,\sinh Nk\,(z_0 - \eta)$

$\text{sink}\ Nk\,(\ell + z - \eta) = \sinh Nk\,(d+z)\,\cosh Nk\,(z_0 - \eta) + \cosh Nk\,(d+z)\,\sinh Nk\,(z_0 - \eta)$

$\text{sink}\ Nk\ell = \sinh Nkd\ \cosh Nkz_0 + \cosh Nkd\ \sinh Nkz_0$

$\cos Nk\,(x - \xi) = \cos Nkx\ \cos Nk\xi + \sin Nkx\ \sin Nk\xi$

$\sin Nk\,(x - \xi) = \sin Nkx\ \sin Nk\xi - \cos Nkx\ \sin Nk\xi$

In the above equations the expressions involving $k\xi$ and $k\eta$ are expanded by series to as high an order as required, and by the process of resubstitution expressions are obtained for $k\xi$, $k\eta$, u/C w/C in the expanded form.

For example, consider the second order solution involving the expansion of equations 6 and 7.

$$k\eta_s = a_1\,kA_0\left[\cos kx + k\xi_s \sin kx\right] + a_2\,(kA_0)^2 \cos 2kx - kz_0 \tag{70}$$

$$k\xi_s = a_1\,X_1\,(kA_0)\left[\sin kx - k\xi_s \cos kx\right] + a_2 X_2\,(kA_0)^2 \sin 2kx \tag{71}$$

For the second order it will be seen from equation 54 $kz_o = \frac{1}{2} Y_1\,(kA_0)^2$ and from equation 50 that $X_1 = Y_1$ and $X_2 = Y_2$. For the third order $X_2 = Y_2$ $X_3 = Y_3$, but

$$X_1 = Y_1\left[1 + (kA_o)^2\ \frac{Y_1^2 - 1}{2}\right]$$

The first order solution of equation 71 is $k\xi_s = a_1 X_1 (kA_o) \sin kX$, and is substituted into equations 70 and 71 to obtain the second order equations.

$$k\xi_s = a_1\,X_1\,kA_0 \sin kx + \left(a_2 X_2 - \frac{1}{2} a_1^2 X_1^2\right)(kA_0)^2 \sin 2kx \tag{72}$$

$$k\eta_s = a_1\,(kA_0) \cos kx + \left(a_2 - \frac{1}{2} a_1^2 X_1\right)(kA_0)^2 \cos 2kx \tag{73}$$

Using $a_2 = A_{22}$ and $a_1 = A_{11} = 1$ as given before, equation 73 for the surface profile becomes:

$$\eta_s / A_0 = \cos kx + \frac{3 - \tanh^2 kd}{4 \tanh^3 kd}\,(kA_0)\,\cos 2kx \tag{74}$$

Which is identical to Stokes' solution

Consider now the second order solution for particle velocity for which from Table I

$$\frac{u}{C} = U - U^2 + W^2 \tag{75}$$

$$\frac{w}{C} = W - 2WU \tag{76}$$

To the second order the expanded forms of U and W (equations 20 and 21) become:

$$U = \frac{a_1 kA_0}{\sinh kd}\Big[\cosh k(d+z)\cos k(x-Ct) + k\xi \cosh k(d+z)\sin k(x-Ct) - k\eta \sinh k(d+z)\cos k(x-Ct)\Big] + 2a_2(kA_0)^2 \frac{\cosh 2k(d+z)}{\sinh 2kd}\cos 2k(x-Ct) \tag{77}$$

and

$$W = \frac{a_1 kA_0}{\sinh kd}\Big[\sinh k(d+z)\sin k(x-Ct) - k\xi \sinh k(d+z)\cos k(x-Ct) - k\eta \cosh k(d+z)\sin k(x-Ct)\Big] + 2a_2(kA_0)^2 \frac{\sinh 2k(d+z)}{\sinh 2kd}\sin 2k(x-Ct) \tag{78}$$

The first order solution for $k\xi$ and $k\eta$ for substitution in the above are obtained from equations 1 and 2, respectively as follows:

$$k\xi = a_1 (kA_0) \frac{\cosh k(d+z)}{\sinh kd} \sin kx \tag{79}$$

$$k\eta = a_1 (kA_0) \frac{\sinh k(d+z)}{\sinh kd} \cos kx \tag{80}$$

Substituting equations 79 and 80 into equations 77 and 78 one obtains the following:

$$U = a_1 kA_0 \frac{\cosh k(d+z)}{\sinh kd} \cosh x + 2a_2 (kA_0)^2 \frac{\cosh 2k(d+z)}{\sinh 2kd} \cos 2k(x-Ct)$$
$$+ a_1^2 (kA_0)^2 \frac{1-\cosh 2k(d+z)\cos 2k(x-Ct)}{2\sinh^2 kd}$$

and

(8

$$W = a_1 (kA_0) \frac{\sinh k(d+z)}{\sinh kd} \sin k(x-Ct) + 2a_2 (kA_0)^2 \frac{\sinh 2k(d+z)}{\sinh 2kd} \sin 2k(x-$$
$$- a_1^2 (kA_0)^2 \frac{\sinh 2k(d+z)}{2\sinh^2 kd} \sin 2k(x-Ct)$$

Substituting equations 81 and 82 into equations 75 and 76, one obtains:

$$\frac{u}{C} = a_1 kA_0 \frac{\cosh k(d+z)}{\sinh kd} \cos k(x-Ct) \qquad (83)$$
$$+ \left[\frac{2a_2}{\sinh 2kd} - \frac{a_1^2}{\sinh^2 kd}\right] (kA_0)^2 \cosh 2k(d+z) \cos 2k(x-Ct)$$

$$\frac{w}{C} = a_1 kA_0 \frac{\sinh k(d+z)}{\sinh kd} \sin k(x-Ct) \qquad (84)$$
$$+ \left[\frac{2a_2}{\sinh 2kd} - \frac{a_1^2}{\sinh^2 kd}\right] (kA_0)^2 \sinh 2k(d+z) \sin 2k(x-Ct)$$

Using $a_2 = A_{22}$ and $a_1 = A_{11} = 1$, as given before, equations 83 and 84 become:

$$\frac{u}{C} = (kA_0) \frac{\cosh k(d+z)}{\sinh kd} \cos k(x-Ct) + \frac{3}{4} (kA_0)^2 \frac{\cosh 2k(d+z)}{\sinh^4 kd} \cos 2k(x-Ct) \qquad (85)$$

$$\frac{w}{C} = (kA_0) \frac{\sinh k(d+z)}{\sinh kd} \sin k(x-Ct) + \frac{3}{4} (kA_0)^2 \frac{\sinh 2k(d+z)}{\sinh^4 kd} \sin 2k(x-Ct) \qquad (86)$$

In general Stokes' equations can be written as follows:

$$-\frac{k\phi}{C} = \sum_1^M a_N' (kA_0)^N \frac{\cosh Nk(d+z)}{\sinh Nkd} \sin Nk(x-Ct) \qquad (87)$$

$$\frac{u}{C} = \sum_1^M N a_N' (kA_0)^N \frac{\cosh Nk(d+z)}{\sinh Nkd} \cos Nk(x-Ct) \qquad (88)$$

$$\frac{w}{C} = \sum_{1}^{M} N a_N' (kA_0)^N \frac{\sinh Nk(d+z)}{\sinh Nkd} \sin Nk(x-Ct) \qquad (89)$$

$$k\eta_s = \sum_{1}^{M} b_N \ (kA_0)^N \cos Nk(x \ \ Ct) \qquad (90)$$

It will be convenient to write

$$\begin{aligned} a_1' &= 1 + a_{13}(kA_0)^2 + a_{15}(kA_0)^4 + \\ a_2' &= a_{22} + a_{24}(kA_0)^2 + \cdot \\ a_3' &= a_{33} + a_{35}(kA_0)^2 + \\ a_4' &= a_{44} + \cdots \quad \cdot \end{aligned} \qquad (91)$$

etc.

$$\begin{aligned} b_1 &= 1 + \beta_{13}(kA_0)^2 + \beta_{15}(kA_0)^4 + \\ b_2 &= \beta_{22} + \beta_{24}(kA_0)^2 + \cdots \\ b_3 &= \beta_{33} + \beta_{35}(kA_0)^2 + \\ b_4 &= \beta_{44} + \end{aligned} \qquad (92)$$

etc

$$\frac{kC^2}{g} = \gamma_1 + \gamma_3 \ (kA_0)^2 + \gamma_5 \ (kA_0)^4 + \qquad (93)$$

The procedure applied to the second order solution has been extended to the fifth order, using also the expanded relationship of tanh $Nk\ell$. The results of this expansion leads to the following relations for the coefficients:

TABLE VII

$$\gamma_1 = t = \tanh kd$$

$$a_{22} = \frac{3}{4} \ \frac{1-t^2}{t^3}$$

$$\beta_{22} = a_{22} + \frac{1}{2t} = \frac{3-t^2}{4t^3}$$

$$a_{33} = \frac{3+t^2}{8t^2} \left[\beta_{22} \frac{1-t^2}{2t} + a_{22} \frac{1-2t^2}{t} \right]$$

$$\beta_{33} = a_{33} + \frac{1}{8} + \frac{1}{2}\frac{\beta_{22}}{t} + \frac{1}{2} a_{22}\frac{1+t^2}{t}$$

$$a_{13} = -\beta_{33} - \frac{6-t^2}{8t^4}$$

$$\beta_{13} = -\beta_{33}$$

$$\gamma_3 = \beta_{13} - a_{13} + \frac{5}{8} - \frac{\beta_{22} t}{2} + a_{22}\frac{1-t^2}{2t}$$

$$a_{44} = \frac{1+t^2}{5+t^2}\Big[\beta_{33}\frac{1-t^2}{2t^3} + \frac{1-t^2}{48t^3} + a_{22}\frac{1-3t^2}{4t^2}$$
$$+ a_{22}\beta_{22}\frac{1-3t^2}{2t^3} + 3a_{33}\frac{1-3t^2-2t^4}{t^3(3+t^2)} + a_{22}^2\frac{1-2t^2+t^4}{4t^3}\Big]$$

$$\beta_{44} = a_{44} + \frac{1}{4}\beta_{22} + \frac{1}{2}\frac{\beta_{33}}{t} + \frac{1}{48t} + \frac{a_{22}}{2} + a_{22}\beta_{22}\frac{1+t^2}{2t}$$
$$+ 3a_{33}\frac{1+3t^2}{2t(3+t^2)}$$

$$a_{24} = 2a_{22}a_{13} + \frac{47-29t^2}{96t^3} - \frac{3-5t^2}{4t^3}(a_{13}+\beta_{33}) + \beta_{22}\frac{7+3t^2}{8t^2}$$
$$+ \beta_{33}\frac{1-t^2}{2t^3} + a_{22}\frac{1+11t^2-6t^4}{8t^4} + 3a_{33}\frac{1-t^4}{t^3(3+t^2)} + a_{22}(a_{13}+\beta_{33})\frac{1+t^2}{t^2}$$
$$+ a_{22}\beta_{22}\frac{1+t^2}{2t} - a_{22}^2\frac{1-t^4}{2t^3}$$

$$\beta_{24} = a_{24} + 2a_{13}(\beta_{22} - a_{22}) - \frac{a_{13}+\beta_{33}}{2t} + \frac{1}{2}\beta_{22} + \frac{1}{12t}$$
$$+ \frac{1}{2}\frac{\beta_{33}}{t} + a_{22} + 3a_{33}\frac{1+3t^2}{2t(3+t^2)}$$

$$a_{55} = \frac{5+10t^2+t^4}{8t^2(5+3t^2)}\Big[\beta_{22}\frac{1-t^2}{16t} + \beta_{44}\frac{1-t^2}{2t} + a_{22}\beta_{22}\frac{1-5t^2}{4}$$
$$+ a_{22}\beta_{33}\frac{1-3t^2}{2t} + a_{22}\frac{7-15t^2}{48t} + 3a_{33}\frac{1-4t^2}{3+t^2} + \frac{3}{2}a_{33}\beta_{22}\frac{1-6t^2-3t^4}{t(3+t^2)}$$
$$+ a_{44}\frac{1-4t^2-9t^4}{t(1+t^2)} + \frac{3}{2}a_{22}a_{33}\frac{1-2t^2+t^4}{t(3+t^2)}\Big]$$

$$\beta_{55} = a_{55} + \frac{1}{8}\beta_{22}^2 + \frac{1}{4}\beta_{33} + \frac{1}{384} + \frac{1}{2}\frac{\beta_{44}}{t} + \frac{1}{16}\frac{\beta_{22}}{t}$$
$$+ a_{22}\beta_{22} + a_{22}\beta_{33}\frac{1+t^2}{2t} + a_{22}\frac{1+t^2}{12t} + \frac{9}{8}a_{33}$$
$$+ \frac{3}{2}a_{33}\beta_{22}\frac{1+3t^2}{t(3+t^2)} + a_{44}\frac{1+6t^2+t^4}{2t(1+t^2)} .$$

$$\alpha_{35} = 3\alpha_{33}\alpha_{13} + \frac{3+t^2}{8t^2}\Bigg[\frac{17}{192} + \frac{\alpha_{13}+\beta_{33}}{8} + (\beta_{24} - 2\beta_{22}\beta_{13})\frac{1-t^2}{2t}$$

$$+ \frac{1}{4}\beta_{22}^2 t^2 + \beta_{33} + \beta_{44}\frac{1-t^2}{2t} + \beta_{22}\frac{11+t^2}{16t} + \beta_{22}\frac{\alpha_{13}+\beta_{33}}{2}t$$

$$+ (\alpha_{24} - 2\alpha_{22}\alpha_{13})\frac{1-2t^2}{t} - \alpha_{22}(\alpha_{13}+\beta_{33})\frac{1-4t^2}{t} + \alpha_{22}\beta_{22}\frac{7+5t^2}{4}$$

$$+ \alpha_{22}\frac{19-11t^2}{16t} + \frac{3}{8}\alpha_{33}\frac{35-39t^2}{3+t^2} + \alpha_{44}\frac{1+12t^2+7t^4}{t(1+t^2)} + 3\alpha_{33}(\alpha_{13}+\beta_{33})\frac{1+3t^2}{3+t^2}$$

$$+ \frac{3}{2}\alpha_{33}\beta_{22}t\frac{1+3t^2}{3+t^2} - \frac{3}{2}\alpha_{22}\alpha_{33}\frac{1+2t^2-3t^4}{t(3+t^2)} + \alpha_{22}^2\frac{1-6t^2+5t^4}{4t^2}\Bigg]$$

$$\beta_{35} = \alpha_{35} + 3\alpha_{13}(\beta_{33} - \alpha_{33}) - \frac{\alpha_{13}+\beta_{33}}{4} + \frac{\beta_{24} - 2\beta_{22}\alpha_{13}}{2t} - \frac{\alpha_{22}}{2}(\alpha_{13}+\beta_{33})\frac{1+t^2}{t}$$

$$+ \frac{\alpha_{24} - \alpha_{22}\alpha_{13}}{2}\frac{1+t^2}{t} + \frac{1}{2}\beta_{33} + \frac{1}{8}\beta_{22}^2 + \frac{5}{384} + \frac{1}{2}\frac{\beta_{44}}{t} + \frac{3}{16}\frac{\beta_{22}}{t}$$

$$+ \alpha_{22}\beta_{22} + \alpha_{22}\frac{1+t^2}{4t} + \frac{9}{4}\alpha_{33} + \alpha_{44}\frac{1+6t^2+t^4}{2t(1+t^2)}$$

$$\alpha_{15} = -\Bigg\{\beta_{35} + \beta_{55} - 3\alpha_{13}(\beta_{33} + \alpha_{13}) - \frac{3}{4}(\alpha_{13}+\beta_{33}) + \frac{\beta_{24} - 2\beta_{22}\alpha_{13}}{2t}$$

$$+ \frac{5}{192} + \frac{1}{4}\frac{\beta_{22}}{t} + \frac{1}{4}\beta_{22}^2 + \frac{1}{4}\beta_{33} + \frac{\alpha_{24} - 2\alpha_{22}\alpha_{13}}{2}\frac{1+t^2}{t}$$

$$- \frac{\alpha_{22}(\beta_{33}+\alpha_{13})}{2}\frac{1+t^2}{t} + 2\alpha_{22}\beta_{22} + \alpha_{22}\beta_{33}\frac{1+t^2}{2t}$$

$$+ \alpha_{22}\frac{1+t^2}{3t} + \frac{9}{8}\alpha_{33} + \frac{3}{2}\alpha_{33}\beta_{32}\frac{1+3t^2}{t(3+t^2)}\Bigg\}$$

$$\beta_{15} = -\left[\beta_{35} + \beta_{55}\right]$$

$$\gamma_5 = 2\alpha_{13}\gamma_3 - \frac{13}{8}(\alpha_{13}+\beta_{33}) + (\beta_{24} - 2\beta_{22}\alpha_{13})\frac{1-t^2}{2t} + \frac{171}{192}$$

$$+ \frac{3}{8}\left(\frac{2-t^2}{t}\right)\beta_{22} + \frac{1}{4}\beta_{22}^2 t^2 + \frac{\alpha_{24} - 2\alpha_{22}\alpha_{13}}{t} - \alpha_{22}(\alpha_{13}+\beta_{33})\frac{1-2t^2}{t}$$

$$+ \alpha_{22}\beta_{22}(2+t^2) + \alpha_{22}\beta_{33}\frac{1-3t^2}{2t} + \alpha_{22}\frac{65+81t^2}{24t} + \alpha_{33}\frac{12+3t^2}{3+t^2}$$

$$+ 3\alpha_{33}\beta_{22}\frac{1-6t^2-3t^4}{2t(3+t^2)} + \frac{1}{2}\beta_{22}(\alpha_{13}+\beta_{33})t$$

$$+ 3\alpha_{22}\alpha_{33}\frac{1+10t^2+5t^4}{2t(3+t^2)} + \alpha_{22}^2\frac{1+14t^2+17t^4}{4t^2}$$

Using kz_o as determined, etc.

$$K = 1 + (kA_0)^2 \frac{t^2-1}{2} + (kA_0)^4 \left[\frac{18+63t^2-72t^4-59t^6+26t^8+24t^{10}}{64t^8}\right] \quad (\text{'}$$

The above coefficients may be solved conveniently in consecutive orde For example, for deep water t = tanh kd = 1, whence,

$$\gamma_1 = 1, \quad \alpha_{22} = 0, \quad \beta_{22} = \frac{1}{2}$$

$$\alpha_{33} = 0, \; \beta_{33} = \frac{3}{8}, \; \alpha_{13} = -1, \; \beta_{13} = -\frac{3}{8}, \; \gamma_3 = 1, \; \alpha_{44} = 0, \; \beta_{44} = \frac{1}{3}$$

$$\alpha_{24} = \frac{1}{2}, \; \beta_{24} = \frac{1}{3}, \; \alpha_{55} = 0, \; \beta_{55} = \frac{125}{384}, \; \alpha_{35} = \frac{1}{12}, \; \beta_{35} = \frac{99}{128}$$

$$\alpha_{15} = -\frac{7}{16}, \; \beta_{15} = -\frac{211}{192}, \; \gamma_5 = \frac{1}{2}, \; K = 1$$

Loss of Accuracy in the Expanded Form

When the exact solution of the wave problem to a particular Mth order is expanded to obtain the Stokes' solution to the same Mth order there will be a loss of accuracy. The greatest errors will be with the higher order terms. The first term will have minimum error. The reason for the errors arises from the fact that the coefficients a_N of the series (either the expanded or the unexpanded form) are evaluated on the basis of the unexpandec form. The above statement appears somewhat difficult to understand if one inadvertently considers Stokes' solution to be in an exact form to the Mth order. If this is the case, then Stokes' form must be expanded along the free surface (which results in the unexpanded form) prior to substitution into Bernoullis' equation. This operation results in an evaluation of the corresponding coefficients based on the unexpanded form, but are then applied incorrectly to the Stokes' or the expanded form.

For example the velocity potential component for the Mth or last term of the Mth order, for the unexpanded form and Stokes' form are respectively as follows:

$$-\frac{k\phi_M}{C} = a_M (kA_0)^M \frac{\cosh Mk(\ell + z - \eta)}{\sinh Mk\ell} \sin Mk(x - Ct - \xi) \tag{95}$$

and

$$-\frac{k\phi_M}{C} = a_M' (kA_0)^M \frac{\cosh Mkd}{\sinh Mkd} \sin Mk(x - Ct) \tag{96}$$

Along the free surface $z = \eta = \eta_s$ and the above equations become respectively:

$$-\frac{k\phi_{M_s}}{C} = a_M (kA_0)^M \frac{\cosh Mk\ell}{\sinh Mk\ell} \sin Mk(x - Ct - \xi_s) \tag{97}$$

$$-\frac{k\phi_{M_s}}{C} = a_M' (kA_0)^M \frac{\cosh Mk(d + \eta_s)}{\sinh Mkd} \sin Mk(x - Ct) \tag{98}$$

For evaluation of the coefficients of the Mth or last term, the expansion of cosh Mk $(d + \eta_s)$ will be cosh Mkd which is the same idea as $z = \eta = \eta_s = 0$. Any consideration of finite η_s for the Mth term of Stokes' Mth order results in M + 1, M + 2, etc. order terms, which are neglected by the mechanics of the solution.

It then follows that the error in the Mth term of Stokes' solution will be in proportion to:

$$\frac{\cosh Mk\,(d + z)}{\cosh Mk\,(d + z - \eta)}$$

Along the free surface the error will be

$$\frac{\cosh Mk\,(d + \eta_s)}{\cosh Mkd}$$

and along the sea bottom there will be no error since the above ratio reduces to unity.

If one considers the last term of the third order wave theory, M = 3, and for example, the wave H = 35 ft., T = 12 sec. and d = 85 ft., then one obtains L = 581 feet, η_o = 22.1 feet at the crest and $\eta_o - H = -12.9$ feet at the trough and from the above ratio:

$$\frac{\cosh Mk\,(d + \eta_s)}{\cosh Mkd} = \frac{16.057}{7.869} = 2.04 \quad \text{at the crest}$$

and

$$\frac{5.225}{7.869} = 665 \quad \text{at the trough}$$

The deviations of the above ratio from unity reflects considerable error. For the unexpanded form the above ratio is always unity.

For the M-1 or next to the last term of the Mth order, the percent error will be less since the expansion of this term for Stokes' solution will be $\cosh\left[(M-1)\,k\,(d + \eta_s)\right] = \cosh\left[(M-1)\,kd\right] + (M-1)\left[k\,\eta_s \sinh (m-1)\,kd\right]$

In view of the above considerations it appears that the use of Stokes' higher order solutions should be limited to low wave steepness, i.e. η_s small compared with d.

With the aid of electronic computors, the unexpanded form given in the present paper can be utilized easily for computing wave properties and thereby obtain greater accuracy theoretically than by utilizing Stokes' equations.

SUMMARY AND CONCLUSIONS

A theory for waves of finite height, presented in this paper is an exact theory, to any order for which it is extended. Two sets of equations are given in an unexpanded form, when upon expansion represents an approximation to the exact theory, and this approximation is identical to Stokes' theory extended to the same order. The waves are irrotational.

Consecutive order of equations are given which can be used, either by the long hand method of computation or by use of high speed computors for computing the wave properties. These equations have been worked out to the fifth order, both in the exact form and also the approximation or Stokes' form.

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APPENDIX

SYMBOLS

A_o = H/2, half wave height

$a_N = a_1\ a_2,\ a_3$ etc. Coefficients of velocity potential series

$B_s = B_{11},\ B_{13}$, etc. Terms for the Bernoulli Equation

C = L/T Wave celerity

d = Undisturbed mean water depth

$F = F_1,\ F_3,\ F_5$, etc. Higher order terms for wave celerity

g = Acceleration of gravity

H = Wave height, vertical distance between crest and trough

k = $2\pi/L$, Wave number

K = Constant for Bernoulli Equation

ℓ = Parameter related to mean water depth

L = Wave length, horizontal distance between two successive wave crests

M = Mth term of the Mth order

N = 1, 2, 3, 4, to M, Consecutive terms of the series

p = pressure

R = Remainder terms in expansion of equation for particle velocity

r = Exponent

s = Exponent

T = Wave period

t = time also used to denote t = tanh kd

u = horizontal component of particle velocity

u_s = u at the free surface

U = A form of notation used related to u for higher order terms

w = vertical component of particle velocity

w_s = w at the free surface

W = A form of notation used related to w for higher order terms

x = horizontal coordinate of particle

$X_N = X_1, X_2$, etc. = $1/\tanh Nk\ell$

$Y_N = Y_1, Y_2$, etc. = $1/\tanh Nkd$

z = Vertical coordinate of particle

$z_o = \ell - d$

η = Vertical displacement of particle from its undisturbed position of rest

$\eta_s = \eta$ for the free surface

ξ = Horizontal displacement of particle from its undisturbed position of rest

$\xi_s = \xi$ for the free surface

ρ = density

$\Theta = k(x - Ct)$

$\Theta^1 = k(x - Ct - \xi)$

∇^2= Operator

∂ = Notation for partial differential

ϕ = Velocity potential

ψ = Stream function